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Professional development for teachers' mathematical problem-solving pedagogy – what counts?

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Problem-solving is of importance in the teaching and learning of mathematics. Nevertheless, a baseline investigation conducted in 2016 revealed that mathematical problem-solving is virtually missing in South African classrooms. In this regard, a two-cycle design-based research project was conducted to develop a professional development (PD) intervention that can be used to bolster Grade 9 South African teachers' mathematical problem-solving pedagogy (MPSP). This article discusses the factors that emerged as fundamental to such a PD intervention. Four teachers at public secondary schools in Gauteng, South Africa, who were purposively selected, participated in this qualitative research study of a naturalistic inquiry. Teachers attended PD workshops for six months where PD activities that were relevant to their context were implemented. Between the PD workshops, teachers were encouraged to put into practice the new ideas on MPSP. Qualitative data were gathered through reflective interviews and classroom observations which were audio-recorded with teachers' consent. Data were analysed through grounded theory techniques using constant comparison. The findings from the study suggested that teachers' personal meaning, reflective inquiry, and collaborative learning are factors fundamental to their professional growth in MPSP. The major recommendation from the study is that facilitators of PD must acknowledge these factors to promote teachers' professional growth in MPSP. If PD processes and activities are relevant to teachers' personal meaning, reflective inquiry, and collaborative learning, teachers find the PD programme fulfilling and meaningful. This study contributes to the PD in MPSP body of knowledge by having worked with teachers in an under-researched context of historical disadvantage.

Keywords: Mathematics education; mathematical problem-solving pedagogy; personal meaning; professional development; reflective inquiry; collaboration.

Introduction

Historically, South African learners perform poorly in mathematics in national tests of achievement like Annual National Assessments (ANA), in regional tests of achievement such as Southern and Eastern African Consortium for Monitoring Education Quality (SACMEQ), and in international tests of achievement like Trends in Mathematics and Science Study (TIMSS). Of the approximately 270 500 Grade 12 learners who wrote mathematics in the 2018 matric examinations, only 37% passed with 40% and above. These results indicate that South African learners' performance in mathematics is inadequate and this crisis has existed for over 20 years (Van Jaarsveld & Ameen, 2017). In this regard, the Department of Basic Education (DBE) in South Africa propounds problem-solving as another way for improving the teaching and learning of mathematics (DBE, 2011). The Curriculum and Assessment Policy Statement (CAPS), which is South Africa's intended curriculum, states that its objectives are to produce learners who are able to identify and solve problems and make decisions using critical and creative thinking. An analysis of the CAPS curriculum shows that its content centres mainly on the proficiency of mathematical facts, procedures, and skills and the term problem-solving refers mainly to solving problems as in performing calculations (see DBE, 2011). On the other hand, in doing mathematics, learners make explorations, construct arguments and formulate proofs and conjectures, and use problem-solving to evaluate and interpret these actions (Schoenfeld, 2013), such that mathematics should be taught from a problem-solving perspective (NCTM, 2007). In addition, learners' active participation in the problem-solving process encourages robust understanding (Schoenfeld, 2014). Nonetheless, given the importance of problem-solving in the teaching and learning of mathematics, the baseline study that I conducted in March 2016 revealed that participant teachers were unclear on implementing mathematical problem-solving pedagogy (MPSP) and required support in implementing it (see Chirinda & Barmby, 2018). The baseline study investigated South African Grade 9 mathematics teachers' teaching strategies, views on MPSP, and the support they required in implementing MPSP. MPSP refers to the practices that mathematics teachers adopt to assist learners to understand and be able to do problem-solving.

Further findings from the baseline study were that problem-solving is virtually missing in South African classrooms. The findings from the baseline study are consistent with Jagals and Van der Walt's (2016) observation that mathematical problem-solving is limited in the South African implemented curriculum and the assessed curriculum (e.g. national examinations and formative assessments). In light of the findings from the baseline investigation, I developed a professional development (PD) intervention for Grade 9 teachers' MPSP. The PD intervention is a framework that can be adapted by the South African DBE and other mathematics education practitioners to support teachers of mathematics in the teaching of problem-solving. I revisit this later to discuss how I designed the PD intervention.

MPSP is of importance since, besides learners' preconceptions, teaching is widely known to influence learning. Consequently, if teachers are to encourage learners to do problem-solving or to improve learners' problem-solving skills, then teachers need to deepen their MPSP. MPSP can enrich teachers' mathematical experiences because it broadens their views of what it means to do mathematics with learners. Teachers' MPSP can be grown by providing them with PD programmes that focus on the teaching of mathematical problem-solving. This kind of PD is commonly known to improve teachers' confidence in implementing MPSP and their interaction with learners. Subsequently, this improves learners' mathematical problem-solving processes and abilities. However, there is no sufficient literature on research on the PD of South African teachers that focuses on MPSP. Over the years, several longitudinal projects that focus on PD of primary and secondary mathematics teachers have been conducted in South Africa (see Biccard, 2019; Brodie & Borko, 2016; Pournara, Hodgen, Adler, & Pillar, 2015; Venkat, 2019) and none of these projects has focused on the PD of teachers in MPSP. Consequently, PD programmes that focus on teachers' MPSP are under-represented in South Africa. The study reported in this article emerged from my continuing efforts to fill in this existing gap in the literature on the PD of South African teachers in MPSP. The purpose of this study was to explore factors that impact teachers' professional growth in MPSP. This study is essential because for PD to be effective, the factors that support teachers' professional growth must be understood. Specifically, I formulated the following research question: What factors are fundamental to a professional development intervention for teachers' mathematical problem-solving pedagogy?

Setting the context of the study

Context relates to the factors external to teachers and learners that may affect the process of teaching and learning (Lester, 2013). In this study, *context* refers to the many factors surrounding the teaching and learning mathematics environment. These include physical aspects (schools, classrooms, class sizes, and resources), economic and political factors, and the curriculum which comprises the objectives and quality of teaching and learning and teachers' daily grind that encompasses the intended, implemented, and assessed curricula. The factors listed here are not exhaustive of the influences on MPSP in this context and there might be other factors equally important. Before developing the PD intervention for MPSP, I conducted a context analysis to identify the areas that needed attention to generate some useful ideas for the development of the PD intervention. I conducted this study at public South African secondary schools positioned in informal settlements. Informal settlements are the areas where people build their temporary housing structures without the approval of the local government. The South African government has often upgraded these settlements to proper residential townships by providing infrastructure. From the context analysis, it was evident that the curriculum was strictly centralised, overloaded and teachers worked under pressure to cover the prescribed material within the stipulated time. In terms of the teaching and learning environment, the analysis revealed that the classes were multilingual, large, overcrowded, and resource-deficient in terms of technology and furniture. Multilingual classes are instances where learners speak more than two different languages in one class. Most learners came from poor homes where there were neither desks nor electricity to do homework after school.

Mathematical problem-solving teaching

This study focused on designing a PD for teaching mathematical problem-solving. Problem-solving is when a problem-solver is confronted by a mathematical problem for which they have no known solution (Schoenfeld, 2013) or are unaware of how to solve it promptly (Newell & Simon, 1972). Several scholars agree that problem-solving is when a goal exists, and the problem-solver does not know how to attain it (Lester, 2013). Polya (1957) suggested a four-step process that a problem-solver experiences during the process of problem-solving. Polya proposed that understanding the given problem prior to solving it is the first step of the problem-solving process. It involves learners familiarising themselves with the given problem and pinpointing the unknowns and the useful data (Barmby, Bolden, & Thompson, 2014). Polya's second problem-solving step is devising the plan, where learners must gather all available information and ponder possible problem-solving strategies to apply. These strategies can be making a table or a chart, problem reformulation, guess and check, etc. The third problem-solving step is carrying out the plan where learners implement the selected problem-solving strategy. The final step is looking back at the answer and problem-solving strategy to assess if they make sense and fit with the original problem. During this step, the teacher requires learners to justify their actions and consolidate what they would have learned. Polya's steps are linear, but the problem-solving process is rarely sequential or straightforward as depicted by the model (Reys et al., 2012).

During the problem-solving process, learners rarely solve problems linearly but frequently skip or revisit steps and move back and forth among the steps. Over the years, I have noticed that when presented with a problem, learners usually get better comprehension during the devising a plan stage and may need to look back at how they understood the problem or devised the plan; therefore, the process of problem-solving is seldom linear.

In this study, Polya's (1957) four steps were used as a guideline during the workshops and implementation of the MPSP by the teachers. Polya's steps only provide guidance on how teachers can teach learners problem-solving, but do not provide ways teachers can identify learners' difficulties in problem-solving or skills they require to do problem-solving successfully. MPSP is complex, as observed by Mason (2016):

> In order for problem-solving to become an integral part of learners' experience in school and university, all aspects of the human psyche, cognition, affect, behavior, attention, will and metacognition or witnessing must be involved. Focusing on only one or two aspects is simply inadequate and very unlikely to lead to full-scale integration into learners' ways of being in the world. (p. 109)

The picture presented by Mason (2016) is intricate in that teachers are on a daily basis faced with the intended curriculum, which forms part of the implemented curriculum that, in turn, is impacted by the examined curriculum.

Schoenfeld (2016, p. 20) recognises teachers' daily grind and notes that 'individuals do not work, or learn, in a vacuum', but the context shapes the MPSP and the problem-solving process. In the South African context, learners are faced with aspects of the language of learning and teaching. English is the language of learning and teaching in South Africa, yet 76% of the population has an indigenous South African language as their first language (Statistics South Africa, 2017). This implies that most South African learners are learning in a language that is not their mother tongue. Polya (1957) did not incorporate language issues in his four-step problem-solving process, yet language and the process of problem-solving are not separate (Chirinda & Barmby, 2017). During problem-solving, learners need to understand the problem first before attempting to solve it, yet the language of learning and teaching is a challenge to most learners in the South African context.

Professional development in mathematical problem-solving pedagogy

Professional development involves the programmes implemented to enhance teachers' knowledge and practices with the key objective of improving learner achievement (Loucks-Horsley, Hewson, Love, & Stiles, 2010). PD in MPSP focuses on supporting teachers to be able to teach problem-solving effectively. Several PD projects that focus on supporting teachers' MPSP have been conducted around the world. In Chile, the Activando la Resolución de Problemas en las Aulas (ARPA) Initiative PD programme, which focuses on integrating problem-solving into teachers' practices, has been running since 2014 (Felmer, Perdomo-Díaz, & Reyes, 2019). In the ARPA Initiative PD programme, teachers attend workshops for a year where they participate in problem-solving activities and how to introduce them to their learners. The major finding from the programme was that a PD in problem-solving pedagogy should be flexible to accommodate different teachers, contexts, schools, etc. (Felmer et al., 2019). In Australia, Widjaja, Vale, Groves and Doig (2017) conducted a lesson study project with 10 elementary school teachers and numeracy coaches. Widjaja et al. examined the participants' PD from implementing structured mathematical problem-solving classes. The findings reported that the project had a positive influence on teachers' MPSP and beliefs in MPSP. An after-school research programme, the Informal Mathematical Learning, was conducted in the United States for three years. This particular project comprised a PD component that focused on assisting mathematics teachers from a local school district to improve their capability to design and enact mathematical problem-solving activities. Findings revealed how the teachers gave guidance to learners on problem-solving (see Weber, Radu, Mueller, Powell, & Maher, 2010). In Israel, an in-service PD programme was conducted with 170 teachers who participated in a mathematical activity 'Solving problems in two different ways' that focused on the notion of symmetry (see Leikin, Berman, & Zaslavsky, 2000). The purpose of the PD programme was to increase participant teachers' problem-solving expertise. Findings were that teachers started implementing various problem-solving strategies in their lessons because they found them helpful to themselves and their learners.

The above programmes are from around the world; however, in South Africa there are few programmes that explicitly support teachers in teaching mathematical problem-solving. To address this identified gap, in the large project, I developed a PD intervention for teachers' MPSP. The study reported in this article focuses on the factors fundamental to such a PD intervention. In the context of the study, the PD intervention was about supporting teachers who worked with mathematical word problems found in the CAPS curriculum. Mathematical word problems are verbal or (con)textual descriptions of a problem situation (Verschaffel et al., 1994) that incorporate mathematics tasks implanted in real-world situations.

The adult learning theory: Andragogy

I adopted andragogy, the adult learning theory, as a theoretical framework for this study. In 1968, Malcolm Shephard Knowles proposed a theory that distinguished adult learning (andragogy) from children learning (pedagogy). Andragogy is the art and science of adult learning. It refers to any kind of adult learning (Kearsley, 2010). Knowles (1984) suggested five assumptions about adult learning: adults have a developed self-concept, previous experience, practical reasons for learning, and are ready to learn since they are internally motivated. The first assumption is that adult learners have an established sense of self (Knowles, Holton, & Swanson, 2005). As human beings mature, they move from being dependent to being independent and form a complete entity called 'self'. They become distinct from those around them and require a sense of independence in learning.

Consequently, they prefer a more self-directed learning approach rather than being teacher-led. In this study, teachers were contributors to their own PD experiences in MPSP: they were actively involved in the workshops' activities and the implementation of the MPSP. Thus, teachers had control over their learning. Secondly, past experiences are important in adult learning (Knowles et al., 2005). As people mature, they accumulate academic and social experience from which they can learn. Accordingly, to develop teachers professionally, facilitators need to build on their existing knowledge. Facilitators need to merge new ideas with teachers' lived experiences. In this regard, I did not go into teachers' contexts as a knowledgeable person on MPSP. Instead, I conducted a baseline study and started the PD intervention from teachers' views and experiences of MPSP. In this way, I created an environment where teachers felt that they had ownership of the PD intervention.

Thirdly, adults are ready to learn when there is a purpose and relevance, for example when it is about professional growth connected to their work. Adult learning experiences must be scheduled so that they are concurrent with their readiness to learn (Knowles et al., 2005). The baseline investigation disclosed that participant teachers were unclear on implementing MPSP and indicated that they required support in implementing it. This suggested that teachers were ready to participate in the PD intervention.

Fourthly, adults are interested in learning practical skills that help them to solve problems. As human beings mature, their orientation towards learning switches from subject-centeredness to problem-centeredness. Teachers were interested in participating in the PD for MPSP because it would help them assist their learners in doing mathematical problem-solving. Finally, adult learners are driven by internal motivation. Adults develop their methods of motivation and are driven to learn for their reasons, for example for promotion or to bolster self-esteem. Knowles (1984, p. 12) observed that as people 'mature the motivation to learn is internal'. Adults are motivated to learn if they believe that what they are learning is relevant to their life and profession (Knowles et al., 2005). During the baseline investigation, teachers complained that they had experienced PD that was not usable in their practice. Teachers' exhaustion from low-quality and irrelevant PD programmes was my primary motivation to engage design-based research (DBR) as a research methodology.

Knowles (1984) also offered four principles that can be applied to adult learning education to make it more effective. The first principle is that adults require to be actively involved in the planning and evaluating of their learning. Secondly, experience provides the foundation for the learning activities. Thirdly, adults are captivated by learning content that is immediately relevant to their job or personal life. Fourthly, adults are inclined towards problem-centred learning rather than content-oriented (Kearsley, 2010). To this end, andragogy was relevant as a lens to design the teachers' PD in MPSP which was self-directed, relevant, contextual, task-based, less theoretical, and more practical. The five assumptions and four principles of andragogy were also used as a lens in designing a semi-structured interview schedule that was used during the reflective interviews.

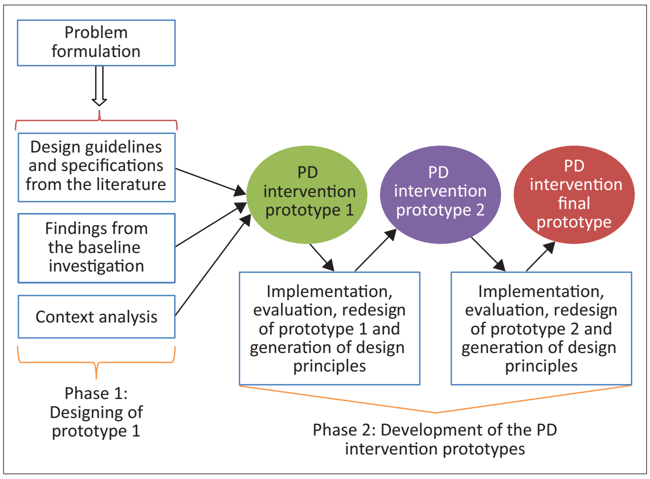
The design-based research project

Sztajn, Borko and Smith (2017) observe that many researchers inadequately recount the design and content of their PD programmes. I avoid the identified pitfall by giving a detailed description of the PD programme for this study in this section. A DBR project was initiated in 2016 and focused on designing, implementing, evaluating, and redesigning a PD intervention that can be used to support Grade 9 South African teachers in teaching mathematical problem-solving. The context of the study was a crucial aspect; therefore, I engaged DBR to develop a PD intervention that was relevant to the teachers' context. DBR is a research design that utilises cycles and was appropriate for this study since, at the beginning, I did not have in mind an example of a PD intervention for Grade 9 teachers' MPSP but wanted to develop one that was sensitive to teachers' context.

After identifying the need to design a PD intervention to support Grade 9 teachers in MPSP, the first phase involved a literature review, context analysis, and the baseline study. The second phase involved two cycles of design-implementation-evaluation-redesign of prototypes of the PD intervention. I designed the first prototype from the design principles generated from the literature review, context analysis, and the baseline investigation.

I implemented the first prototype in the first cycle. Next, I designed the second prototype from the design principles generated from the first cycle and I implemented it in the second cycle. From the design principles generated in the second cycle, I redesigned the second prototype to develop the third prototype. Figure 1 illustrates the study's research design.

In the first cycle, three PD workshops were carried out with teachers on the last Wednesday of the first, third, and fifth months of a six-month period. In the second cycle, the workshops were on the last Friday of the first, third, and fifth months of the six-month intervention period. Each workshop was three hours long and this amounted to nine hours of training for teachers in each cycle. The first workshop had five main activities, which were 30 minutes long, with a 30-minute break between the third and fourth activities. The activities



Source: Adapted from Mafumiko, F. (2006). *Micro-scale chemistry experiments as a catalyst for improving the chemistry curriculum in Tanzania*. Unpublished PhD thesis, University of Twente

FIGURE 1: The research design of the study.

were theoretically informed by the literature, the context, my personal preferences as a researcher, the adult learning theory, and the intended, implemented, and examined curricula that the teachers were dealing with on a daily basis.

Activity 1: Teachers watched two short videos either from Japan, Singapore, or the United States on MPSP. This study supported teachers' mathematical problem-solving pedagogy using Polya's (1957) problem-solving framework as a guideline; therefore, the videos clearly showed teachers implementing Polya's four problem-solving processes. The videos, although from different contexts, were to initiate discussions about MPSP, and inspire and model the teaching of mathematical problem-solving to participant teachers. I discussed the videos with the teachers, focusing on what MPSP entails and how to apply Polya's four steps of problem-solving in their practice. At the beginning of the workshop, I had provided teachers with Polya's *How to solve it* book that focuses on mathematical problem-solving. We used it as a guideline during the discussions.

Activity 2: Teachers collaboratively solved at least two word problems relating to the topic they were teaching at the time and simulated how to teach these problems to their learners. Teachers worked through the problems by going through Polya's (1957) problem-solving steps, that is, understanding the given problem prior to solving it, devising the plan, carrying out the plan, and looking back at the answer. Teachers shared ownership of the problem, its solution strategy, and the final solution. Teachers expounded on ways of introducing or posing these problems so that learners understood the given problems. The following are examples of Grade 9 mathematics word problems that were used during the workshops:

- Elizabeth is going to her sister's wedding this morning. She began driving at 7 am on the N1 highway at a constant speed of 80 km per hour. At 7:30 am, Elizabeth's mother followed her and began driving along the same highway (N1) at a constant speed of 95 km per hour. At what time will Elizabeth's mother catch up with Elizabeth?

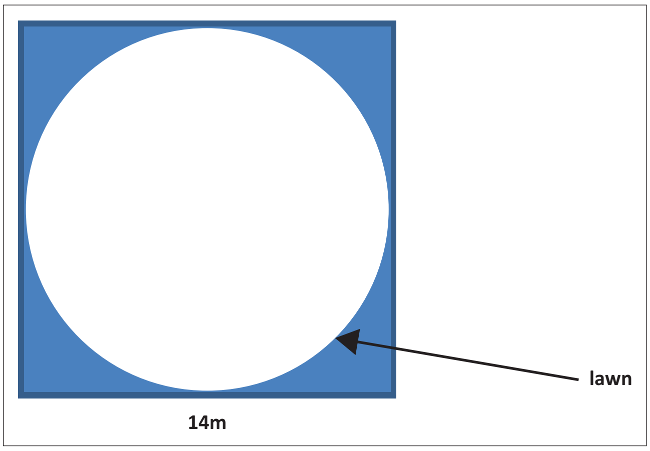


FIGURE 2: Circular patio fitted in a square yard.

- Sipho built a circular patio that perfectly fits inside his father's square yard measuring 14 meters on the sides. If he needs to plant a lawn in the four corners of the square, how many square meters of the lawn does he need to buy?

The following is a demonstration of how discussions of this problem unfolded in the workshop.

Understanding the problem: The question requires us to find the amount of lawn needed to be planted on the four corners of the yard. This area is inside the square but outside the circle. The square's sides are 14m long. The circle's radius is not given, but we know the circle fits perfectly inside the square (see Figure 2).

Devising a plan: We can find the area of the square and circle by using formulas. Area of square = s^2 and area of a circle is $A = \pi r^2$. By finding these two areas, we can get the amount of lawn by subtracting the area of the circle from that of the square.

Carrying out the plan: Side of square = 14m:

$$A_{square} = s^2 = 14^2 = 196m^2 \quad \text{[Eqn 1]}$$

The circle fits perfectly inside the square; therefore, the radius of the circle is half the side of the square, $r = 7$m:

$$A_{circle} = \pi r^2 = \pi 7^2 = 49\pi = 153,86m^2 = 154m^2 \quad \text{[Eqn 2]}$$

Sipho must buy $A_{square} - A_{circle} = 196 - 154 = 42m^2$.

Looking back: Estimation. The shaded region looks like a quarter of the square area therefore $42m^2$ is reasonable.

Activity 3: We discussed a structure of implementing the MPSP and came up with guidelines that we modified from Polya's (1957) problem-solving steps:

- The lesson starts with a problem (preferably word problem) in the textbook, or the teacher poses a problem related to those found in the CAPS curriculum.

- Learners read the problem as individuals or in pairs and the teacher determines if all learners understand the problem. If some learners do not understand the problem at hand, the teacher assists learners to understand by using approaches like re-reading the mathematical problems carefully with the aim of understanding before solving them. I encouraged teachers to ask learners questions like: Do you understand what the problem is looking for? Do you know all the words? Can you repeat the problem in your own words?
- This stage involves learners devising strategies to solve the given problems. We discussed with teachers how they could help learners to create a plan to solve a given problem. For example, during problem-solving teachers could ask learners questions like: What operation are you going to engage? Do you need to draw a picture? A table? Would you use an equation?
- This stage involves learners finding a solution to the problem independently, in pairs or as a group. I discussed with teachers how they could assist learners in carrying out the plan, persisting with a chosen plan, or if a plan does not work to discard it and choose another one. The teacher, during this process, circulates in the classroom, asking learners questions about their work, clarifying misunderstandings, giving suggestions, and helping or giving hints to learners who get stuck. Moreover, the teacher may repeat the understanding process with the stuck learners if necessary. Concurrently, the teacher looks for learners who have interesting solutions with the objective of asking them to explain their solutions in a certain order during the whole class discussions. After getting a solution to the given problems, I persuaded teachers always to help learners review their answers by reflecting on what would have worked and did not work and try to find alternative solution strategies.
- At this point, the class conducts a whole class discussion where learners present their ideas and solutions as well as listen to others. Learners are required to look at the similarities and differences among the solutions being presented by their peers and to realise that there are multiple solution strategies to some given problems.
- The teacher sums up the solutions presented by learners, highlights the main points, and concludes the lesson or, if there is still time, learners solve another problem.

Activity 4: During the fourth activity, I supported teachers on implementing MPSP in a multilingual context. In the first cycle, we focused on code-switching, and in the second cycle we focused on translanguaging. Code-switching is generally known as the utterance of two or more languages in the same conversation.

Translanguaging is 'the flexible and meaningful actions through which bilinguals select features in their linguistic repertoire to communicate appropriately' (Velasco & García, 2014, p. 7) and, in this context, it involved learners receiving information in one language and repeating it in another.

Activity 5: In the last activity of the workshop, we discussed the importance of reflection. I coached teachers on how to continuously reflect on self during the implementation of the MPSP. I concluded each workshop by asking teachers to summarise what they were taking from the PD.

Between the PD workshops, teachers implemented the MPSP for two months. During the implementation of the MPSP, I conducted classroom observations and guided teachers as was required. Following the initial and second implementation, the second and third workshops were conducted, where the objective was for teachers to listen to the audio-recordings of the observed lessons and reflect on how they had implemented MPSP. After the reflective process, teachers watched two more videos showing MPSP and collaboratively solved mathematical word problems similar to those they were currently teaching. Teachers also simulated how to teach mathematical word problem-solving. I discussed with teachers their experiences with learners' language during the implementation of MPSP and linguistic strategies appropriate to support learners in their context. Finally, I coached teachers on how to reflect on their MPSP and classroom practices incessantly.

Data gathering and study design

The study reported on in this article is a qualitative research approach of a naturalistic inquiry (Salkind, 2010) which was conducted within a DBR project. The qualitative research approach permitted interaction with teachers in their natural settings, namely classrooms and schools (Denzin & Lincoln, 2011). The study was naturalistic in that I encapsulated and interpreted teachers' experiences, thinking, and actions about a phenomenon, that is, problem-solving and MPSP (Salkind, 2010). The five assumptions and four principles of andragogy were used as a lens in designing the PD programme for teachers' MPSP, and the reflective interview schedule.

Participants

Thirty-one Grade 9 mathematics teachers (19 female and 12 male) from 20 different South African public secondary schools were conveniently selected to participate in the baseline study. For the PD intervention, two schools (school A and school B) were purposefully selected out of the initial 20 schools based on their accessibility and representativeness of South African public secondary schools that are resource-deficient, multilingual, and have a high learner to teacher ratio. Furthermore, during the baseline study, the Grade 9 teachers of the two selected schools revealed that they misunderstood MPSP, were utilising traditional methods of teaching mathematics, and had shown interest in the project.

Two female teachers (with pseudonyms Mary and Bertha), aged between 31 and 40 years, from school A, participated in the first cycle. One female and one male teacher (Ms N and Mr M), aged between 25 and 30 years, from school B, participated in the second cycle. Both teachers in the first cycle had bachelor's degrees in Mathematics Education. Mary had 13 years' experience and Bertha had 19 years' experience in teaching secondary school mathematics. Mr M had a diploma in Mathematics Education and Ms N had a master's degree in Mathematics Education. Mr M had six years of experience teaching secondary school mathematics and Ms N had one year of experience of teaching secondary school mathematics.

Research instruments and data collection procedure

I employed a mathematics teacher open-ended questionnaire to conduct the baseline study to establish Grade 9 teachers' views on MPSP (see Chirinda & Barmby, 2018). I gathered data in both cycles using classroom observations and reflective interviews which I audio-recorded with participants' consent. Only data from classroom observations and reflective interviews were relevant for the study reported in this article. I conducted pre-observations before the implementation of the PD intervention and observations during the implementation of the intervention. I conducted pre-observations and observations while teachers were delivering lessons and recorded information on the spot, as it occurred. Information was recorded with teachers' permission on the observation comment card. The purpose of the observations was to observe teachers' practices and establish the difficulties that occurred during implementation of the MPSP. The reflective interviews were conducted with each teacher on the lessons observed and were facilitated by playing the teachers selected audio recordings of the observed lessons. A semi-structured interview schedule was used to facilitate the reflective interviews.

Data analysis

I engaged grounded theory data analysis techniques using constant comparison to analyse the data, such that the process was both ongoing and retrospective. Grounded theory data analysis techniques involve systematic guidelines for analysing qualitative data to generate codes, themes, and theories grounded in the data (Chamaz, 2014). After completion of data collection, I conducted the retrospective analysis by going through 'comparative levels of analysis' (Chamaz, 2013, p. 295). Comparative analysis permitted uninterrupted interaction with the data since I could continuously ask analytic questions about these data and emerging themes (Chamaz, 2013). To develop major categories, I compared data with data, data with emerging codes, codes with codes, and raised noteworthy codes to provisional categories based on the frequency of occurrence. I then compared data and codes with the provisional categories to establish major categories (themes). I created codes, for example, by pinpointing significant statements on mathematical problems, problem-solving, and MPSP made by teachers during classroom observations, PD workshops, and reflective interviews. To increase the credibility of the findings, the codes and themes were verified by an independent coder who either validated the findings or eliminated codes or themes based on disconfirming evidence.

Ethical considerations

Permission to conduct research was granted by the overseeing university and Gauteng Department of Education (GDE) with reference numbers 2016ECE002D and D2016/373AA. Regarding informed consent, teachers freely participated in the study without implicit or explicit coercion and were given assurance that they could withdraw from the study at any time without being disadvantaged. I upheld issues of beneficence throughout the research by ensuring that the study was beneficial to mathematics teachers and mathematics education. The teachers were not subjected to any possibility of harm. I viewed teachers as autonomous beings and respected them throughout the study. I did not divulge participants' information that they disclosed during the study to others or institutions. The audiotapes and transcripts were stored in password-protected files on a password-protected computer. The data or reports did not reveal the participants' names or their schools.

Findings and discussion

In this study, I was interested in identifying the factors fundamental to the PD intervention for teachers' MPSP. The research question was: What factors are fundamental to a professional development intervention for teachers' mathematical problem-solving pedagogy? The major findings from the study were that teachers' personal meaning, reflective inquiry, and collaborative learning are factors fundamental to their PD in MPSP. In the next sections, I discuss the findings and present the themes with direct quotations to 'give life' to the data analysis and to make teachers' voices audible.

Teachers' personal meaning

Personal meaning is what the teacher desires, feels, thinks, considers to be truthful, and includes what meaning they give to certain routines or conflicts (Elbaz, 1990). Howson (2005, p. 18), referring to learners, explained that personal meaning relates to learners' personal sense 'relating to relevance and personal significance (e.g., "What is the point of this for me?")'. This explanation is homologous to teachers in this study since personal meaning emerged as a factor fundamental to their PD in MPSP.

Teachers' personal meaning in the previous PD programmes in a particular context

During the pre-observations, all four teachers indicated that they had previously participated in several PD programmes and viewed PD as not improving their pedagogy or promotion. Teachers had developed negative attitudes towards these PD programmes, which they indicated as short-term, often lasting not more than one day. Teachers believed that these programmes were designed and conducted by PD facilitators without teaching experience in their contexts. Teachers were frustrated that the PD programmes were unrelated to their profession and real classroom challenges. As noted by Mary:

> 'I have attended short workshops before, where facilitators spoke about other trivial things in maths. However, I really want to attend workshops based on different strategies for teaching mathematical problem-solving. These workshops should also teach me to design problem-solving activities that can shift the learners' mindset to love mathematics.'

The above quote implies that teachers' disposition towards past PD programmes impacted on their professional learning. This was not surprising because personal meaning is known to be a foundation for structuring one's actions in the world. Kilpatrick, Hoyles and Skovsmose (2005, p. 2) observe that 'we may claim that an activity has meaning as part of the curriculum, while learners might feel that the same activity is totally devoid of meaning'.

This is comparable to what teachers felt about the PD programmes that the school required them to attend. The reflective interviews revealed that the teachers viewed PD facilitators as authorities who came to impart new knowledge to them but seemed not to understand who they were, what context they were working in, what they experienced every day when they were working in overcrowded and multilingual classes, and how they were supposed to learn to enhance their practice. This was put across by Mr M:

> 'The school sometimes requires us to attend workshops to improve on mathematics teaching. Unfortunately, these workshops do not cover what happens in our classrooms. The facilitators always speak of models that are helpful overseas and not to me.'

In this regard, I engaged DBR for this study, such that the teachers' input assisted in developing a PD intervention that was relevant to their lived experiences and context. As the intervention progressed, the reflective interviews disclosed that the teachers had reshaped their personal meaning in professional learning and were interested in the PD intervention because it had relevance and personal significance to their MPSP and classroom practices.

The PD intervention provided teachers with prospects of learning from their MPSP on a daily basis, with and from their colleague. A key factor in PD for teachers' MPSP was giving teachers the opportunity and voice to pinpoint what practices would best support their strengths in MPSP. This observation suggests that, before implementing the intervention, the PD facilitators should first investigate the characteristics of PD in MPSP that teachers need through informal observations, surveys, focus groups, or discussion groups. This is the only way PD can effectively address teachers' personal meaning in MPSP. If teachers find the PD programme irrelevant to their personal meaning and non-aligned to their concerns on MPSP, it rapidly disappears from their memory.

Teachers' personal meaning in MPSP

At the beginning of each cycle, teachers exhibited what I would term 'traditional' approaches to teaching mathematical problem-solving. From the pre-observation lessons, I could not see the teaching of problem-solving in mathematics classrooms (see Chirinda & Barmby, 2018). Instead, learners listened attentively as the teachers demonstrated the algorithm to solve the given problems and then learners worked to practise the given algorithm until they acquired the skills. Teachers' reasons for adopting the traditional methods of teaching were that they felt that learners lacked basic problem-solving processes to do problem-solving in Grade 9 and that the overloaded curriculum left no time for problem-solving. In the workshops, we discussed that when teaching problem-solving, it is not mandatory to get answers for the given tasks, but it is important to help learners to do problem-solving.

Initially, teachers did not recognise that doing problem-solving enhances learners' mathematical understanding of problem-solving processes that can be applied to different problems in the future. In the interviews, teachers also revealed that they taught procedurally because they assumed that learners did not understand since they struggled with the language of teaching and learning (Chirinda & Barmby, 2018).

After attending the PD intervention, classroom observations revealed that the teachers were no longer drilling procedures but implementing the problem-solving strategies they learned in the workshops. I discuss, in a vignette (Figure 3), how Mary discussed with her learners each step of Polya's (1957) four-step problem-solving processes while solving the following word problem:

> Reverend Joseph of the Methodist Church of South Africa has passed on, and his wish before he died was that people should wear only white, red, and pink hats at his funeral. 171 women and 93 men attend the funeral. His wife made 13 dozen white hats, five dozen red hats, and three dozen pink hats for the guests. At the end of the funeral, 11 hats were not worn. How many hats were worn by the guests?

As the intervention progressed, I observed that teachers began to emphasise doing of mathematical problem-solving rather than focusing on learners finding the correct answers. Learners were now given opportunities to participate in problem-solving and contribute meaningfully during lessons. Teachers began to promote classroom discussions, by providing learners opportunities to discuss solutions or solution strategies in pairs or groups. Learners were required to monitor their work instead of relying on the teachers' guidance. During problem-solving, I observed that teachers frequently probed learners to be open-minded by asking them to explain their solutions and solution strategies. Learners were given opportunities to present their solutions on the board so that their classmates could learn different solutions.

Teachers' personal meaning in giving learners full ownership of problem-solving

At the start of each cycle, teachers in the study did not believe that their learners could learn mathematical problem-solving because they did not have basic problem-solving processes

Understanding the problem

The first step was for learners to understand the problem. Mary established first if learners understood what was being asked for in the problem by asking them to do the following:

* State the problem in their own words.
* Identify the unknowns.
* Decide what information was important and irrelevant to the problem.

In the given problem I observed that it took all the learners time to realise that they did not need the information about the colour of hats and gender of the guests because it was irrelevant. Mary worked with learners to understand that the problem being asked in the question was: How many hats were worn by the guests?

Devising a plan

After understanding the problem, Mary required learners to devise a plan to solve the problem by considering various strategies like:

* Making a table, diagram or chart
* Trying a simpler form of the problem
* Writing an equation
* Guessing and checking
* Looking for patterns

In the given problem Mary and her learners had to figure out the total number of hats made before determining the number of hats worn by the guests. After trying so many strategies Mary agreed with the learners that writing an equation to show the unknown was the best way to solve the problem:
(13 dozen + 5 dozen + 3 dozen) - 11 = number worn

Carrying out the plan

During carrying out the plan, Mary required learners to implement the chosen strategy from the second step. I observed that half of the class struggled to find how many hats were made in total by the Reverend's wife because they did not have the prior knowledge that 1 dozen equals 12. After Mary assisted learners in finding the initial hats, learners needed to know how many were worn if only 11 remained after the funeral. To find out, Mary worked with the learners to write an equation that would be used to solve the problem and find the answer:
So, (13 dozen + 5 dozen + 3 dozen) - 11 = number worn.
13 × 12 = 156; 5 × 12 = 60, and 3 × 12 = 36,
so we have (156 + 60 + 36) - 11 = number worn.
252 - 11 = 241
The number of worn hats = 241.

Looking back

Looking back implies that the learners must check their solution strategy and the solution itself. I observed that learners in Mary's class preferred to skip this step and felt that it was unnecessary. As per our discussions in the workshop, Mary explained to the learners that looking back was important in that sometimes you can add when you are supposed to subtract which results in a wrong answer. She stressed to learners that checking the answer would help to identify the error. Mary then discussed with the learners if the answer to the given problem made sense. The above problem required learners to find the hats worn out of a total of 252. Mary explained that getting 241 as the number of the worn hats made sense since it was less than 252 and it would not have made sense if they had got an answer bigger than 252. After her explanation, I observed that learners understood that guests would not have worn more hats than those that were made and therefore the answer was reasonable.

FIGURE 3: Implementation of the mathematical problem-solving pedagogy by teacher Mary.

from prior grades. Participant teachers believed that they were not supposed to give learners full ownership of problem-solving, but rather demonstrate each procedure step by step and require learners to practise the procedures (Chirinda & Barmby, 2018). Teachers reflected that they believed that they had the responsibility of re-explaining procedures in response to learners' questions. They felt that learners were underprepared from previous grades and had difficulty grasping Grade 9 concepts. Subsequently, during PD workshops, I worked with teachers on how they could implement MPSP with underprepared learners. Furthermore, I proposed that teachers explain to learners how to understand the given problem and how to devise problem-solving strategies. After that, the teachers could discuss the different problem-solving strategies applicable to the given problem, eventually solve it, and reflect on the solution. This illustration could then be followed by a class exercise for learners to practise the problem-solving processes with the teacher moving around as a facilitator assisting learners as was necessary. As teachers found relevance and personal meaning in implementing the new ideas on MPSP from the workshops, I observed that they gave learners full ownership of PS. For example, in one lesson, I observed Ms N incorporating open-ended word problems that encouraged learners to develop a sense of ownership. The following is an example of a scenario that Ms N gave to her learners and required them to pose problems:

> Meghan's old grandmother starts walking at 8am to go to the mall for shopping. Meghan must clean the house; therefore, she follows grandma after a few hours. They both arrive at the mall at the same time.

I observed the learners working in pairs trying to understand the given problem prior to solving it. I could hear most of the learners planning and formulating problems with excitement and looking back at the answers after finding solutions. Nonetheless, a few learners wanted to be shown how to pose the problem; nevertheless, I observed that Ms N identified meaningful ways to motivate learners to persist in problem formulation. This seems to suggest that Ms N had established self-understanding in personal meaning in giving learners full ownership of doing mathematical problem-solving.

Teachers' reflective inquiry

Reflective inquiry is the teacher's act of looking back at the teaching and learning of what has transpired and re-constructing or re-capturing the events' occurrences, emotions, and experiences (Schon, 1987). In both the first and second cycles, I conducted the second and third workshops whose chief objective was for teachers to reflect on their implementation of MPSP collaboratively. Teachers conducted the reflective process by reviewing and reflecting on the audio-recordings of the observed lessons. I also used reflective interviews to assist teachers' reflective inquiry on the implemented lessons and learners' problem-solving processes. During the workshops and reflective interviews, I noticed that as teachers intentionally and methodically reflected on their participation in the PD activities and implementation of MPSP, they made sense of their actions and extracted meaning from those actions. Reflective inquiry assisted teachers to consciously improve their MPSP as put across by Mr M during one discussion in the workshop:

> 'In my class, I usually give learners problems to work on as individuals. When they get stuck, I quickly give them hints or show them how to get the answer before the bell rings. Now I understand, I will not be giving learners opportunities to engage in problem-solving when I do this. From now on, I will not rush to provide learners answers. Learners need to take their time. They need time to think about the problems and find ways of how to get answers by themselves.'

Bertha also indicated this in one of the reflective interviews:

> 'I often show my learners the meaning of a problem before they begin solving it. I now see that I am taking away from learners, the chance to learn to analyse problems on their own. Going forward, I will let learners take the lead in understanding the problems of the day.'

The above excerpts seem to demonstrate that self-inquiry evoked meaningful learning in the teachers. Reflective inquiry as a means of teacher PD is recommended by Muir and Beswick (2007), who advocate that to improve teachers' classroom practice, emphasis should be placed on teachers' learning and self-reflection. When teachers engage in reflective practice, they become cognisant of their own and others' thinking and assumptions (Drago-Severson, 2007). In this study, this awareness resulted in participant teachers' growth. Reflective thinking shifted teachers away from routine thinking towards reflective action that involved learning from doing (Dewey, 1916) and allowed teachers to build knowledge that guided their MPSP (Schon, 1987). This finding seemed to imply that teachers' reflective inquiry was a factor fundamental to their PD in MPSP. I observed that self-reflection during the workshops and reflective interviews forced teachers to confront and become critical of their personal meaning in mathematical problem-solving and MPSP. I also discovered that as teachers became critical of their personal meaning in what the teaching of problem-solving implied, they were able to self-correct and improve their MPSP. During the reflective interviews, I realised that teachers did not reflect in a vacuum, but that context was an important aspect. When teachers reflect, they adapt ideas to their contexts (Zaslavsky, Chapman, & Leikin, 2003), and for this study, it allowed teachers to function better in their context and become more confident in implementing MPSP. In this context, self-reflection was an important aspect of PD that reshaped teachers' personal meaning and promoted their professional growth.

Teachers' social participation

Teachers were required to work collaboratively during PD workshops and the implementation of the MPSP. Collaboration means teachers are working and learning together to address challenges they grapple with in their profession (Robutti et al., 2016). Introducing collaboration at the beginning of each cycle was challenging as teachers were used to working as individuals and not sharing expertise with their colleagues. Teachers felt that their practice they had kept personal and invisible for a long time was now being made visible to their colleagues.

After explaining the characteristics of collaboration, teachers in each cycle rose to the challenge. They accepted the joint responsibility of working on the word problems and other activities during the workshops and implementation of MPSP.

It is well documented in the literature that collaboration brings teachers together to implement new teaching strategies and reflect on their practices (Robutti et al., 2016). During these interactions, teachers can observe, emulate, and advance their cognitive functions (Vygotsky, 1978). This implies that learning is located in practice and happens as one actively participates in activities (Lave & Wenger, 1991). This was true for this study when teachers worked collaboratively during workshops. As teachers worked together during the intervention, I observed that they were able to look back and reflect on their teaching. This observation suggests that collaboration assisted teachers to easily accept the new ideas on MPSP that they were learning in the workshops since they tried them out through the support of a colleague during the implementation stage. This can be noted from Ms N:

> 'It was a great privilege to work with my colleague during this programme. Having real conversations with him before and after the observations of the lessons benefitted me professionally in that he is familiar with the subject, CAPS curriculum, and mathematics problem-solving teaching strategies.'

Teachers observed each other's successes, challenges, and failures. My observations during the workshops were that as teachers collaboratively solved the word problems and planned how to teach these problems to their learners, they reached a shared understanding of how to implement Polya's (1957) four steps of problem-solving processes in their lessons. In addition, I observed that when teachers worked on activities collaboratively during the PD workshops it resulted in the advancement of their MPSP.

The reflective interviews gave insight into teachers' perspectives about the process of collaboration during the PD intervention. Teachers in both cycles embraced collaboration and valued the discussions they carried out with their colleague on how to teach specific mathematics problems to learners. Teachers reflected that working cooperatively with a colleague strengthened feelings of positive interdependence between them. Positive interdependence, which means the success of one teacher was determined by the success of their colleague, is acknowledged to deepen learning (Drago-Severson, 2007). In this study, I observed that positive interdependence reinforced teachers' understanding of new ideas on MPSP that they learned in the workshops, which in turn improved learners' problem-solving processes (see Chirinda, 2018). The outcome of collaborative learning was not a surprise since it concurs with Loucks-Horsley et al.'s (2010) observations that teacher collaboration results in effective PD and improved teaching quality and learning outcomes.

In the second cycle, Ms N, a novice teacher with one year of teaching experience, reflected that collaborating with her colleague during the PD intervention had vastly improved her MPSP and eliminated feelings of loneliness. As a newcomer both in the profession and in the school, Ms N explained that she had felt isolated. This implies that collaboration eradicates feelings of isolation that beginner teachers usually find themselves experiencing. Ms N valued collaborating with Mr M as she felt she had someone to work with and no longer felt isolated. This finding seems to imply that there is value in pairing new teachers with the experienced members of the teaching staff. The finding agrees with Vygotsky's (1978) notion of scaffolding, where the more experienced help the novice progress to higher levels of understanding. This was evidenced by Ms N's reflection that collaborating with Mr M during the workshops made her understand that Grade 9 learners normally have the same misconceptions in problem-solving. She elaborated that having discussions with Mr M on using problem-solving to teach some word problems helped her see how she could address these misconceptions. As she indicated in this quote:

> 'As a beginning teacher, the discussions I conducted with my colleague helped me understand several aspects on the teaching of problem-solving including misconceptions learners usually have in problem-solving and how to handle them.'

The above quote suggests that collaboration supported Ms N's professional growth. This growth resulted in her teaching being accessible and effective to learners as she was now able to identify and handle learners' misconceptions in mathematical problem-solving. This finding suggests too that collaboration is a factor fundamental to PD intervention for teachers' MPSP in the study. Nonetheless, my observations were that teachers' collaboration in the context seemed limited because of the prescribed curriculum and a tight school timetable. This observation agrees with Cookson (2005), who perceives that an overloaded curriculum like in the context of this study limits teacher collaboration and hampers the possibilities of teachers working together. The reality that participant teachers did not have time to work together implies that collaboration should be embedded in the school curriculum so that teachers have enough time in the school day to meet regularly and focus on supporting each other's professional growth.

Final thoughts

This study explored factors that are fundamental to a professional development intervention for teachers' MPSP.

This study's findings were that teachers' personal meaning, reflective inquiry, and collaborative learning are the factors fundamental to their professional growth in MPSP. This study's focus on teachers' personal meaning in the previous PD programmes they had attended seemed to have effectively addressed their personal meaning in PD in MPSP. In addition, the PD facilitator's ability to recognise teachers' personal meaning in MPSP seemed to have facilitated professional growth in their MPSP. The reflective inquiry process, in turn, gave teachers insight into their personal meaning in the activities and processes of the PD intervention. Clarke and Hollingsworth (2002, p. 947) posit that 'if we are to facilitate the PD of teachers, we must understand the process by which teachers grow professionally and the conditions that support and promote that growth'. In this study, teachers' personal meaning, reflective inquiry, and social participation supported and promoted their growth in MPSP. The primary recommendation from the study is that facilitators of PD must take into account these factors if they are to encourage teachers' professional growth in MPSP. In this article, I argue that if PD processes and activities are relevant to teachers' personal meaning, reflective inquiry, and collaborative learning teachers find the PD programme fulfilling and meaningful.

This study contributes to the PD in the MPSP body of knowledge by having worked with teachers in an under-researched context. A study in this context is valuable since research conducted in contexts of advantage is ubiquitous in the mathematics education field and few studies originate from countries and contexts of historical disadvantage (Skovsmose, 2011).

Because of inadequate monetary resources, the study focused on teachers in one South African district. The limitation of focusing on one district is that I could not explore how a PD intervention for MPSP could be designed and implemented to support Grade 9 teachers in other districts. However, the objective of this study was not necessarily to explore several participants but to design a PD intervention to support teachers' MPSP. Nonetheless, I recommend that further research can be done with a larger, more diverse sample of South African teachers to determine how personal meaning, reflective inquiry, and collaborative learning promote their growth in MPSP. I feel that there has not been enough critical examination of this aspect in the South African context.

This study revealed that collaboration is a factor fundamental to PD intervention for teachers' MPSP; however, further research needs to be done to establish how best collaboration can be conducted in contexts where teachers feel that there is not enough time to collaborate because of prescribed school curriculums and tight school timetables.

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Author's contributions

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Data availability

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