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Authentic Investigation Activities for Reasoning Ratio Concept

Pınar Anapa SABAN, Merve Koçyiğit GÜRBÜZ

Abstract: The aim of this research is to examine the effect of authentic investigation activities on the reasoning process of ratio concept which has been regarded as the milestone of mathematics and science teaching. In line with this purpose, the research has been carried out with eight 7th grade students attending a public school located in Central Anatolia Region in Turkey. The success levels of these students in mathematics are in a heterogeneous structure ranging from higher and moderate to lower levels. The data of this qualitative study has been collected through authentic investigation activities, researcher logs, and interviews and has been analysed via descriptive techniques. At the end of the study, it has been concluded that the teaching environment containing authentic investigation activities has a positive effect on the reasoning process of students on ratio concept.

Key words: Ratio, Authentic Investigation Activities, Constructivist Approach, Conceptual Reasoning.

1. Introduction

In today's world, the concept of number is composed of natural, rational, real and complex number concepts although this has not always been the case. Euclid (330-270 BC) used to deal only with whole numbers while he made use of ratio and proportion instead of real numbers (fractions and irrational numbers) in his efforts to explain the numeric relationships between such geometric shapes as line segments, triangles, circles, and spheres. Even though we have all the number systems today, the ratio and proportion concepts are still useful in comparing quantities (Snow J.E & Porter M.K, 2009). In addition, ratio is one of the fundamental concepts in mathematics used extensively in teaching such subjects as probability, slope of a line, similarity and trigonometric ratios. These subjects provide a basis for important topics in science teaching such as calculation of chemical equations, use of ideal gas laws, identification of density, velocity, and acceleration, and analysis of electric circuits. In this regard, the concept of ratio is one of the most important subjects in the mathematics curriculum of secondary schools (Akar, 2009). However, the studies in the literature put emphasis on the fact that there are serious problems in reasoning and teaching of ratio (Post, Behr & Lesh, 1988).

Students is Turkey formally come across with the concept of ratio firstly in 6th grade and in higher grades, they are taught the similarities of geometric shapes, slope calculations, linear equations and inequalities, the subjects designed to improve their discernment. It will be wrong to expect from students having issues in making sense of ratio in 6th grade to effectively use their proportional discernment. Consequently, it is of very importance to carry out studies that aim to identify learning environments facilitating better understanding the ratio concept and activities and strategies to be used within these environments.

This study has analysed the effect of authentic investigation activities developed by Ben-Chaim, Ilany & Keret (2012) on the process of reasoning of ratio concept.

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1.1 The Concept of Ratio

Ratio is the number that compares the two quantities or measurements in a multiplicative relationship (contrary to that in subtractive or additive relationship) in a given situation (Van de Walle, 2013). Freudenthal (1983) has proposed three different methods for comparing the two quantities; with:

- The comparison of one whole with its own part: That is the comparison of one part with the whole that it exists in.
- The comparison of one part with another part with which it has a relationship: That is the comparison of parts of different wholes or different parts of a same whole.
- The comparison of one part with another part with which it has an indirect relationship: That is the comparison of two quantities or measurements having indirect relationship by dividing them into each other.

These three methods created by Freudenthal actually indicate two different types of ratio. The first one of these different types compares the same-type quantities. There are two types of ratio and these are part-to-whole comparisons and part-to-part comparisons. For instance, the ratios in a box containing 5 brown and 7 white eggs are as follows: 5 and 7 (brown-white part-to-part comparison), 7 and 5 (white and brown part-to-part comparison), 5 and 12 (brown part-to-whole comparison), and 7 and 12 (white part-to-whole comparison).

The second type of ratio compares different types of quantities and it is called ‘rate’. However, the ambiguity in the use of ‘rate’ term is an additional assumption of the fact that the comparison describes a quality which is common in many situations (Lamon, 1995). For instance, 3 Turkish liras per pencil is a ratio between the cost in terms of Turkish liras and the number of pencils. The calculation goes as 6 liras per 2 pencils, 24 liras per 8 pencils, 54 liras per 18 pencils, etc. There are two different types of quantities as the number of pencils and Turkish liras.

Even if ratio, rate and the difference between them could be exactly defined, these definitions could not fully reflect the real meaning behind the idea. The nature and real meaning of ratio and rate could be seen in problem situations. The examination of relevant contexts containing the semantics and applications of rate and ratio could enable students to have critical reasoning they should have about rate and ratio concepts. Teachers should prepare activities for students that will enable them to separately analyse each situation in order for them to make a sensible reasoning about ratio and rate (Lamon, 1995).

2. Theoretical Framework

2. 1. Authentic Investigation Activities on Ratio

Authentic research activities present non-routine open-ended problem situations that will activate students’ ability to evaluate and use the information necessary to solve daily life problems. As these activities require original and autocritic answers, they provide a basis for deeper researches. Within this context, it is thought that authentic investigation activities will improve high-level thinking and problem-solving skills of students.

Authentic investigation activities on ratio contain multiple-answer problem situations prepared by considering the difficulties and misconceptions experienced during the reasoning of rate and ratio concepts. These problem situations are presented with spiral tasks having many stages. The unknown quantities and ratios are usually given through qualitative and quantitative comparisons along with whole numbers, fractions, decimals or percentages. The tasks are given to the students on papers.

With the goal of forming a basis for rate and ratio concepts, each activity has an entry level and a higher level for advanced learning. These activities enable understanding of many other concepts related to rate and ratio. While rate concept is related to density, ratio is related to part-to-part or part-to-whole comparisons.

2. 2. Conceptual Reasoning

The term of concept could be described as the reflection of objective reality in the human brain (Hançerlioğlu, 1967) or knowledge representing the common volatile features of different objects or phenomena gaining sense in the mind (Ülgen, 2004). Çepni (2005) indicates that concepts start to develop in the mind from birth and as the concepts develop, they become constituents of knowledge. Understanding or creating of new knowledge could be enabled by combining or bringing together these constitutes. This process is described in the literature as conceptual reasoning.

Conceptual reasoning contains a procedure related to making sense of mathematical concepts, principles and descriptions within the mind. In other words, this process means that the individual converts new knowledge into a correlational and systematic structure after evaluating it along with their current knowledge and understanding (Skemp, 1978).

Conceptual reasoning could not be realised within the learning processes in which students are passive and they are only taught by being dictated with descriptions. In order that conceptual reasoning could be realised, students should be active learners within learning environments during which they are given the opportunity of discovering the main characteristics of the taught concept on their own.

Meir Ben-Hur (2006) has indicated that there are five main basics of teaching process during which conceptual reasoning could be realised in mathematics. These are as follows:

1. Practice: Sufficient amount of application for the concepts to be taught.
2. Decontextualisation: Students experience different applications in order to learn a concept.
3. Verbal generalisation: Students explain the concept or the meaning of concept in their own words.
4. Recontextualisation: Students discover new situations or match up their past experiences with the recently learnt concept.
5. Realisation: Teacher transfers the concepts in the curriculum into new situations or facts from daily life.

Authentic ratio investigative activities configure the meaning of rate concept as a tool for estimating the total number of objects in a given area containing the concept of ‘density’ while they configure the concept of ratio with part-to-part or part-to-whole comparisons. In addition, these activities enable recontextualising the concept with daily-life practices and transferring each acquisition in the curriculum related to ratio into daily life situations. Thanks to these features of activities, students have the opportunity of experiencing different types of application in various situations of ratio concept. In this context, it could be stated that the structure of authentic ratio investigative activities are compatible with the five steps created by Meir Ben-Hur (2006) with the aim of realising the conceptual reasoning.

2.3. Problem statement and research questions

Lamon (1995) indicates that the insensible use of rate and ratio terms instead of one another in daily life causes students to be exposed to incorrect use and terminology. This situation can be seen as a factor in the formation of learning difficulties in students regarding the concepts of ratio and rate. The activities used in the teaching process have an important effect on eliminating learning difficulties but it is no easy task to reconcile precise mathematical ideas with informal, colloquial usage. In this context; the aim of the study is to determine the effect of authentic investigative research activities on the process of making sense of the concepts of rate and ratio. For this purpose, answers to the following questions were sought:

1. What kind of perception did the students develop regarding the concepts of rate and ratio with the activities in the school mathematics textbook?
2. How did authentic investigative activities affect students’ perceptions of rate and ratio?

3. Method

Qualitative research method was used in this study which aim to reveal the effect of authentic investigative activities on the making sense of rate and ratio.

3.1. Participants

While identifying the participants, the maximum variation principle of purposeful sampling method has been applied. The participants of the study are eight 7th grade students who are all students of one of the researchers from the 6th grade. The reason why the participants have been selected from 7th grade is that the study will reveal the effect of authentic investigation activities on the reasoning of ratio concept by first identifying how well the concept has been learnt and structured in 6th grade. The selected students could express themselves easily and their level of success in mathematics ranges from higher and moderate to lower level. The success levels have been identified by considering the grades the students gained in 6th grade. The students whose grades are between 0-50 have been regarded as having 'lower level success', those between 50-80 and 80-100 have been regarded as having 'moderate level success' and 'higher level success', respectively.

Table 3.1: *Code Names Given According to Success Levels*

Success Levels	Code Names
Higher Success Level	S8, S7
Moderate Success Level	S4, S5, S6
Lower Success Level	S1, S2, S3

The number of participants has been identified as nine before the study; however, since one student from the higher success level group has not attended to the activities, the research has been carried out with eight students coded as in Table 3.1.

3.2. Collection of Data

The most important data-collecting tool of the study is authentic investigation activities. All problems situation of each activity have been derived from real-life situations. The data collection tools have been presented in Table 3.2 below.

Table 3.2. *Data Collection Tools*

Data Collection Tool	Application	Purpose
Authentic investigation activities related to ratio, voice records	Voice records have been taken while applying five authentic investigation activities related to ratio	The most important data collection tool of the study is five authentic investigation activities on ratio. Apart from submitting written answers, it is of much importance that students provide verbal explanations. For this reason, voice records have been taped during the application.
Research products	Activity papers of students written during the application of authentic investigation activities	The most important products of students are the written documents on which they have provided their answers related to authentic investigation activities related to ratio
Observation notes	Notes taken by the observer after each practice	The researcher has aimed to take notes related to his/her observations at the end of the day
Interview questions	Interviews during the application	It is aimed to interview with the students during the application about the themes of authentic investigation activities related to ratio

Five authentic investigation activities for teaching ratio prepared in compatibility with secondary school mathematic curriculum have been applied within the study. While the students have been directed to group-work, they have also been given the opportunity of working individually. Each problem situation within the authentic investigation activities for teaching ratio and the feature of ratio concept presented within each problem situation have been evaluated and analysed by student-student and student-researcher interaction. The content of each authentic investigation activity for teaching ratio has been presented in Table 3.3 below.

Table 3.3. *Authentic Investigation Activities for Teaching Ratio Concept*

Type of Activity	Purpose of the Activity
Starter Activity	It has been aimed to find the meaning in the students' mind related to ratio and whether they are able to differentiate additive and multiplicative operations.
Who's right?	It has been aimed through this authentic investigation activity that students make sense of rate concept causing to form new units by comparing different quantities as 'kilometre per litre'. The students have been provided rich research and discussion environment within this activity. They have been given the opportunity of developing new solution strategies and testing and discussing each strategy in order to successfully solve the problems within each task of the activity.
Which stationery store is more profitable?	It has been aimed through this activity that students make sense of rate concept causing to form new units by comparing different quantities as 'unit price'. The students have been provided rich research and discussion environment within this activity. They have been given the opportunity of developing new solution strategies and testing and discussing each strategy in order to successfully solve the problems within each task of the activity.
How many beans in the bag?	The aim of this activity is to identify whether ratio indicates a homogeneously distributed feature within the given context. In this discussion related to homogeneity of ratios, the ratios are associated with measurements. The main point is to identify whether the student has the ability to specify the eligible smaller part of a whole in order to evaluate the general situation when asked whether the ratio is homogenously distributed in a specified area. Within this context, this authentic investigation activity correlates the rate with 'density' as a tool for estimating the number of objects in a given area. The main aim of this activity is to teach students density is the number of specified objects per unit volume.
Pizza Party	The aim of this authentic investigation activity is that students could make sense of 'ratio' concept through the tasks within problem situations presented via pizza sharing theme. Students compare ratios, apply the features of ratio, find the part of a given whole or the whole of a given part within this activity. Therefore, both the knowledge portfolio and proportional reasoning skills of the students could be improved.

3.3 Analysis of the Data

The data collected through authentic investigation activities has been analysed in accordance with the steps of descriptive analysis (Yıldırım&Şimşek, 2003) in parallel with the themes identified before. While analysing, the answers provided by the students in response to the problems presented within the tasks in each of the related authentic investigation activity have been themed by considering the epistemological structure of ratio concept. These themes are as follows:

Unstructured Reasoning: Students having unstructured reasoning about ratio has explained the concept of ratio as the additive relationship between two measurable quantities. These students calculate the ratio of two quantities by an algorithm in which they divide the first quantity by the second and by thinking this situation as a formula, they reach to the solution by directly writing the given numbers into the appropriate places.

Semi-Structured Reasoning: Students having semi-structured reasoning about ratio has could explain the concept of ratio as a multiplicative relationship between two quantities. These students have internalised the fact that these quantities relatively change at the same time within a multiplicative relationship. These students could identify how many units the given quantity will be equal to when measured through another quantity; however, they lack the ability to use this knowledge in other problem situations.

Structured Reasoning: Students having structured the ratio concept are aware that ratios with different numeric values are conversions of one another. Therefore, they could expand or reduce the fractional expression by multiplying or dividing it with the same number. Within two equal fractional expressions, they could determine how much the denominator will change on condition that the equation will be preserved while the value of numerator will undergo a fair amount of change.

The collected data have been independently coded under these themes by two field researchers while the conflicts have been settled by discussions and then a common consensus have been obtained over coding system (Lincoln and Guba, 1985).

4. Findings

4.1. Findings Related to the Starter Activity

The starter activity has been prepared with the aim of determining the level of apprehension about ratio concept that they firstly learnt in 6th grade. Within this activity, the students have been given a rice recipe and they have been asked to explain the "relationship between the amount of rice and water" through Table 4.1. Besides, they have been asked to investigate whether the ratio between the amount of rice and water will change depending on the number of person.

Table 4.1. *The Relationship Between the Amount of Rice and Water*

	Serves 4	Serves 8	Serves 12	Serves 16	...
Amount of rice (water glass)	1	2	3	4	...
Amount of water (water glass)	2	4	6	8	...

The comments of students about the relationship between the amount of rice and water given in Table 4.1 have been presented below.

S1: "The amount of water doubles as the amount of rice increases."

S2: "There is a pattern."

S3: "It doubles."

S4: "The amount of water doubles."

S5: "The relationship increases doubly. In other words, if we put 5 water glasses of rice in the next level, I add '2' into '8' and find that '10' glasses of water are necessary."

S3: "The ratio will increase because as the amount of rice increases, the amount of water increases more."

S4: "The amount of rice increases one glass and the water increases two glasses. Hmm… I think it won't change."

S6: "It says the ratio between rice and water. Namely, we have said water is twice as much as rice. I think it won't change."

S7: "The ratio doesn't change because the relationship always doubles."

Apart from S6 and S7 coded students, all others have failed to see the multiplicative relationship between the amount of rice and water by only focusing the difference between the amounts of water and they have indicated the ratio of rice amount to the amount of water will change depending on the number of persons. Although the S4 coded student have determined that the relationship between the amount of rice and water is an additive one, he/she has asserted that the ratio of rice amount to that of water will not change depending of the number of persons.

With the aim of identifying the level of perception related to ratio concept, semi-structured interviews have been carried out with students within the second part of the starter activity. The students have been asked the question of "What the word of ratio evokes in your mind? Please explain with examples from your daily life. (You can use figures, diagrams, etc.)". The answers given to this question have been provided in Table 4.2.

Table 4.2. *Connotations of Students Related to Ratio*

Students	Connotations Related to Ratio
S1	For example, if you use 3 boxes of milk for making 1 pudding, then you use 4 boxes of milk for 2 puddings.
S2	Ratio between two things.
S3	For example, it is necessary to put two glasses of water for one glass of rice.
S4	Equation. For example, the ratio of girls to boys in a classroom.
S5	Proportion, in other words, adjusting two things…
S6	Dividing.
S7	We could express it in fractions. For example, 3/5, 2/3, etc.
S8	Comparison, difference and calculations.

When the answers in Table 4.2 have been examined, it could be observed that S8 has coded the ratio concept in his/her mind as a comparison based on additive relationship while S6 has coded with dividing and S7 has coded with fractions.

With the aim of identifying the level of apprehension about 'rate' and 'ratio' concepts, the researcher has asked the question of "What do rate and ratio concepts mean to you?". The responses for this question have been provided in Table 4.3.

Table 4.3. *Student Statements Regarding the Concepts of Rate and Ratio*

Students	Statements Related to Rate and Ratio
S1	I don't remember.
S2	I don't know.
S3	For example, hmm, rate has units.
S4	I don't know exactly but according to my estimation, rate has units and ratio doesn't have units. Do they?
S5	Hmm... I think rate has units.
S6	Rate has units but ratio doesn't have.
S7	Ratio does have no units while rate has.
S8	Rate has a unit in the end. For example, distance slash time, I mean m/sc. But, we set up a proportion with ratios. 5 apples slash 3 apples, 5/3.

The answers provided in Table 4.3 indicates that except S8 coded student, there are no perceptions in the students' mind related to rate and ratio concepts. It is thought that students have only commented on lexical meaning of 'rate' and 'ratio'.

It has been determined through the starter activity that all students have set up an additive correlation between the amounts of rice and water within the rice recipe and they think the amount of rice will increase along with the increasing amount of water. By considering these determinations and with the purpose of configuring the concept of rate by comparing units in the students' mind, Activity-1 and Activity-2 have been developed.

4.2. Findings Related to Activity-1

Students have been anticipated to analyse and solve the problem situation given below.

"The bus of 7/C class has covered 52 kilometres with 4 litres of fuel and the bus of 7/B class has covered 45 kilometres with 3 litres of fuel. Each driver claims that his bus is more fuel efficient. Who's right in this situation?"

The individual solutions of the students have been provided in Table 4.4.

Table 4.4. *Individual Solutions of Students for the Activity-1 Problem*

Students	Choice of Answer	Reason of Choice
S1	7/C	Used less fuel.
S2	Indecisive	Both can be right.
S3	7/B	He has covered less distance.
S4	Indecisive	The bus of 7/B class has covered less distance with less fuel but that of 7/C has covered more distance with more fuel.
S5	7/B	He has reached more quickly.
S6	7/C	7/B: 45/3 = 15 7/C: 52/4 = 13, 13 is less.
S7	7/B	The bus of 7/B has covered 15 kilometres with 1 litre, it is more.
S8	7/B	When kilometres are equalled, the bus of 7/B is more economical.

When the data in the Table 4.4. has been examined, it has been observed that S6 and S7 coded students compare the distance covered with 1 litres of fuel while S8 coded student has compared the amount fuel used within the equal distance. Some of other students has commented as '*less distance with less fuel*' while the rest have remained indecisive.

It could be observed from the answers in the Table 4.4. that except S6, S7 and S8 coded students, all the rest have had difficulty in developing the necessary reasoning skill for the solution of the problem. For this reason, the researcher has divided the students into four groups and asked *"whether the bus of 7/C will use more fuel again if it goes from the same route as of 7/B bus"*. The discussion among the students after this question has been provided below.

S3 and S6: *First of all, we have divided 52 kilometres into 4 to find out the bus has covered 13 kilometres with 1 litre of fuel. Then, we have divided 45 kilometres into 3 and calculated that the bus has covered 15 kilometres with 1 litre of fuel. Since covering 15 kilometres with 1 litre of fuel is more economical, we think that the bus of 7/B is more fuel efficient.*

S2 and S5: *We have also calculated how much distance the bus has gone with 1 litre of fuel. Our way of solution is the same as that of our friends'.*

S4 and S8: *Firstly, we have tried to equalise the kilometres and tried to find the lowest common multiple of 45 and 52. Then, we would find how many litres the buses have used but since we failed to find the common multiple, we failed. Then, we have found the distance covered with one litre of fuel, as our friends have done. Because 15 kilometres is more distance, we have chosen the bus of 7/B.*

Although there has been the condition of covering the same distance within the question directed to students, only the group composed of S4 and S8 coded students have paid attention to this rule. However, these students have failed to calculate the least common multiple of 45 and 52 and therefore, they couldn't find the solution that they have to find depending on the condition. Here, since the students have failed to make a comparison by expanding the fractions representing the $\frac{52}{4}$ and $\frac{45}{3}$ ratios, they opted to make a comparison between the covered distances with 1 litre of fuel, instead.

Since S7 coded student who has failed to come up with a solution has not understood the solutions of his/her friends, he/she has asked to the researcher *whether the amounts of fuel used could be equalised with the same logic.* Then the researcher has asked this question to other students as, *"Your friend asks if the covered distance is not equalised, could the amount of fuel be equalised. What do you say? Could we make a comparison?"* and started another discussion. The conversation between S3 and S8 coded students are as follows:

S3: *We have found the distance covered with 1 litre of fuel anyway. In other words, we have already found out who could cover more distance with equal amount of fuel.*

S8: *Yes, but we can equalise litres without dividing them. The bus using 4 litres of fuel in 52 kilometres, 52+52+52, 3 times of 4 litres in three 52 kilometres, that is 12 litres of fuel. Likewise, for the bus using 3 litres of fuel in 45 kilometres, if we multiply 3 with 4, that is 12 litres of fuel and then, we can make a comparison.*

Having noticed that some of the students could not understand this way of solution, the researcher tells that the bus of 7/B class is more fuel efficient as it covered 220 kilometres with 12 litres of fuel while that of 7/C has covered 156 kilometres with the same amount of fuel.

The groups of S3-S6 and S2-S5 coded students have used the strategy of comparing rates while the other group composed of S8 and S4 coded students have tried to make use of rate expanding (fractions) strategy with the aim of equalising the distance covered, they have failed to do so, though. When they have noticed they could not find the least common multiple, they have converted into another strategy. However, S8 coded student has successfully implemented rate expanding strategy while he/she has been comparing the amount of covered distance with equal amount of fuel.

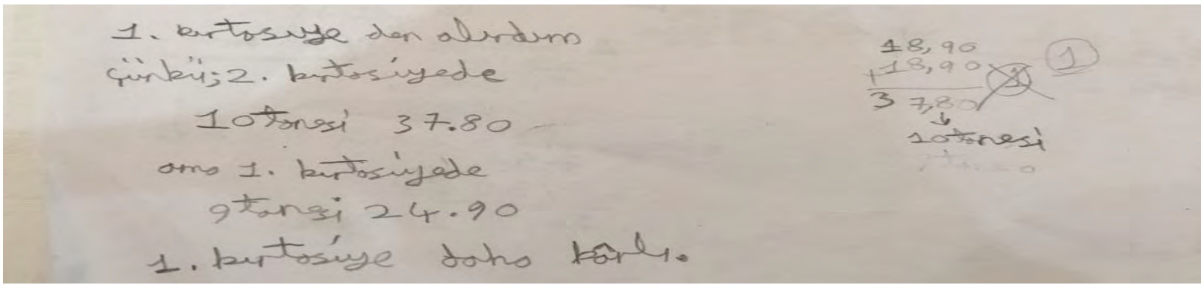
4.3. Findings Related to Activity-2

Within this activity, the students have been given a problem scenario containing two different situations related to an advertisement poster and they have been demanded compare rates.

Scenario: Your teacher has wanted you to buy board-markers to use in the classroom. You have gone shopping and seen two stationery stores written 'Discount' on its shop window. You have to make the best choice. The poster of one of the stationery stores says, 'Big Discount! 9 Board-Markers is only for 24.90TRY!'. The other poster of the second store says, 'Discount in Board-Markers. 18.90TRY for 5!'. How do you make the best choice? Please provide explanations for your answers."

The solution of S3 coded student has been given in Figure 4.1 below.

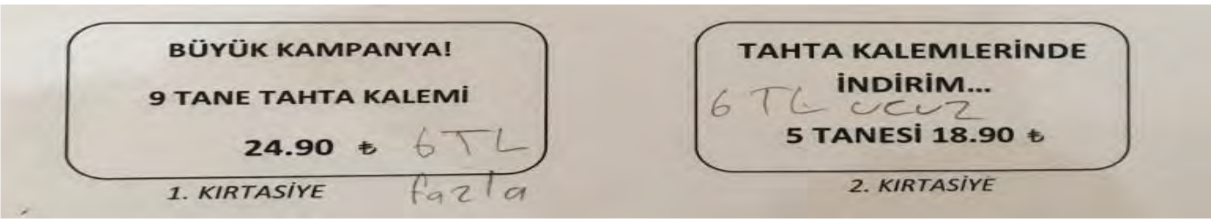
Figure 4.1. *The answer of S3 Coded Student for the 'Which is more profitable?' Activity*



S3 coded student has explained the way of solution as *"I find the price for 10 board-markers by buying twice from those whose 5 pieces is 18.90TRY"*. The expression of "twice" and the operation of $18{,}90 + 18{,}90 = 37{,}80$ presented in Figure 4.1 has been accepted as an indicator of the fact that the student has made use of additive reasoning.

The solution of S2 coded student has been given in Figure 4.2 below.

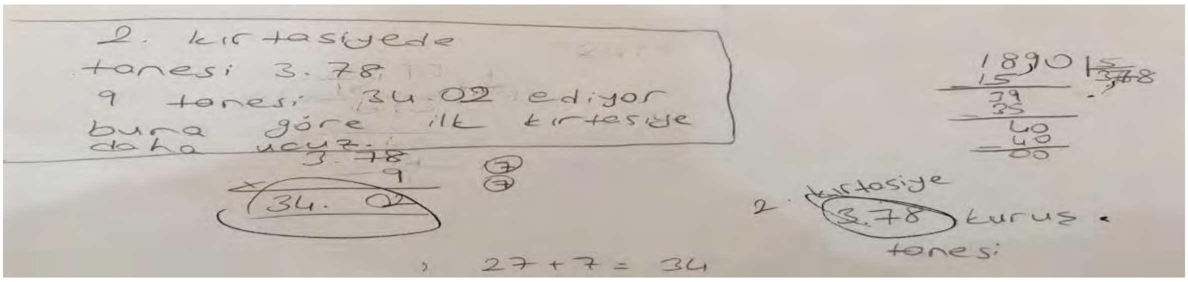
Figure 4.2. *The answer of S2 Coded Student for the 'Which is more profitable?' Activity*



When Figure 4.2 has been examined, it is observed that the S2 coded student has compared the prices within two stationery stores as ***"6 TRY cheaper"*** and ***"6 TRY expensive"***. This clearly shows that the student has used the additive reasoning while trying to solve the problem.

The solution of S8 coded student has been given in Figure 4.3 below.

Figure 4.3. *The answer of S8 Coded Student for the 'Which is more profitable?' Activity*



The data in the Figure 4.3 indicates that the S8 coded student has made a comparison between the prices of two stores by firstly finding the unit price of the board-markers of second stationery store and then by multiplying it with 9. The S8 coded student has made use of rate identification and expansion strategies to solve the problem.

The answers of all the students given to these problems have been provided in Table 4.5.

Table 4.5. *Individual Solutions of Students for Activity-2*

Students	Choices and Reasons
S1	Store 1 (Ne reasons.)
S2	Store 2. Because the price of Store 1 is 6TRY more.

S3	Store 1 because it is cheaper.
S4	Store 1 because 9 board-markers in Store 2 is more expensive.
S5	Store 1 because the price for 10 board-markers in the other store is 37.80TRY and it is more expensive.
S6	Store 1 is cheaper because 24.90/9=2.80TRY. The price in the other store for one board-marker is 18.90/5=3.90TRY approximately.
S7	Store 1 is more convenient. We can find one of the store is cheaper by dividing the prices.
S8	Store 1 because the price of one board-marker in the Store 2 is 3.78 and if we buy 9 board-markers from this store, we will pay 34.02TRY. This price is expensive from that of Store 1.

When the data in the Table 4.5 is examined, it could be observed that while S6 and S7 coded students have made use of the strategy of unit rate comparison, S8 coded student has benefitted from unit rate and ratio expansion strategies. S1 coded student has not come up with an explanation related to the solution. S2, S3, S4 and S5 coded students have made use of additive reasoning.

After the Activity-2, the researcher has made an unstructured interview with the students about whether and if so, where they have come across with a situation like this. The responses of the students from this interview have been provided below.

S8: *On one occasion, I went to the market with mom to buy a scouring pad. There were 3-piece and 5-piece packs. Mom directly bought a 5-piece pack by thinking it would be cheaper. But I looked at the prices of both packs and divided the prices into 5 and 3 and found out that one pad in the 3-piece pack was cheaper than those of in the 5-piece pack. Then I immediately told mom to change the packs and we bought 3-piece pack. I learnt that multiple piece packs are not always convenient.*

S4: *I have noticed that we use mathematics in many situations in daily life. I have two siblings and we all like to eat chocolate spread bread. For this reason, chocolate cream doesn't last longer at home and so, our dad buys large size chocolate creams. There were two chocolate spreads in the market within 750 grams and 500 grams of jars. There was a price tag on the larger size and dad directly bought it by thinking it would be more profitable. But actually, 500-gram spread was more convenient.*

Researcher: *So, how could you find out that the 500-gram spread was more convenient?*

S4: *I multiplied 500 with 2 but couldn't reach a solution by this way. Then, I divided 500 by 2 and added the quotient over 500. That is 750. Then, I divided the price by two and added the first price again and I found out that it was cheaper.*

When the results obtained from Activity-1 and Activity-2 and the student performances in the unstructured interview are taken into consideration, it could be stated that S8 coded student has a structured reasoning about rate concept while S6 and S7 coded students have semi-structured reasoning about rate concept. By having no strategy and commenting by adding the price of 250-grams spread onto 500-grams price instead of determining the price of 750-grams spread through multiplying 250-grams price with three, S4 coded student could be said to remain at the additive reasoning than multiplicative reasoning. Therefore, S4 coded student could be stated to have unstructured reasoning about the rate concept. Additionally, not having an exact explanation and solution while trying to solve the problems, S1, S2 and S3 coded students could be said to have unstructured reasoning about the rate concept.

4.4. Findings Related to Activity-3

The students have been divided into two groups. The researcher has given to students a pouch containing half kilogrammes of white beans and a cup of shell beans and told them to designate a representative for the groups. Then, the students have been instructed to replace 100 white beans in the pouch with 100 shell beans and to stir the bag.

Task-1: The representative students have formed sample groups containing 25, 50, 75, and 100 beans and identified the number of shell beans in each sample group. The representative students then have told their friends to fill in the blanks in the tables related to each sample group with the approximate number of shell beans and the total number of beans. The students have identified the ratio of shell beans in each sample group to the total number of beans through this way. In other words, they have determined the ratio of a part (the number of shell beans) to a whole (total number of beans in the pouch) for each sample group in the case of the division of a whole (beans in the pouch) into two parts (white and shell beans). This task has been carried out by all the students.

Task-2: Each student has been asked to estimate the total number of beans in the pouch by examining the data they have written during Task-1. The discussion between the students and the researcher has been presented below.

S3: *We, first, drew 25 and then 50 beans. The number increased but how could we find the total number? I think we couldn't…*

Researcher (R): *Is there anybody thinking differently? I, think no. So, is there any quantity in the pouch that you are sure about its number?*

S4: *Yes, there are 100 shell beans.*

R: *What others think about this opinion?*

S1: *Yes, we put 100 beans in the pouch (The others approve of this statement).*

R: *Then, has the total number of beans in the pouch increased? You say you have put 100 beans.*

S1: *No, teacher. We have removed 100 white beans, as well. The total number has not changed.*

R: *Then, if you are sure about the fact that there are 100 shell beans in the pouch, could you predict the unknown quantity in the pouch, that is the total number of beans?*

Buy the help of directives of the researcher, the students in the first group have tried to reach the solution by expanding the ratios they obtained during Task-1. However, they have noticed their mistake soon afterwards and then determined the total number of beans by considering the number of shell beans (100). In order to reach the total number of beans in the pouch, they firstly, have identified that they have to multiply 7 (the number of shell beans in the first sample) with 14 to reach 100. Then, they have multiplied 25 (the total number of beans in the first sample) with 14 and concluded that the total number of beans in the pouch could be 350.

The operations and calculations implemented for the first sample group have been applied to the second, third, and fourth sample groups. At the end of each application, they have found the total number of beans in each sample group as 500, 493 and 500, respectively.

The same operations have been carried out by the students in the second group and the total number of beans calculated by the second group students have been found as 420, 490, 500 and 485 for the first, second, third, and fourth sample groups, respectively.

The researcher has asked the students to *'find out a single numerical value for the total number of beans'* by taking four different values into consideration. The students have stated that they could specify a single value for the total number of beans in the pouch by calculating the arithmetic mean of four different values they have obtained during the tasks. After these calculations, the first and second group have specified the total number of beans as 460 and 473, respectively.

The students have been asked to explain the reason of different values between predictions. Both groups have stated the first reason of these differentiations as the chance factor. For other reasons, one of the groups has stated that they have made rounding-off after calculations and the other group has indicated the number of sample groups caused the differentiation between the predictions and remarked that as the number of experiments increase, they could find more approximate values.

The students have noticed that the value of $\frac{7}{25}$ ratio they have found in the Task-1 expresses the total number of shell beans relative to the total number of beans. They have also realised that the ratio related to the number of shell beans specified for each experiment is so close to that of the total number of beans. Within this context, they have concluded that "the ratio doesn't change as the number of samples increases or decreases".

4.5. Findings Related to Activity-4

Within this activity, the researcher and the students have gone to a pizzeria and the students have been divided into two groups. One group consists of three students while the other is composed of five students. Two pizzas for the three students in the first group and four pizzas for the five students in the second group have been ordered.

Task-1: The students have been asked to identify the number of persons and pizzas (by drawing) of their table and they have written the number of persons and pizzas without any mistakes. The students have drawn the pizzas in the form of a circular figure, like a pie chart, having eight slices but the slices are not equal in size. After being asked the reason of their choice in drawing the pizzas in a circular figure, two of them have stated they have drawn it by imaging the shape of a pizza while five students have remarked they have drawn the figure by looking the pizzas on the table. These explanations have indicated that the students have drawn the figures through non-mathematical affective reasoning and by taking the shapes of pizzas on the table into consideration.

Task-2: The students have been asked whether they accept to switch places with another student from the opposite group when they are hungry and would much like to eat pizza. They have also been requested to provide explanations about their choice.

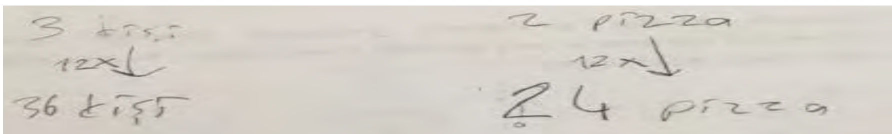
Each student has determined and compared the number of pizza slices that could be eaten by each one of their friends by considering the number of persons and pizza slices in each table. However, they opted to use such expressions as "5 or so" or "6 or so" rather than such fractional expressions as $\frac{16}{3}$ or $\frac{32}{5}$ when talking about the number of pizza slices to be eaten by one person. This choice indicates that students tend to use approximate whole numbers instead of fractions while expressing ratios such as $\frac{16}{3}$ or $\frac{32}{5}$.

Task-3: Each student has been asked to think themselves as the organiser of the next pizza party. They have been told to order pizzas beforehand in order that the pizzas could be ready when all the students have come up to the pizzeria. The main question asked to them has been to identify the number of pizzas to be ordered for each student coming to the party to be able to eat the same amount of pizza that the organiser is now eating.

The answers from the first group have been provided below.

S1 coded students has stated that 2 pizzas have been ordered for 3 persons in his/her table and starting from this point of view, S1 has drawn the schema provided in Figure 4.4 in order to identify the number of pizzas to be ordered for 36 persons. Since the number of students attending the next party (36) is 12 times as much as the number of students in the current group (3), S1 coded student has thought that the number of pizzas to be ordered for the next pizza party should be 12 times higher than the number of pizzas ordered for the current group. Therefore, S1 has found the number of pizzas to be ordered as $2\times 12 = 24$.

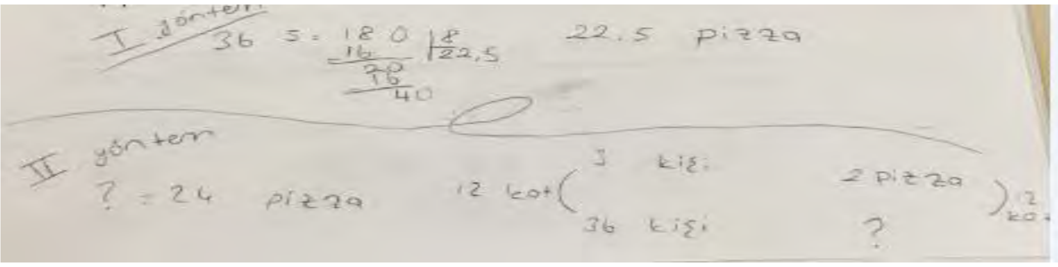
Figure 4.4. *The Solution of S1 Coded Student*



S8 coded student being in the same group with S1 has proposed two different ways of solution to the problem. In the first way of solution, S8 has determined that since each student in his/her group has eaten 5 slices of pizza, the number of pizza slices to be ordered for 36 students would be 180. Then, by thinking that each pizza consists of 8 slices, S8 has concluded that 180 slices of pizza would be 22.5 pizzas. The second solution has been the same as that of S1 coded student.

The researcher has asked S8 coded student whether the first way of solution is correct or not. Then, he/she has stated that the way of solution "is not completely wrong"; however, since there are *"many long and challenging"* operations within this way, he/she has indicated there may be calculation errors. The researcher then has pointed out the fraction of $\frac{16}{3}$ that is the number of pizza slices each student in his/her group has eaten. Through this information, S8 coded student has been reminded that each student in the group has not eaten 5 whole slices and 1 leftover slice has been shared among three of them. Therefore, S8 coded student has indicated the value of 22.5 that he /she has found in the first way of solution is wrong due to not considering 1 leftover pizza slice. Additionally, S8 coded student has remarked that the way of solution in which he/she has tried to find out the number of pizza slices eaten by each one of his/her friend is illogical according to the way of solution that is based on scalar relationship among the same quantities. The way of solution of S8 coded student has been given in Figure 4.5 below.

Figure 4.5. *The Solution of S8 Coded Student*



The second group has had difficulty in finding a solution to the question of *"How many pizzas would be ordered for 36 persons if 5 persons ate 4 pizzas?"*. Only S7 coded student has proposed a way of solution based on the number of pizza slices. Within this solution, roughly 7 groups have been identified consisting of 5 persons out of 36 students. Since there are exactly 32 pizza slices in each group of 5 and there are 7 groups, $7 \times 32 = 224$ pizza slices are necessary for the next party. Due to the fact that each pizza consists of 8 slices, it is necessary to order $224 \div 8 = 28$ pizzas.

Since there are no extra ways of solution from the second group, the researcher has provided the way of solution based on the number of pizza slices with S7 coded student. In this way, the attention of the students has been drawn to the fact that 1 student will stay out of the groups if 7 groups with 5 students (4 pizzas for each group) have been formed for 36 students. Besides, because the number of students (36) for the next party is 1 more than 7 times as much as the total number of groups of 5, the number of pizzas to be ordered for the next party should be 1 more than 7 times as much as the number of pizzas ordered (4) for the group of 5. Therefore, the number pizzas to be ordered has been identified as $(4\times 7) + 1 = 29$.

It has been concluded at the end of Task-3 that the students have opted to use rounded whole numbers while identifying and comparing rates instead of fractional numbers representing the related rate value and that this choice has caused them to make faulty interpretations and wrong calculations about the problem.

5. Discussion and Conclusion

The results of this study which examined the effect of authentic investigation activities on the reasoning skills of students while learning the concept of ratio have been identified by considering the epistemology of ratio concept. The reason why the participants have been selected from 7th grade is that the study will reveal the effect of authentic investigation activities on the reasoning of ratio concept by first identifying how well the concept has been learnt and structured in 6th grade. Within this context, a starter activity has been prepared by considering the acquisitions within 6th grade mathematics curriculum related to ratio concept.

It has been identified at the end of the starter activity that apart from two of them, all students have set up an additive reasoning while comparing the relationship between the amounts of rice and water. This result indicates that students have developed additive thinking ability by using additive reasoning during elementary school while they don't have the ability to use multiplicative thinking skills since they have not set up a multiplicative reasoning ability in their mind. The studies in the literature have proposed the existence of additive universe before forming of multiplicative reasoning and they have even stated "the transition from additive reasoning to multiplicative reasoning" as the biggest hurdle in learning mathematics (Jacob and Willis, 2003; Lamon, 2007; Lamon and Lesh, 1992; Siemon et al., 2006).

The purpose of the starter activity has been to identify the connotations of ratio concept within the students' mind and it has been identified that the ratio connotes fractions and dividing for the students having higher and moderate level of success. The reason why the ratio concept connotes dividing in the mind of students could be that class books of mathematics in the secondary school teaches "ratio" as "the comparison of two quantities by being divided to each other". In addition, the expression of "while writing ratios of two quantities, the first number is written as the numerator and the second is written as the denominator" within the same books could have induced students to relate the concept of ratio to fractions. Another reason of this connotation could be that the concepts of ratio, fraction and dividing have all the same symbolic representations.

Another conclusion of the starter activity is that students do not have structured reasoning about the concept of ratio since they have all tried to explain the concepts of rate and ratio only by considering the lexical meanings. In order for students to have structured reasoning about ratio concept, activities including different contexts but having the same mathematical point of view should be developed (Lamon, 1995). The use of contexts comparing ratios will definitely help students develop and improve multiplicative or rational reasoning skills. Through considering these points, Activity-1 and Activity-2 have been prepared with the aim of the fact that students make sense of rate by making a comparison between rates.

The students have been anticipated to identify the necessary criterion by determining which bus is more fuel efficient in Activity-1. In order to identify this criterion, they are expected *"to compare the distance to be covered with 1 litre of fuel" or "to compare the amount of fuel to cover a specified distance"*. S6 and S7 coded students have compared the distance covered with 1 litre of fuel. Only the student coded S8 has compared the amounts of fuel to cover a specified distance; however, as he/she has failed in *"equalising kilometres"* in his/her own words, he/she has not reached a solution through this way. S8 coded student has failed to think the amounts of fuel used per 1 kilometre, in other words, the rate values of $\frac{4}{52}$ and $\frac{3}{45}$. Instead, he/she has opted to compare the rate values of $\frac{52}{4}$ and $\frac{45}{3}$, the distance to be covered with 1 litre of fuel. The reason why S8 coded student has opted to use the values of $\frac{52}{4}$ and $\frac{45}{3}$ could be that they are equal to whole numbers, 12 and 15, respectively.

The Activity-2 has been prepared in order for the students to interpret the concept of *unit price* by using rates. The students have been expected to find that the amount of money to be paid for 9 board-markers is 9 times as much as the unit price by using the unit price as the rate. The students coded S5, S6, S7 and S8 have easily gone through this activity.

It has been aimed through Activity-3 that students could expand the concept of rate into density concept as a tool in order to predict the number of objects (the number of cats, people or beans) in a given area. The students have tried to predict the total number of beans in a pouch containing 100 shell beans through considering the fact that the number of shell beans in each sample they have taken is one fifth of the total number of beans in the pouch. By the help of this activity, the students have learnt that quantities in a ratio change as folds of each other.

The students have carried out the necessary operations by comparing, expanding and reducing the rates. However, as observed throughout the study, the inadequacy of students in arithmetical operations has revealed during this activity, as well. The students have opted to provide an approximate answer through whole numbers without calculating decimal numbers. Since the calculation requires decimal operations, they have failed to notice 1 leftover slice has to be equally divided by three students in the first group and 2 leftover slices have to be equally shared by five students in the second group. Instead of this, they have tended to solve the problem by setting-up a scalar relationship between two quantities in the same measurement space. The scalar relationships in the same measurement space have been correctly set-up in the case that quantities in the same measurement space are whole numbers while the relationship has not been set up in the contrary case.

The results and findings of Activities 1, 2, 3, and 4 have clearly indicated that the students opt to use rates containing whole numbers. This result supports other findings in the literature stating that students more easily interpret the relationships between quantities containing whole numbers than those having no whole numbers (Fernandez et al., 2011; Tourniaire&Pulos, 1985). The inadequacy of students in arithmetical operations has been clearly observed within problem situations composed of decimal number ratios. This finding could be supported by other results in the literature indicating students provide wrong answers to the problem situations containing non-whole number ratios between quantities by using wrong additive strategies (Cramer et al., 1993; Misailidou&Williams, 2003; Van Dooren et al., 2010 a).

Through the authentic investigation activities for teaching ratio, the students coded S5, S6, S7 and S8 having higher level of success in mathematics have successfully structured the concept of ratio as the multiplicative relationship between two quantities. These students have also internalised the fact that quantities relatively, in other words within a multiplicative relationship, change at the same time. While they are not able to formally identify how many units a quantity will be equal to when measured through another quantity, they are informally able to make this calculation by the help of their experience from daily-life. Within this context, authentic investigation activities could be said to have positive effects on the reasoning process of students about the concept of ratio.

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Authors

Pınar Anapa Saban, Eskişehir Osmangazi University, Eskişehir/Turkey. E-mail: panapa@ogu.edu.tr

Merve Koçyiğit Gürbüz*,* Milli Eğitim Bakanlığı, mrvkocyigit@yandex.com