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TREATMENTS AND CONVERSIONS IN STUDENT WRITTEN SOLUTIONS TO WASAN GEOMETRY PROBLEMS

Özlem ÇEZİKTÜRK, Ahmet Şükrü ÖZDEMİR

Abstract: Cognitive difficulty arises from two types of cognitive processes: treatments; within the same, conversions; between different types of representational registers. Conversions are difficult since they ask for understanding of two representations. Direction and the choice of first register could be a threshold for the student. Wasan geometry is used as a thought provoker and pre service mathematics teachers' written solutions to those problems for usage of cognitive processes: treatments and conversions are analyzed by content analysis method of qualitative research. Descriptive data regarding number of conversions and treatments are given. As much as correct / partially correct answers, the relation between cognitive processes and answers were analyzed. Students' written answers to three Wasan geometry problems on tangent circles are qualitatively analyzed for specific examples of treatment processes and conversion processes. Neither problem type nor correct answer is a reason for the choice of representational registers. One treatment may be worth many conversions. More usage is related to students' conformity. Direction of the conversion may be misleading. None of the students stays in mono-functional registers. Top and lowest grade students use more treatments and conversions. Usage of more conversions is not related to success and result in less number of treatments.

Key words: Wasan geometry, Representations, Cognitive processes, Treatments, Conversions

1. Introduction

Wasan geometry flourished in the Edo seclusion period of Japan by the hang Wooden blocks from the roofs of temples as challenging mathematical problems. These did not consist any solutions. Monks, samurais, house-wives or even students would solve them (Hidetoshi & Rothman, 2008; Rubin, undated). Although most of these problems were in geometry, some were in algebra. Due to its systematic structure, Wasan geometry was chosen to promote mathematical thinking in this study.

Foundations of concepts, thinking and the abilities as much as problem solving strategies and the way that mathematics is part of daily life are important constituents of mathematical learning (MEB 2013). Same list includes different representational registers and using them to demonstrate concepts. Mathematical thinking includes thinking procedures with mathematical concepts and structures such as guessing, thinking via shapes or without shapes, induction, deduction, generalization, giving examples, thinking with concepts and lines of mathematical thinking, proving, approaching by patterns, data gathering, etc. (Schoenfeld, 1992; Wares, 2016; Alkan & Bukova-Güzel, 2005; Kahramaner & Kahramaner, 2002; & Yıldırım, 2004; Stacey, 2006). Thinking becomes easier with the representational registers. Choice of the mode of representation depends not only to the problem but also to the concepts, thinking procedures and with respect to ease of use. Mathematical thinking asks for an artifact, some uses graphics, diagrams, and some use only verbs. These all end up with the representational choices, which build the structure of the solution. Stacey (2006) gives Fermat and his last theory as an example to this. "To prove asks for passing through many dark rooms to enter into a lighted one" as in his words. To think like a mathematician is the aim and by solving problems, by inquiring mathematical projects, by modeling, and by relating, it is possible to be a part of this intellectual promenade (Stacey, 2006; Devlin, 2012).

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2. Representations

Representations are physical realities that ideas, concepts, and processes can be changed by learners (Lesh, Post & Behr, 1987, as cited in Akkuş & Çakıroğlu, 2006). Multiple representations can be used as tools for understanding mathematics and developers of new mathematics that students ask but prefer one representation over the others, frequently. Most preferable and mathematical representation mode is symbolic mode (Akkuş & Çakıroğlu, 2006). It is followed by tabloid and algebraic modes. Schematic mode is the least preferable at least if not given by the question developer since it is like modelling which asks for high level cognition. Most preferred translation between representational modes is from tabloid to symbolic conversion. The least found is algebraic to symbolic mode conversion. Students prefer what they like, what they can manage and what they are emotionally attached to. If they feel close to a single representation mode, they choose unconsciously. Students should be shared positive and negative sides of each representational mode to overcome this fact in daily schooling (Akkuş& Çakıroğlu, 2006).

In mathematics learning and problem solving, following five representational modes are seen in general (Lesh, Post, & Behr, 1987): real world examples, static snapshots, models that can be manipulated, symbols and daily language. Problems happening within conversions, affect both mathematics learning and problem solving performance. For a student to understand an idea it takes a) identifying the idea in multiple representations, b) manipulating in the given system, and c) conversion from one system to another. Conversions can be thought in parallel to the modelling processes. Teachers can ask like : "use your own words, show a graphics which show the situation", "describe a problem in similar situation, think with real objects", "write some expressions for a line of word problems", etc. Good problem solvers are the ones who can find related representational modes and structures and who can translate. Representation mostly includes some sides of a situation, and some connections between different representations. Each partial relation includes only a way of representing, not from partial to whole (Lesh, Post & Behr, 1987).

2.1. Misconceptions due to representations

Some misconceptions may evolve while learning and should be predicted. If they are due to weak and consistent concept model, they may hinder learning. Duval (2017a) points to a cognitive paradox within the duality of representation of mathematical object without the representation: "Can someone approach to the mathematical object as intended, by the help of the representation?" The answer is: not always. Treatments and conversion processes and specialties of the concept may enable this closeness if used properly. Different types of experiences may affect each other's effectiveness and limits (D'Amore, 2002). For example, to show an angle with an arc around the angle may flourish some problems since, showing arc far away and close to the vertex would seem the angle differs by measure even though it does not (Santi & Sbaragli, 2007).

D'Amore (2002) points to a duality. If the student can only face a semiotic representation; how can the student know the mathematical object as a whole? Representation carries one or a few types of characteristics and this in turn would result in the limited version of the knowledge of the object. If the student is aware of this fact, the representations may be increased. But if unaware of this fact, may be filled up with misconceptions. D'Amore (2002) names conceptual attainability as noetics and attainability with symbols as semiotics. If teachers know semiotics and noetics as same, then they would not give attention to treatments, conversions or in general representations. In result, semiotics become more complicated.

2.2. Semiotics and Wasan geometry example

With respect to Berger (2010), doing Mathematics and learning Mathematics are semiotic cooperation. Signs are tools for thinking, comprehension, reasoning and learning. Each sign has a triad structure: an object, an idea and an interpretation. (Peirce, 1998, as cited in Berger, 2010). For example, if derivative is the object, ideas reflect the representations of the object symbolic, descriptive and graphical, and interpretation may be the concept limit in relation. With a semiotic viewpoint, (Rotman,1993, as cited in Berger,2010) the idea and the object complete each other. Duval (2017) argues that the origin of the mathematical activity lies within identification of the same object between different types of representational registers. Here, two contents should not match.

Another characteristic of a semiotic representation is its transformability. Each semiotic representation is unique in itself, and many semiotic representations establish the mathematical objects up to a degree. Duval's test (2017): A) Can we juxtapose object and its representations? B) Can we decide if two representations belongs to the same object or not? Or in short; are these two representations are of the same thing even if the content differs a lot. Literature shows related problems with the representation of the number line. Some students are having problems with identifying the distance between two opposite signed numbers. This can be seen in analytical geometry courses even in the university level. Duval (2017) renames this problem as zero barrier. Not understanding the concept of zero in the number line prohibits the formation of the concept of negative numbers. Wasan geometry has mostly schematic semiotic representations. While solving, some symbolic and algebraic semiotic representations can be seen.

2.3. Theories of semiotic representations

Duval's theory of semiotic representations points top three important ideas: same mathematical object has many semiotic representations as semiotic modes. Each different semiotic representation identifies one property of the object, semiotic representations should not be mixed with mathematical objects. Finally, each semiotic representation should be based on the semiotic mode it is supported. Treatments and conversions are two cognitive activities and former takes into account within same mode translations and the latter takes into account different mode translations (Pino et al., 2015). Conversion creates problems such as not matching content between source and destination. Sometimes treatments may be problematic with the use of natural language who lets demonstrations. For a system to be named as semiotic: there should be 1) a single or line of footprints of the object, 2) transforming representations into more meaningful representations, 3) transforming so that new meanings can develop. Any look should infer rom mathematical objects, processes, meanings and representational modes (Pino et al., 2015).

3. Treatments and Conversions

Teachers should be educated for mathematical thinking by problem solving as a dynamic endeavor to feel like mathematicians (Tataroğlu-Taştan, 2013; Fraivillig, Murphy, & Fuson, 1999). Time seems an important factor here, hence students should be let longer times while solving problems to achieve freedom in mathematical thinking procedures. Which representational modes were chosen in the answer is important as much as how these modes work together towards a meaningful solution process. Duval (2006) gave two obstacles while cognition taking place: treatments and conversions. Treatments are transformations of thinking within the same register while conversions work in the translation from one type of register to another.

Duval (2006) argues that mathematical processing always involves substituting some semiotic representation for another. (p.107). Here, important part is transformation but not representation itself. One needs to understand that the only way to access mathematical objects are through semiotic representations that is hard to identify apart from the mathematical object. Some processes are easier on one semiotic representation while others choose one and only one representation type. In most cases though, a system of representations involves two types of representations. Three or more representational transformations are problematic due to the possibility of cognitive overload (Sweller, 1988). Ability to change from one representation system to another is mostly a critical threshold in learning and mathematical problem solving. In geometry for example, verbal / graphical or symbolic / diagrammatic transformation of representations is a need.

Two types of transformations of semiotic representations to recall are treatments and conversions. Carrying out a calculation can be a treatment, on the other hand, while seeing the graphical visualization of a function $y = 2x + 3$ is a conversion from symbolic to graphical. Because, as we already explained treatments are supposed to be within the same register as from graphic to graphic or from symbolic to symbolic. In conversion,on the other hand, the objects are not changed only the mode changes. In other words, a conversion can be from symbolic register to graphical or from symbolic to tabloid for example. Since, it requires to function in two very different representation systems, conversion is more difficult. Duval (2006) argues that problematic mathematical understandings may happen due to this. Interestingly, semiotic representations can be transformed into other semiotic representations. Treatments need some conversions to happen in parallel (Duval, 2017).

Problem of seeing a perpendicular angle when there is not one can be a problem source. Student identifies the angle as a perpendicular one although it is either slightly smaller than or slightly greater than a perpendicular angle. Here, the process is conversion from graphical to symbolic mode. And, the faulty conversion process may cause to identify the problem as Pythagorean Theorem problem although it is a cosines theorem application.

3.1. Problems with conversions

A conversion can be misleading for example: If the multiplication of two variables is positive, than the numbers are both positive or both are negative, but two being positive and two being negative are not equal! (Duval, 2006; p.113). Hence, more than one conclusion can come out of the one proposition.

“Whatever the level and whatever the field, the non-congruent conversions (previous examples) are for many students an impassable barrier in their mathematics comprehension and therefore for their learning.” (Duval, 2006; p.123). The direction of the conversion may form the problematic part. One understandable way is mostly used more frequently but for other way, careful choice of necessary and sufficient conditions is needed. How can we make our students to be aware of these? Or can we make them differentiate among these? The problem of comprehension in learning mathematics is a problem both of recognition and discrimination. (Duval, 2006)

Students need to dissociate of represented object and content of the object first represented. Students must discriminate between the relevant part of the representation and irrelevant part for the problem. Two sources of incomprehension can occur: 1) treatments in multifunctional registers: to see in geometry, one needs to identify possible full configurations or sub-configurations. One can stay in mono functional registers; hence it turns into an algorithm. 2) In case of conversion, one to one mapping between the source and target representation is important and helpful. The analysis of target representation is needed. The organization happens within the source and target representations. Finally, direction of conversion acts like a threshold (Duval, 2006).

Cezikturk (2019) investigates Interactive Diagrams (abbreviated as ID) and the value of the direction of conversions in them. Student, first deduce meaning from the static representation. Then, she or he interacts with the ID, meanwhile, translation types and representation modes are analyzed. Individual parameters are detected for their prospective effects on the system. This in turn leads to identification of mathematical relationships to be synthesized. Patterns are analyzed with pattern ends. Student, uses the insight that she /he receives from this example, on different but same topic IDs. Even though, the idea that semiotic representation as much as student count is supported, students with a good understanding of Instrumental Genesis has a much higher degree of success in the process. Since conversions are difficult to understand, students use more treatments as more convenient. In case of the direction of conversions, if against intuition, students get perplexed and have difficulty.

Mathematical thinking processes depend on cognitive synergy of registers of representation. As if there is an extension of mental capacity, different representation systems should coordinate. In reality, conversions do not function in separated regions. They happen to be around treatment processes. Problem solving is a series of representations by whom we proceed from source to target registers. In order to make students notice the basic visual features of oppositions that are mathematically relevant and cognitively significant, representation discrimination tasks have to be integrated into a conversion task. It is only by investigating representation variations in the source register and target register, students can realize these (Duval, 2006). Sometimes, a conversion cannot be separated from treatment because it is the choice of treatment that makes the choice of register relevant. (Duval, 2006).

Following research questions are investigated:

1. What is the cognitive synergy of registers in Wasan Geometry?
2. How these different types of transformations add up to thinking and prohibit thimking in Wasan Geometry?
3. How can stundent decide upon what part of the conversions and treatments are relevant and irelevant in Wasan Geometry?
4. How to recognize and discriminate suitable transformations of modes of representation in Wasan Geometry?
5. How direction of the conversion affects thinking in Wasan Geometry?
6. How non congruent conversions problem affects student understanding in Wasan Geometry?

4. Methodology

4.1. Research Design

As methodology of the study, *content analysis* was thought. Content analysis as a method roughly generates from communication research. It is about analysis of the contents of communication to study human behaviors. In qualitative research, student work can be analyzed through in-depth analysis. Approach involves defining terms, explaining analysis units, explaining coding approach and finally presenting results. Researcher is supposed to determine the categories before the analysis begins. Hence, the codes are important. This approach is mostly discussed for possible validity and reliability issues. Coding generator (researcher) may add subjective opinion. In addition, approach is criticized for being limited to recorded data as in the case of student work as home works or exams. Mayring (2000, as cited in Kohbacher, 2006) gives some solutions to most of these problems by his specific approach to content analysis method. He proposes following content analytical rules and step by step models without rash quantification. Each unit of code analysis must be allocated to specific categories. Theory guided analysis is proposed. Inclusion of quantitative steps of analysis and triangulation can be used. Method should be systematized. From three distinct analytical procedures; *structuring*: filtering out particular structures from the material, was used in this study. Dimensions of structuring was taken based on the representations theory of treatments and conversions processes of Raymond Duval (2006).

4.2. Participants

As final work of the mathematics and art course, we asked 27 students' solutions to three Wasan geometry problems (Figure 1) not on voluntary basis. Students enrolled within a public university, in department of Primary Mathematics Teacher Education and are preschool mathematics teachers.

4.3. Research Instrument and Procedure

Data gathering tool was a homework assignment on three Wasan Geometry Problems related to the tangent circles involving circles, squares, symmetry, Pythagorean Theorem, polygons, etc. The difficulty level was not too high. Writing mathematics is a hard way of communicating.

Problem 1: If the big radius is R and small radius is r, and the side of the square is a, how are they related to each other?

Problem 2: If the two sides of the rectangle are related by $\sqrt{2}$, how are big half circles with radius R and smalle circles wirh radius r are related?

Problem 3: If the side of the square is a, and the circles' radius are r, and R , how are they related?

Some answers are without an in-detail explanation of the thinking behind it. These problems were chosen also due to their openness to different registers of representation and to possible transformations between them (Figure 1). Diagrammatic to symbolic or symbolic to diagrammatic conversions and treatments within diagrammatic, symbolic representations occurred.

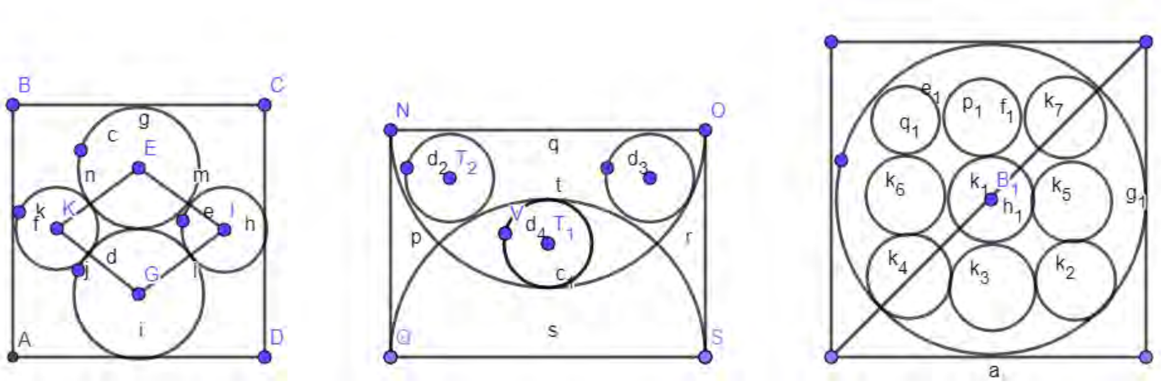

Figure 1. *Three Wasan geometry questions used in the study*

First question asked for the relation of radiuses of the tangent circles to one side of the square which they reside (the quadrilateral is from a students' answer). Second question gave only side to side ratio of the rectangle as ($1\sqrt{2}$) and asked for relation between radiuses of five small circles and radiuses of big circles. Third question asked for the relation between radius of small circle and big circle and the square. All three questions consist of tangent lines, circles tangent to circles, Pythagorean Theorem, quadratic equations, diagonals to squares and rectangles. We asked students to write all they think for the solution (Fraivillig, Murphy, & Fuson, 1999). Student answers were collected on paper. We taught Wasan Geometry by the examples we have used in our previous article (Çezikturk-Kipel & Özdemir, 2015). The problems were a little bit more difficult. However, here thinking lines were more important than the solutions to the problems. Also, student answers were categorized as full correct answer, partially true explanation and false explanations as in Yeşildere and Türnüklü (2007). By this way, stds' answers were categorized for the way they used conversions and treatment processes. In figure 2, a full correct answer figure that a students drew can be seen.

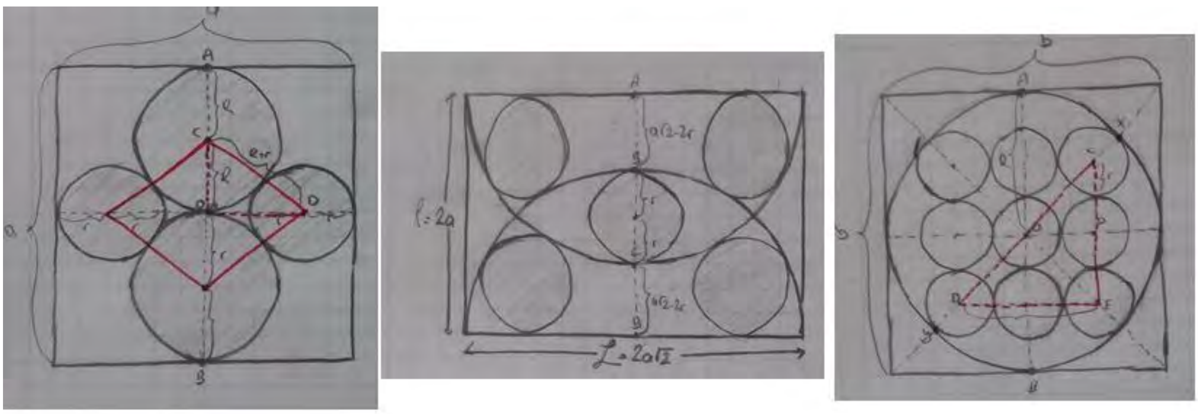

Figure2. *Three Wasan geometry questions used in the study (a students' solution to all)*

All three questions involved some symmetry within. First problem included with respect to y and x axes, vertically. Second problem included symmetry again with respect to y and x axes but horizontally. Third problem included symmetry with respect to y = x, y = - x, and with respect to y and x axes. Symmetry turned out to be an important factor in thinking steps, but it will be analyzed in another paper.

4.4. Data Analysis

If students have used diagrammatic to verbal, verbal to symbolic, symbolic to diagrammatic translations in their answers to problems, those were noted as conversion. If they have showed some transition within diagrammatic representation, verbal or symbolic, then it was noted as treatment process. Their problems with these kinds of translations were counted and analyzed in depth. Written answers were included as instances of thinking lines. The sources of difficulty within these transitions were investigated. Duval (2006) argues that these processes are the main source of difficulty in problem solving and mathematical thinking.

4.5. Results

Some students used diagrams actively while others used them only for reference. They needed to translate from diagrammatic to symbolic and to verbal what they think. Some did those without any hesitance but others had some difficulty in the way Duval explained. Unsystematic exploration, lacking logical mathematical activity and procedural thinking without connection to meaning were all searched for, from the student answers (Henningsen, & Stein, 1997).

Table 1. *Analysis of Students' answers to three Wasan problems*

	No of totally true	**No of partially true**	**No of wrong argument**
Question # 1	22	3	1
Question # 2	17	8	4
Question # 3	19	4	3

As can be seen from the Table 1, students' answers were mostly totally true. Wrong argumentation in the questions one by one; 1, 4, and 3. Hence, question 1 was much easier to understand and did not cause any confusion from students. From these results, it can be concluded that the difficulty level of the Wasan problems goes from easy to difficult but q2 being the most difficult. 1st problem was solved by most students; 22 out of 26. Q3 was solved by 19 students out of 26. Q2 turned out to be the most difficult of the three questions since 17 could solve it while 5 solves partially. In table 1, raw sums result in more answers than expected is due to students' multiple answers to some questions.

Table 2. *Mean grades of the students from the assignment*

Students with all answers true (syntax problems, calculation faults, etc.)	88.6
Students with partially true answers	82.47
Students with wrong arguments	73.83

Students' grades from the task also gives a spectrum of task difficulty level. 7 students graded as 100; full grade. Also, 2 students got 99 and, 1-97 and 2-95. 9 students' mean grades were 88.46 and these students answered all three questions with totally correct explanation. Mean Grades of partially true answered students, was 82.47. 22 out of 26 students answered at least all problems partially correct with no wrong explanations (Table 2).

Table 3. *Student numbers with All True, Partially True and Wrong Argument*

	Students with totally true answer	**Students with partially true**	**Students with wrong arguments**
Question #1	All except…	16, 23,26	18
Question #2	All except…	1,6***,7,9,10*,12**,24,25	6***,10*,12**,**17**
Question #3	All except…	6,7,8,13	2,5,**17**

*, **, *** students who gave two answers for the same question

In Table 3, student answers to each three questions were cross tabulated for partially correct explanations or for wrong explanations. Numbers on the right referred to the student paper numbered while analyzed. One star referred to student number 10 giving two different answers to the same question. This also adds up to the complex thinking of the students. Similarly, student numbered 6 and student numbered 12 are the ones who gave separate explanations for the same question. Student numbered 17 was special in the way that he/she was the one with the lowest grade from the homework as 50. Although he /she answered first question as true. For the second problem, answer included a perpendicular angle although it did not exist. In the third question, answer included an approximation which did not exist. It may be rush to conclude that this student was unsuccessful. It is important to acknowledge of his/her usage of treatments exceeded his/her usage of conversions especially for the second and third question. It may be a good example of problem with conversion, hence choosing a safer side as treatments process. (Q1>3C,4T; Q2>1C,5T; Q3>1C,6T) data from Table 4 indicates this preferable distinction of the student 17. Here Qi stands for the question number, #C is for number of conversions and #T is for number of treatments within that question for that particular student.

Table 4. *Exceptional students' results for conversions and treatments (C for conversions and T for treatments)*

	s3	s4	s11	s14	s15	s19	s20	s21	s22
Grades >99 and 100	Q1>3C,5T Q2>3C,2T Q3>2C,2T	Q1>1C,4T Q2>1C,3T Q3>1C,2T	Q1>1C,2T Q2>3C,5T Q3>2C,1T	Q1>2C,2T Q2>3C,7T Q3>2C,3T	Q1>3C,3T Q2>2C,3T Q3>2C,4T	Q1>3C,3T Q2>3C,5T Q3>2C,2T	Q1>5C,4T Q2>2C,4T Q3>1C,4T	Q1>3C,3T Q2>4C,3T Q3>1C,2T	Q1>2C,3T Q2>5C,2T Q3>3C,3T
Lowest grades ≤70	S17 (50)					Q1>3C,4T Q2>1C,5T Q3>1C,6T			
	S18(70)-different solution					Q1>2C,2T Q2>3C,3T Q3>1C,4T			
	S7(70)					Q1>2C,3T Q2>2C,3T Q3>5C,2T			

Table 4 indicates numbers of conversion and treatment processes used by the student numbered s*, and especially for exceptional students. The most important finding here is that no student from the Table 3 and Table 4 coincides especially for the top part of the table or i.e. for students with grades 99-100. Hence, we can conclude that exceptional students used more treatments and conversions. Students numbered 4 and 11 are interesting in a way that their grades were high but they have used less number of conversions. Student number 4 graded with 100 but for each question he/ she used only one conversion process. So, we can say that usage of more conversion processes or less number of conversions is not a problem for success. It is a very important finding that although conversion processes can act as a threshold, they are not a must for success. In other words, they are not properties of good problem solvers.

In table 4, we also see s20 with highest number of conversions for question 1. But some other students solve the same problem with only one conversions. This may be a sign that use of conversions have nothing to do with the problem type. It is mostly a choice of the solver instead (Table 4). It is also interesting to note that this student saw a need for 4 treatments in the same question.

When we look into paper of student numbered 4 in detail (table 4), all conversion processes were from diagram to symbolic transformation. In answer, systematic and clear approach to all three questions can be seen in clear. Treatment processes were from symbolic to symbolic. We may decide that student (4) was safer with the symbolic representation, although usage of diagrams was reasonable. However, no treatments within diagrams were identified in the paper of this student.

S21 (table 4) solved the third question with an unexpected method. He /she used diagonal of square made by the circle with r radiuses. Not only tangent square to circles but also square made up with the centers of these circles. Properties of the diagonal of a square, tangent circles and radius addition, etc. was used by this student to give 100 graded paper. For the first two problems, 3 treatments and 3 conversions were counted. But for the last problem, only one conversion and two treatments were recorded. This was parallel to the students numbered 4 and 11 with less number of conversions. Hence, showing also less number of conversions is not a problem for good problem solvers.

Last but not least s16 and s25 gave also exceptional answers to the last and the first problems respectively. S16 preferred verbal register to answer third question diagonal of square, its length being √2 times of the side, hence used symmetry of the figure with respect to y = x and y = -x as much as y and x axes. s25 also included symmetric properties of the problem with respect to y and x axes. Verbal and symbolic registers were used together. But note that, these two students did not get either the highest grades or the lowest grades of the exam (table 4).

There are more points we need to stress such as s21 and s22; for question number 2 these students used more conversions than others, as much as 4 and 5. They both were high achievers with grade 100. S21 used conversions from diagrammatic to symbolic. But s22 used conversions not only diagram to verbal but also diagram to symbolic conversions. S22 used a very special solution to question number 1. As part of the treatment process within diagrammatic register, he/she seen transporting the lower half circle to upper part of the frame of the question to build a full circle with 6 tangent circles, hence again some sort of symmetry occurred again (Figure 3). Unfortunately, there is only this picture of his resulted symmetry that he somehow detected.

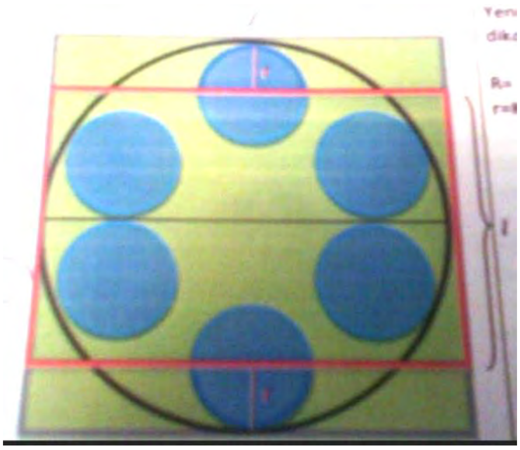

Figure 3. *Answer of s 22 to question number 1 with a special form of symmetry and intuition*.

With question number 3, students mostly used less number of conversions 2-3. This may show something about the specifics of the problem. But it is not clear. Again, s18 graded with 70 but his /her last solution was exceptional. 1 conversion was followed by four treatments. The third question was in which rotational symmetry could be seen i.e. if we rotate the figure for 90 degrees, the figure does not change. It could be said that complexity of the figure causes usage of less number of conversions than treatments since, conversions are more complex than treatments as a process.

There is a tendency for lower number of conversions with higher number of treatments (see student 17). But we need to be careful while stating a result like this since, s14 with second question showed 3 conversions and 7 treatments. And that was the highest number of treatment processes used in one answer throughout the study. But it is still less than as can be seen.

Problem representation gave a partial information about the question. Students needed to translate the representation into symbolic mode. Most students did have some problems with this. None of the students in the study stayed in a mono-functional register, they all used both conversions and treatments.

5. Discussion

5.1. What is cognitive synergy of registers?

From figures 4 to rest, we see student answer papers. All are student made but only specific notes about conversion or treatments are made by teacher as in red color. Unfortunately, again we only have these somehowpartially clear photos of student papers since they were photographed from student papers.

Representational synergy is a must. Duval (2006) mentioned about conversions being around treatment processes. Actually, a solution required a synthesis of treatments and conversions as in the example paper of s2 in Figure 4, s1 in Figure 5, etc. Even when the least number of conversions were used in solution, they tend to be after or just before a treatment process. As in the example paper of Figure 5, this synergy can act like an art piece, such that each part of the picture adding up to the whole picture as a juxtaposition. Besides this synergy, there is a tendency to have more number of treatments while less number of conversions.

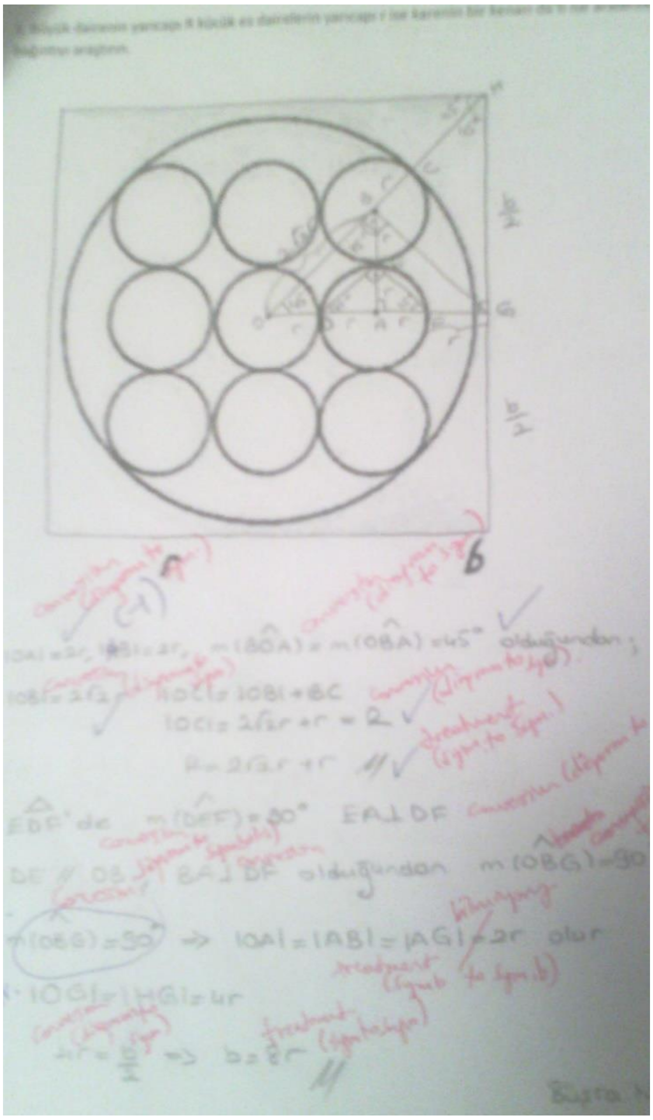

Figure 4 *Direction of conversion is important (s2, q3)*

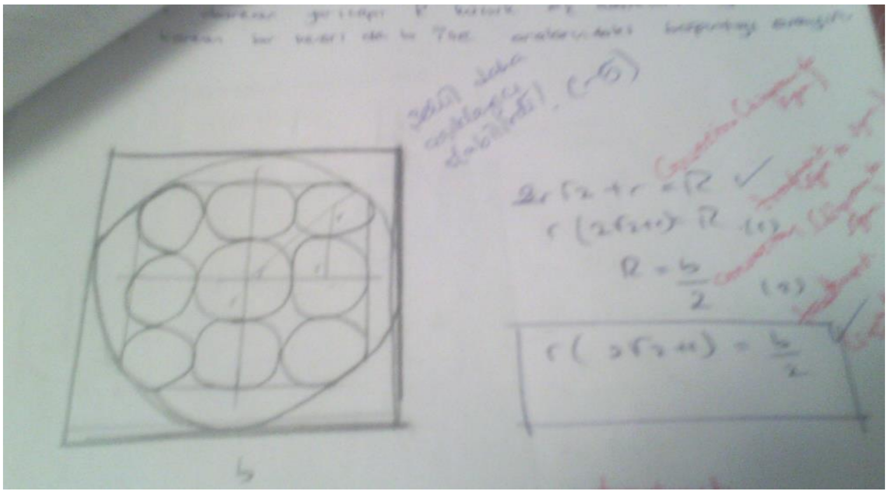

Figure 5. *A figure can say thousand words or none*

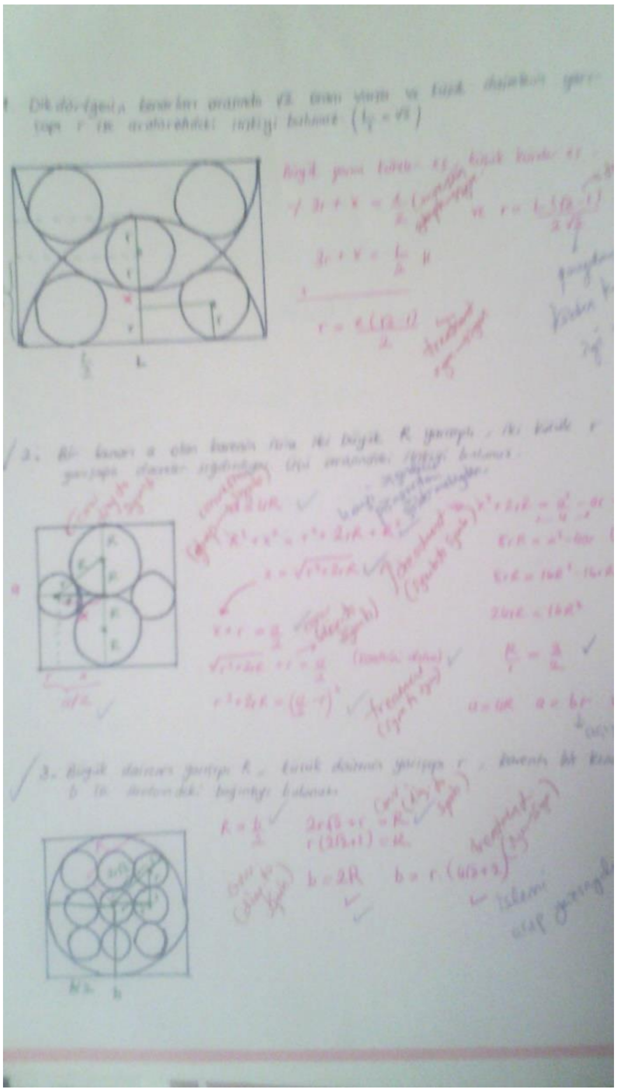

Figure 6 . *s10*

5.2. How these different types of transformations add up to thinking and prohibit thinking?

Some explanations start from either a conversion or a treatment process. If the start is not related to the solution as if student feels frustrated and either not continues or continues with hesitation and imbalanced rigor. If we recall Figure 3, student is fully aware of the semiotics of the problem and uses it for his/her purpose. Connects two schemas to build a unique and more meaningful one. As can be seen from first figure of Figure 1, some students have seen rhombus and named as a square as a misconception. Actually the angles are not perpendicular. These are all paralel to D'Amore (2002), Pino et al. (2015) , Cezikturk (2019) and Duval (2017).

5.3. How can a student decide upon what part of the conversions and treatments relevant and irrelevant?

Good diagram, control in each step, good start (figure 5), and attention (figure 6) are possible ways students use in overcoming relevancy problem. Good start helps building intuition, control checks the relevancy to the problem, attention assures not losing one's way in solution towards the right answer, and good diagram frames the problem for conversion and treatment processes to sail within. (Figure 7). These all adds up to what Duval (2006) state on relevancy.

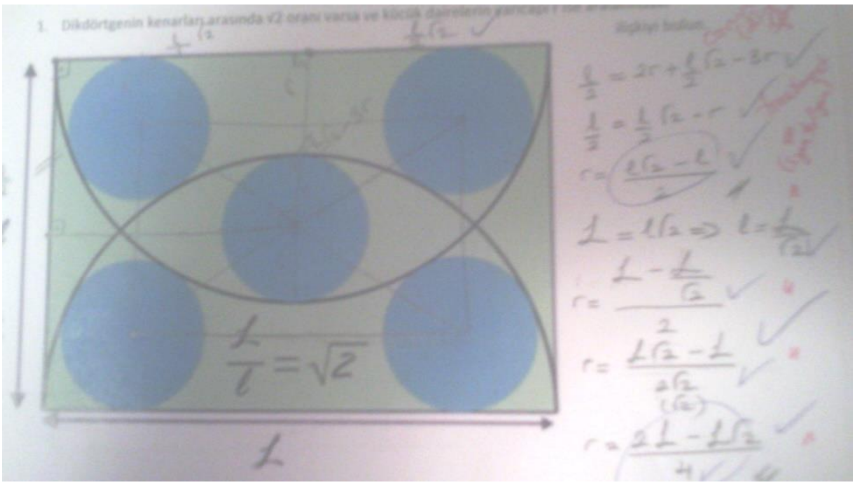

Figure 7 *What is relevant what is irrelevant (s2)q1*

5.4. How to recognize and how to discriminate suitable transformations of registers?

A good treatment can worth a couple of conversions (figure 8). And a bad one can be misleading. Sailing in a mono-functional register, easies the problem (Figure 9). A one to one mapping between the source and the target is searched for and if found, it proceeds similar to what Cezikturk (2019) state. If not, students stay in mono-functional register as Duval (2006) states.

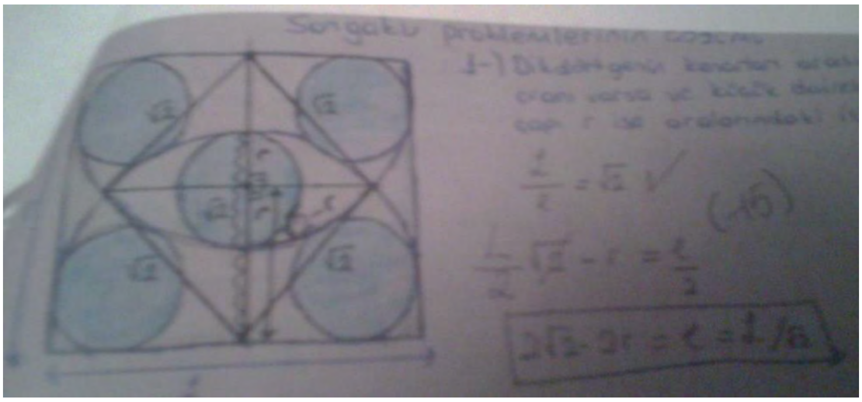

Figure 8. *s26 q1*

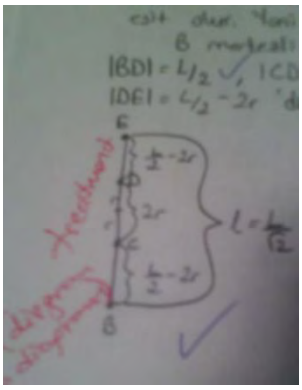

Figure 9. *s3, standing in mono-functional register q1*.

5.5. How direction of the conversion affects thinking?

Direction of conversion is important (Figure 4). S2 saw the right triangle on the left inside the top right corner of the figure of q3 and think that right hand side is also the same triangle. Although it is perpendicular, if one makes calculations symbolically, it turns out to be different than the left triangle. Because, from its center to right hand side of the square is 2r (√2-1) and not 2r as in left. It can be misleading. Some students see the help of diverse thinking or going from answer to problem. However, some ways are one way! (Figure 9) Students needs experience to deal with this problem. Duval (2006) names this as a problem of recognition and discrimination. Cezikturk (2019) points to lack of intuition due to less number of experiences.

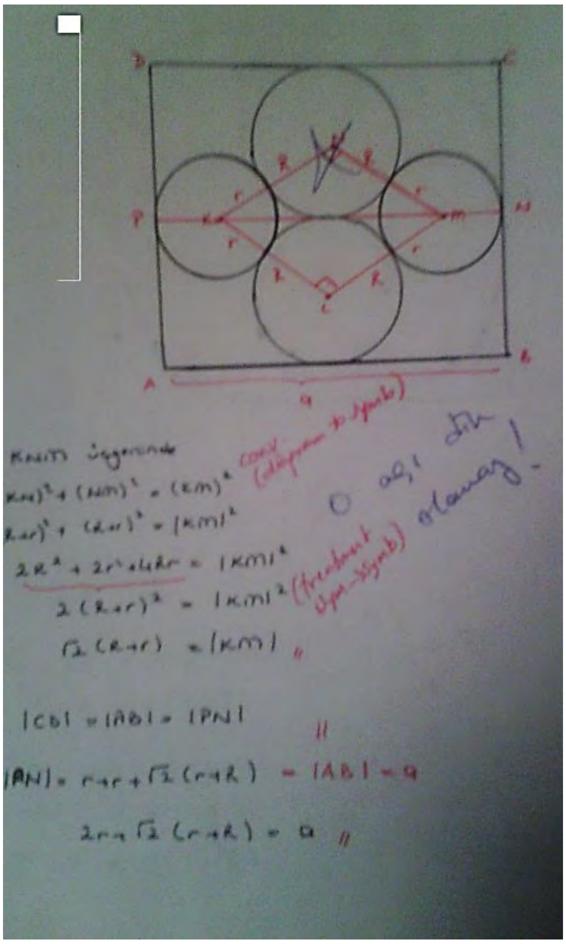

Figure 10. *A conversion can be misleading (s 6), question number 2, 2nd way*

5.6. How non congruent conversions problem affects student understanding?

Non congruent conversions lead to wrong roads, and results in either lost way or results in not relevant explanation or a wrong intuitive judgment as in the examples of a not existing right triangle. (figure 9). Hence intuition plays an important role here. Experience is necessary to fullfill right judgement as can be seen. When they were missing this experience with non congruent conversions, problem becomes much more difficult to solve as Cezikturk (2019) points.

6. Conclusion

Cognitive difficulty from two cognitive sources is not a new idea: treatments and conversions. These two types of translations are important in different senses. Conversions are due to their need for understanding of two different structured representations. Direction as well as choice of first register could be a threshold for the student. Wasan geometry is a thought provoker and pre service mathematics teachers' written solutions to those problems give a source for these two types of cognitive processes. In this study, preservice mathematics teachers' answers on a Wasan problem on tangent circles is qualitatively analyzed for specific examples of treatment processes and conversion processes. Neither problem type nor correct answer is found to be a reason for the choice of representational registers. One treatment sometimes may be worth to many conversions. More usage is related to students' conformity with those processes. Direction of the conversion may be misleading. Students may get perplexed which way it goes if they are not experienced in that conversion before. Because, intuition needs at least a slight experience. None of the students stay in mono-functional registers in case of complex systematic problems like Wasan. Top and lowest grade students use more treatments and conversions. This may be due to their need for crystallized knowledge on those processes. Usage of more conversions is not related to success and result in less number of treatments. To understand and analyze more, studies with Wasan geometry examples needs to be increased.

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Authors

Özlem ÇEZİKTÜRK, Marmara University, İstanbul (Turkey). E-mail: ozlem.cezikturk@marmara.edu.tr

Ahmet Şükrü ÖZDEMİR, Marmara University, İstanbul (Turkey). E-mail: Ahmet.ozdemir@marmara.edu.tr

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