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Mathematical Connections Activated in High School Students' Practice Solving Tasks on the Exponential and Logarithmic Functions

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Abstract
The current paper aims to identify the mathematical connections activated by 10 Mexican high school students while solving mathematical tasks that involve the exponential and logarithmic function. We used the Expanded Mathematical Connections Model (EMCM) and the Onto-Semiotic Approach of Cognition and Mathematical Instruction (OSA) as theoretical frameworks. Task-based interviews were used to collect data that was analyzed using thematic and onto-semiotic analyses. It was found that the connection of reversibility is essential for achieving students' full understanding of the existent relationship between the exponential and logarithmic function; however, this requires a network of connections.

Keywords: exponential function, logarithmic function, mathematical connections, mathematical practice

INTRODUCTION

Mathematical connections activated by Math students and teachers represent a scope of action that claims more and more attention by the international community of researchers specialized in Didactics of Mathematics and the educational administrations at a global scale (e.g., Jaijan & Loipha, 2012; Koestle et al., 2013; Mhlolo, 2012). The main reason for this, according to diverse researches (e.g., Bingölbali & Coşkun, 2016; Businskas, 2008; Eli et al., 2011, 2013; García-García & Dolores-Flores, 2018), is that mathematical connections are inherent to mathematics and, therefore, students should acquire the ability of making connections when mathematics is taught. Apart from being useful for daily life in general, establishing mathematical connections contributes to the development of abilities such as reasoning and communicating, and consequently, they are the basis of mathematical understanding (Mhlolo, 2012).

The significance of mathematical connections for the teaching-learning process of Mathematics has been taken into consideration in the curriculum of countries such as the United States (National Council of Teachers of Mathematics [NCTM]), 2000), Colombia (Ministerio de Educación Nacional [MEN], 1998), Mexico (Secretaria de Educación Pública [SEP], 2017), Spain (Generalitat de Catalunya, 2015), among others. The NCTM (2000) declares, among its principles and standards, that students must be capable of reorganizing and establishing connections among mathematical ideas. According to this curriculum, making mathematical connections enhances students' possibility to develop a better understanding of a given problem, as well as a better understanding of mathematics as a body that comprises interrelated objects (NTCM, 2013). The current research is settled in this research agenda and aims to study the mathematical connections activated in Mexican high school students' practice when solving tasks regarding the exponential and logarithmic functions.

The following reasons have led us to set out our aim: 1) By consulting literature, we found no previous research focused on the particular study of mathematical connections related to the exponential and logarithmic function (only studies related to the co-variational reasoning such as the one from Gruver (2018), and Ferrari-Escolá et al. (2016)), and 2) the study of functions is a key subject in Senior high-school curriculum. In particular, the study of exponential and logarithmic functions is essential since a) they are necessary concepts for Mathematics courses at the university such as

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Contribution to the literature

- The study identifies the mathematical connections activated by 10 Mexican high school students while solving mathematical tasks that involve the exponential and logarithmic function.
- The study analyzes the mathematical connections using the articulation of two methods, which are based on the Expanded Mathematical Connections Model and the Onto-Semiotic Approach of Cognition and Mathematical Instruction: thematic analysis and ontosemiotic analysis, respectively.
- The study shows that the articulation of the two methods of analysis makes it possible to identify the mathematical connections and to analyze the reason for their establishment through a network of semiotic functions.

Calculus, Differential Equations and Complex Analysis (Weber, 2002); b) they are considered pivotal for learning several physical and social processes; c) they are used for modeling several phenomena from real life and other sciences and, d) these two kinds of functions represent a great difficulty to the students.

Some reported difficulties are explained in the following lines. While facing situations or tasks, students usually tend to think immediately in lineal mode and rarely in the logarithmic-exponential mode (Sureda & Otero, 2013). Most university students have difficulty understanding the exponential laws and linking them to the logarithmic laws (Campo-Meneses & García-García, 2020; Weber, 2002). In addition, there is a tendency to work with the natural logarithmic function as the inverse of the natural exponential, since its manipulation is simpler, thus limiting the general use of this function in algebraic contexts (Escobar, 2014). Therefore, it is necessary to develop some research on how the establishment of determined mathematical connections may (or not) explain students' difficulties to achieve a proper level of understanding regarding these mathematical concepts. All those aspects have allowed us to formulate the research question on the following terms:

What mathematical connections are activated in Mexican high school students' practice solving tasks on the exponential and logarithmic functions?

To this end, the use of the Task-based Interview (Goldin, 2020) has been decisive for gathering certain data which have been furtherly analyzed using the Expanded Mathematical Connections Model (EMCM) and the Onto-Semiotic Approach of Cognition and Mathematical Instruction (OSA): thematic and onto-semiotic analyses respectively. The contribution of this research is the identification of mathematical connections and the analysis carried out using both methods, since methods are articulated to identify mathematical connections and to analyze the reason for their establishment through a network of semiotic functions.

THEORETICAL PERSPECTIVE

This section provides a summary of the way mathematical connections are understood in the two theories formerly settled as referents.

Expanded Mathematical Connections Model (EMCM)

In this research, we applied the Expanded Model of Businskas (2008) on mathematical connections proposed by García (2018), and García-García and Dolores-Flores (2018) because it is mainly oriented to student work. In this sense, mathematical connections are considered true (mathematically consistent) relations established by a person between two or more ideas, concepts, definitions, or theorems, other disciplines, or real life; and they come to fruition through written, oral or gestural arguments that students demonstrate while solving given tasks (García-García & Dolores-Flores, 2018).

EMCM considers the following categories of mathematical connections:

Procedural: it has the form of "A is a procedure used while working with object B". In this case, the mathematical connection includes all the rules, algorithms, or formulae established within a semiotic register used to obtain a result.

Different representations: is the relationship between two different semiotic registers referring to the same mathematical concept, but also the relation between two different forms of representing the concept in the same register.

Part-whole: has to do with the logical relations established among mathematical concepts performed by students. These may be generalization (between general and specific cases) or inclusion (a mathematical concept within another one).

Meaning: is the relationship between a mathematical concept and the sense given by students (making it different from another). This can be seen in two different ways -as the relationship between concept and the definition generated by the student or as the relationship between concept and its contexts of use.

Reversibility: has to do with the bidirectional relation between two mathematical operators, that is to say, departing from a concept A to B and at the same time reversing the process departing from B to A. It is needed for students to be capable of departing from a final point and following the course of single reasoning until they arrive at an initial point and vice versa.

Feature: is the relation established by students among a concept and its invariable attributes that distinguish it from others. It also includes the common elements among two or more concepts, procedures, or different representations.

Implication: A implicates B, an object logically leads to another, and they tend to be in the *if…then* way.

It should be noted that the metaphorical connection proposed by Rodríguez-Nieto et al. (2020) is not included, because this connection is directed mainly to the instructional process and the focus of this article is the student's practice.

Some Constructs of the Onto-Semiotic Approach (OSA)

The OSA considers that, in order to describe the mathematical activity from an institutional and personal point of view, it is essential to have in mind the objects involved in such activity and the semiotic relations between them (Borji et al., 2018; Breda, Pino-Fan, & Font, 2017; Breda et al., 2021; Font, Godino, & Gallardo, 2013; Godino, Batanero, & Font, 2019; Rondero & Font, 2015).

Mathematical activity is modeled in terms of practices, a configuration of primary objects, and processes that are activated by practices. Mathematical practice is conceived in this theory as a sequence of actions, regulated by institutionally established rules, oriented towards a goal (usually solving a problem).

In the OSA ontology, the term 'object' is used in a broad sense to refer to any entity which is, in some way, involved in mathematical practice and can be identified as a unit. For instance, when carrying out and evaluating a problem-solving practice, we can identify the use of different languages (verbal, graphic, symbolic, and others). These languages are the ostensive part of a series of definitions, propositions, and procedures that are involved in the argumentation and justification of the problem's solution. Problems, languages, definitions, propositions, procedures, and arguments are considered as objects, specifically as the six mathematical primary objects. Taken together they form configurations of primary objects. The term configuration is used to designate a heterogeneous set or system of objects that are related to each other. Any configuration of objects can be seen in both ways, from a personal and an institutional perspective, leading to the distinction of cognitive (personal) and epistemic (institutional) configurations of primary objects.

Depending on the language game in which they are involved, the primary mathematical objects that are part of the mathematical practice "may be considered in terms of how they participate, and the different ways of

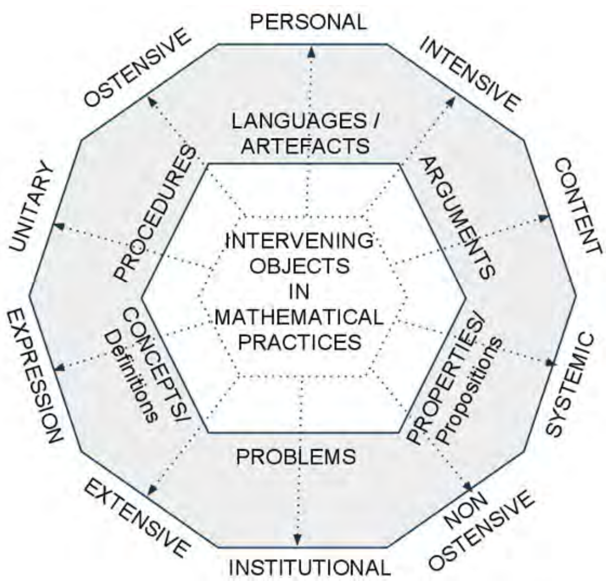


Figure 1. Ontology for an Educational Mathematics Philosophy

doing so may be grouped into dual facets or dimensions" (Font et al., 2013, p. 111). These are: Personal/institutional (see previous paragraph), unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive.

In the OSA, semiotic functions are conceived, metaphorically speaking, as correspondence among sets that comprise three components: a plane of expression (initial object); a plane of content (final object), and a criterion or rule of correspondence (Godino, Batanero, & Font, 2019).

According to Rondero and Font (2015), the following work hypothesis is formulated in the OSA: the semiotic functions are the linking instrument among the primary objects that are activated during mathematical practices carried out in the classroom within a language game. This relation may be understood as a unique semiotic function or as the result of a semiotic interconnection (a network of semiotic functions).

The OSA's theoretical tools have been used to generate what has been called hybrids conceptual maps. This has to do with the representation of the mathematical practice related to solving a problem that allows representing the primary objects activated during the previously mentioned practice and their organization (Moreno et al., 2018). The hybrid conceptual maps have been adapted to develop an analysis of the mathematical practice carried out by students solving tasks from the questionnaire.

Networking of Theories Between the EMCM and the OSA

The networking of theories was carried out in Rodríguez-Nieto et al. (2021), between EMCM and the OSA. Especially, concordances and complementarities are identified in the respective conceptions of mathematical connections along with the methodologies used by both approaches to analyze the empirical data.

These authors conclude that the method used by the EMCM analyzes the mathematical activity carried out by a person to find mathematical connections among several elements related to this activity. However, this mathematical activity is analyzed by an observer who knows the mathematical rules that regulate the mathematical practice and therefore, can provide (or not) meaning to the observed performances of the student. Consequently, it partially coincides with the onto-semiotic method of analysis of the mathematical activity developed by the OSA. These authors also conclude that the methods of both theories complement each other to carry out a deeper and detailed analysis of the mathematical connections.

One of the keywords in these two models (EMCM and OSA) is meaning. In the OSA, the systematic elemental duality is applied to the notion of meaning. The elemental view leads to understanding the meaning of a term as a definition. However, if we ask ourselves what the understanding of a definition is, the answer provided by the OSA is that to understand the definition, a student must put into practice a network of semiotic functions. The systematic view leads to understanding the meaning of a term as something stated by its given usage. If we understand the meaning as the usage, we will say that a person comprehends, understands, knows, etc. the meaning of exponential function when she/he can use it in a competent way for diverse mathematical practices. At the same time, several primary objects get involved in these mathematical practices (forming configurations of primary objects) and establish a relation through semiotic functions.

To the EMCM, meaning is conceived as a typology of mathematical connection that refers to two kinds of relations: first, among the mathematical concept and its definition, and second, among the concept and its diverse contexts of usage. The first relation is similar in a way to the elemental view of the OSA, while the second one only identifies contexts in which a person successfully uses the concept and there is no work concerning terms of the mathematical practice that is why this relationship is limited to the systemic vision of the OSA.

METHODOLOGY

This is a qualitative research, specifically, it is a case study. We used a tasks-based interview to collect data. This includes: 1) the design of a sheet of tasks contained in the instrument given to the student, 2) a protocol for the interview, 3) applying the tasks-based interviews to the participants and 4) the transcription of the resulting interviews. The process of filling the sheet of tasks and giving answers are produced simultaneously, allowing the researcher to formulate questions to the student while solving each task, to get to know the reasons why the student develops (or not) certain mathematical practices.

For instance, this is one of the questions formulated during the interview in task 3: Does a relation between the two resulting functions exist?

Instrument

The instrument (a sheet of tasks and the protocol for the interview) was validated first by experts in mathematical connections (experts' triangulation) and then, through a pilot test carried out on four high school students that had concluded their studies on Differential Calculus and had worked with exponential and logarithmic functions. As a result, the second version of both, the sheet of tasks and the protocol for the interview, was created (Table 1).

Case Study

The study cases were 10 Mexican high school students (4 women and 6 men) coursing the 4th semester (11^{th} grade's second semester). The students took part voluntarily and gave their consent to further usage of the data in this research. The students were taking a course in Differential Calculus and had already worked with exponential and logarithmic functions. From now on, they will be referenced as S1, S2, S3…, S10.

In the Differential Calculus subject, according to the technological baccalaureate curriculum (SEP, 2017), the exponential and logarithmic functions are taught. However, their teaching begins in the first semester (in algebra) when basic concepts of exponents and their properties are introduced. As it is contemplated, the group of algebraic and transcendent functions are taught in the first period of the Calculus course and, in order to do this, there is a total of 15 hours in which only a few are allocated for the exponential and logarithmic functions. According to the expected skills and learning (SEP, 2017), the teaching of these functions is aimed at characterizing these functions as useful prediction tools in models to predict change, analyze sequences, analyze increasing and decreasing regions and make graphs.

Example of the Analysis of a Case

To analyze the data, the thematic analysis used by the EMCM was employed first, and then, the onto-semiotic analysis model proposed by the OSA. This section works as an example of how the data were analyzed using both models and taking the answers provided by S10 to task 3d (since it is the task that stimulates the bigger amount of mathematical connections).

Written production

Figure 2 shows student S10's written production regarding task 3.

Table 1. Task description

Task | **Description**

1 Complete the following table and explain the strategy used.

-3	-2		0	1	2	3		5		7		9
$\frac{1}{8}$	$\frac{1}{4}$		1	2	4	8	16	32				

a. Find a mathematical expression that relates the term of the second row to the term of the first row.

b. What terms in the first row correspond to 1024 and $\frac{1}{6384}$ of the second?

2 An infectious disease begins to spread in a group of 10,000 students in a University. After t days, the number of infected people is modeled by the function:

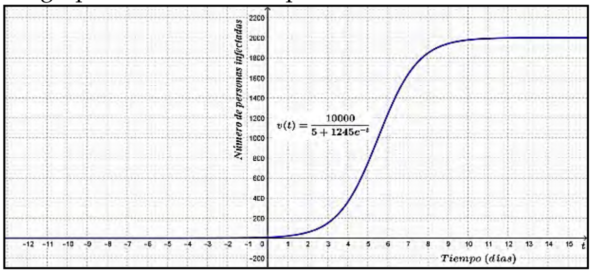
$$v(t) = \frac{10000}{5 + 1245e^{-kt}}$$

Where t means time measured in days and k is a constant.

a. How many infected people are there initially?

b. If after 2 days, there are 54 people infected (Suggestion: with this data you can find the value of k). Considering the value of k you found, how many people will be infected in 5 days?

c. Considering $k = 1$, the graph that models the number of infected people is the one shown below. Look at the graph and answer the questions that arise.



- How long will it take till 1000 people be infected?
- What is the domain of the function $v\ (t)$? What domain makes sense in the situation?
- Which days does the infection spread slowly?
- Which days does the infection spread rapidly?

3 Ehrenberg's relation $\ln w = ln\, 2.4 + (1.84)h$ is an empirical formula that relates the height h (in meters) to the average weight w (in kilograms) for children from 5 till 13 years old.

a) If an 8-year-old girl weighs 28.8 kg, what is her average height?

b) What is the average weight for a 10-year-old boy whose height is 1.5 meters?

c) What function allows to know the average weight for any children whose age vary from 5 till 13 years? Plot a graph of the function.

d) d) What function allows to know the average height for any children whose age vary from 5 till 13 years? Plot a graph of the function.

4 Define what exponential function and logarithmic function are. Is there any relation between them? Of what kind?

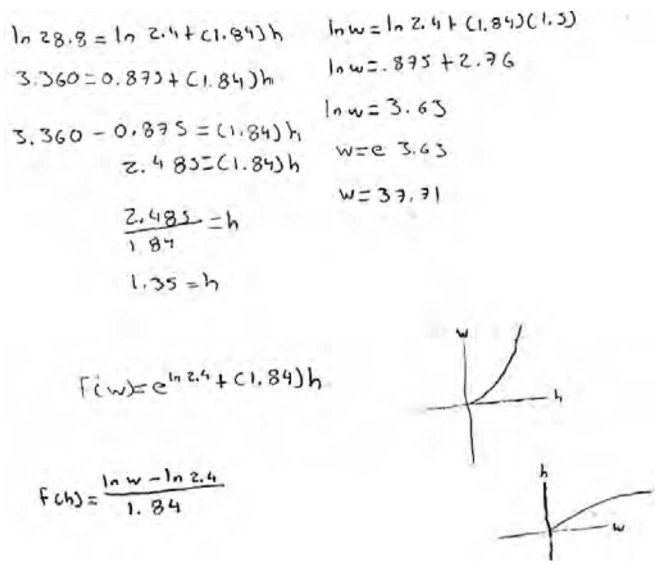


Figure 2. Written production of S10 in task 3

Thematic analysis

We used the thematic analysis method proposed by Braun and Clarke (2012) in this research, which aim is to identify patterns of meanings (themes) through a set of data. This method offers several advantages, for instance: to answer one or more research questions, to analyze less or more data, to produce an analysis based on data or based on theory (such as this article), among others. This method is structured into 6 phases. Below, it is explained how those phases were carried out, using the mathematical connections performed by S10 as an example. In this case, the themes were given a priori (the seven categories of mathematical connections considered in the theoretical framework).

Phase 1. Familiarizing with data: in this phase, a general reading of the interview's transcriptions and the questionnaires answered by each student was carried out several times, to know in detail each production, as well as the language used by the student and to note

Table 2. Examples of data coding for mathematical connections

Excerpt	Code	Remarks
S10: the present graph is similar to the previous one, but with a lower arc (he plots the graph). Interviewer: Why? S10: *Because the height doesn't surpass 2 meters. It will be the inverse graph, a lower one, if that one looks up this one looks down.*	C26 the graph from $f(h)=\frac{\ln w-\ln 2.4}{1.84}$ is inverse to the one from $f(w)=e^{\ln 2.4+1.84h}$	Making a mistake by writing $f(h)$, when it should have been $f(w)$.
Interviewer: Then, what type of function is $f(h)=\frac{\ln w-\ln 2.4}{1.84}$? S10: It will be a logarithmic function because it is inverse to the exponential function.	C29. $f(h)$ is a logarithmic function C30. The logarithmic function is inverse to the exponential function.	

Table 3. Example of subthemes creation by using codes from phase 2

Code	Subthemes (St)
C26	St10. The graphs of $f(h)$ and $f(w)$ are inverse.
C29	St14. Example of logarithmic functions.

Table 4. Example of themes definition

Themes	Subthemes	Explanation
Mathematical connection of reversibility	St10	It establishes the relation of inverse among functions (among graphic representations).
Part-whole mathematical connection	St14	It relates the symbolic expression of functions with the family they belong to.

down significant remarks regarding the mathematical connections that seem to emerge.

Phase 2. Generating initial codes: the data were organized in groups of the same meanings and a code was developed considering the mathematical connections suggested in the EMCM. To do so, phrases indicating a relation between two or more ideas, concepts, representations, or procedures (that were synthesized into codes) were identified in the answers given to the task and questions of the interview. Table 2 shows some of the codes inferred from S10 production.

Phase 3. Looking for themes and subthemes: During this phase, the identified codes were reviewed to collect those sharing a common feature (subthemes). Table 3 shows some subthemes.

Phase 4. Reviewing subthemes: the recoding and discovery of new subthemes consistent with the research question is carried out, in case data demands it. In this case, no new subthemes were shown.

Phase 5. Defining and naming themes: the subthemes obtained from previous phases were categorized (Table 4) using the seven categories of mathematical connections taken from the EMCM.

Phase 6. Producing the report: conclusions obtained from previous phases are presented in written form. In this case, 6 types of mathematical connections emerged from the answers provided by S10.

Model of the onto-semiotic analysis

This model aims to analyze the student's mathematical activity through the identification of objects that emerge from mathematical practices and to establish links among them, through semiotic functions. Moreover, it allows the performance of microscopic analysis, metaphorically speaking, to specify in detail a possible cause for the establishment (or not) of a mathematical connection. This analysis was carried out in accordance with the following phases:

1. Checking of questionnaires with the respective interviews and then, reporting a "mathematical" narrative of each student production.

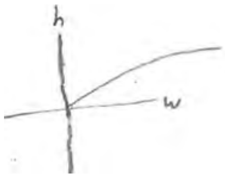
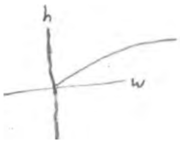
In task 3d, S10 was asked about the function that allows to know the average height of any kid from 5 to 13 years and to plot a graph. S10 got the expression following the same procedure used in task 3a, but not substituting variable *w*, and plotted the function graph. When asked about the method used for plotting a graph, S10 stated that the resulting graph was inverse to the previous one and that the higher value of the range was 2. When inquired about this function's domain and range, S10 answered that they were the domain and range from the previous one because both functions are inverse. S10 explained that the function's graph is a logarithmic function because it is the inverse of the exponential one (put into a graph in task 3c).

Dealing with a temporal narration in terms of mathematical actions makes the second phase easier, consisting of the identification of the mathematical practices developed by S10. The temporal narration in terms of mathematical actions also facilitates elaborating the configuration of primary objects in the third phase. While those actions are taking place, certain key elements (representation, proposition, procedure, etc.) emerge, metaphorically speaking, and those elements are the primary objects of the configuration.

2. Identification and description of the practices recorded in the narrative (for example P12T3 refers to practice 12 from task 3):

P12T3. The student generalizes the procedure of solving first-degree equations (used to find h), the

Table 5. Primary objects identified at S10's productions in task 3d

Objects	
Task	Task 3. Item *d*
Definitions	D1: Formula, D2: Magnitudes D3: Parameter, D4: Variable, D5: Logarithm, D6: First-degree equation, D8: Logarithmic function, D9: Domain, D10: Range.
Representations	Symbolic 1: The relation of Ehrenberg $\ln w = ln\, 2.4 + (1.84)h$ Symbolic 2: *h, w* Symbolic 10: Representation of the domain and range at intervals Symbolic 11: $f(h) = \frac{lnw - ln\, 2.4}{1.84}$ Natural language 3: Logarithmic function Graph 1: Graphic representation of $f(w) = e^{\ln 2.4 + 1.84h}$ Graph 2: Graphic representation of $f(h) = \frac{lnw - ln\, 2.4}{1.84}$
Procedures	Pr3: To solve first-degree equations by transposition of terms. Pr7: To plot a graph for the inverse function (logarithmic function).
Propositions	Proposition 6: the symbolic expression of the function that models the relation between "weight" (independent variable) and "height" (dependent variable) is $f(h) = \frac{lnw - ln\, 2.4}{1.84}$, Proposition 7: the graph of the function $f(h) = \frac{lnw - ln\, 2.4}{1.84}$ is inverse to the graph from $f(w) = e^{\ln 2.4 + 1.84h}$ and  this is it Proposition 8: the exponential and the logarithmic functions are inverse functions. Proposition 9: the domain and the range of the exponential function are the domain and the range of the logarithmic function. Proposition 10: the domain of *f*(h) is the interval (1, 60) and its range (1, 2). Proposition 11: the function that models the relation between "weight" (independent variable) and "height" (dependent variable) is a logarithmic function.
Arguments	Argument 4: Thesis (Proposition 6): the symbolic expression of the function that models the relation between "weight" (independent variable) and "height" (dependent variable) is $f(h) = \frac{lnw - ln\, 2.4}{1.84}$, Reason: this expression is the result of the generalization of the procedure of resolution of the first-degree equations in order to find *h*. Argument 5: Thesis (Proposition 7): the graph of the function $f(h) = \frac{lnw - ln\, 2.4}{1.84}$ is inverse to the graph of $f(w) = e^{\ln 2.4 + 1.84h}$  and this is it Reason 1 (Proposition 11): the function that models the relation between "weight" (independent variable) and "height" (dependent variable) is a logarithmic function. Reason 2 (Proposition 8): the exponential and the logarithmic functions are inverse functions. Argument 6: Thesis (Proposition 9): the domain and the range of the exponential function are the domain and the range of the logarithmic function. Reason: (Proposition 8): the exponential and the logarithmic functions are inverse functions.

student finds the function's symbolic expression that models the relation between "weight" (independent variable) and "height" (dependent variable).

P13T3. The student plots the graph of the inverse function.

P14T3. The student justifies how the graph from the inverse function is constructed from the function graph.

P15T3. The student finds the domain and range from the range and domain of the inverse function.

P16T3. The student identifies the function that models the relation between height and weight as a member of the family of the logarithmic function.

3. Identification of primary objects activated (or emerged) from mathematical practices (Table 5).

4. Stating the relations between the primary objects of the configuration (Figure 3), analyzing which semiotic functions are established among them.

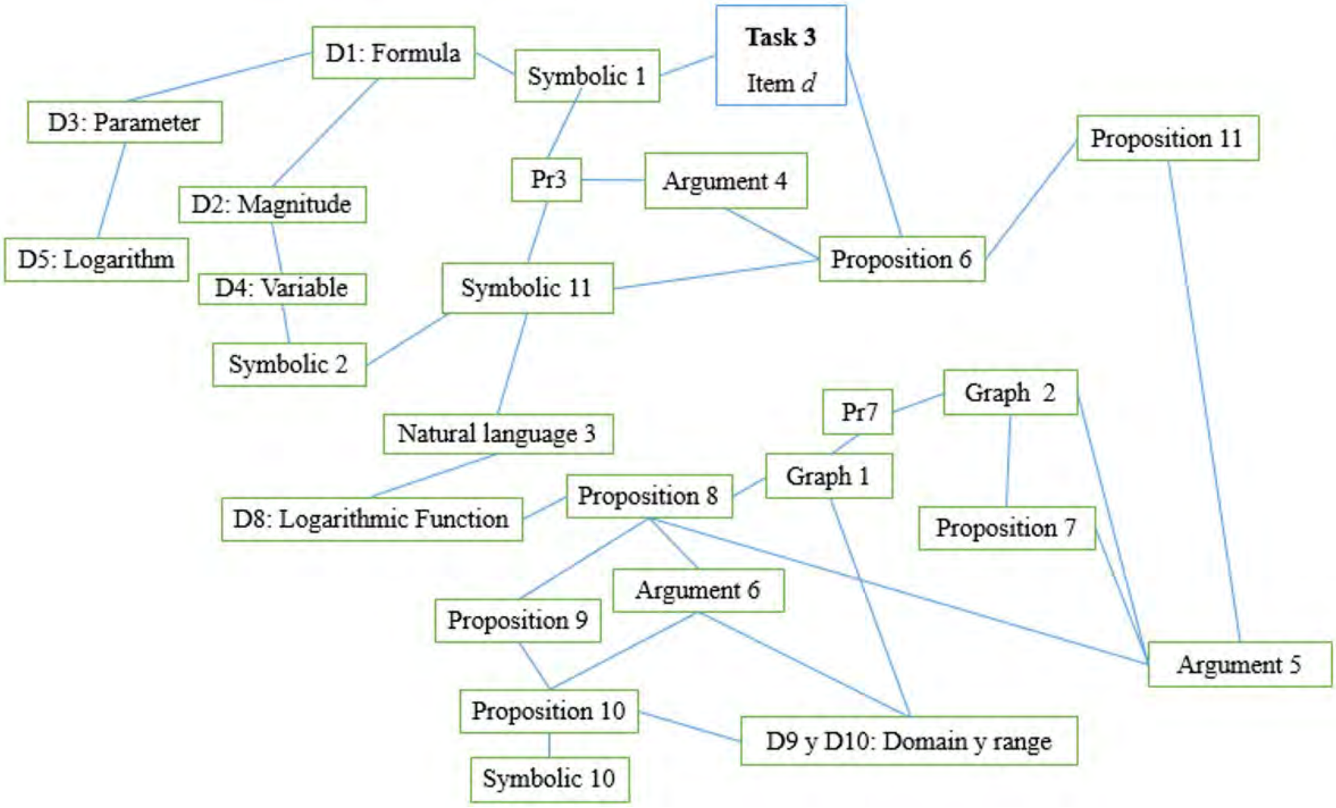
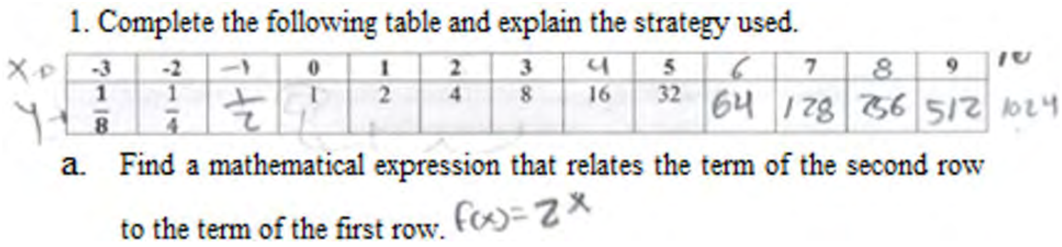


Figure 3. Configuration of primary objects and network of semiotic functions for item *d* by S10


1. Complete the following table and explain the strategy used.

X	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
Y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512	1024

a. Find a mathematical expression that relates the term of the second row to the term of the first row. $f(x)=2^x$

Figure 4. Answer provided by S8 to task 2

In order to solve task 3d and answer questions from the interview, S10 must recognize the symbolic expression 1 as a formula involving parameters and magnitudes that are mathematical variables (symbolic 2). The generalization of procedure 3 (according to argument 4) allows to find the expression of height concerning weight (symbolic 11) and to establish the proposition 6 as an answer to that item. He performs the graphic plot of the function (graph 2) through procedure 7 from graph 1, assuming that both graphs are inverse (proposition 7) and also using his knowledge regarding height and weight estimation corresponding to children from 5 to 13 years. When we asked S10 which kind of function the resulting graph is, the student answers that it is a logarithmic function (natural language 3 and proposition 11) and justifies his answer by arguing that these graphs are inverse (argument 5). Moreover, he represents through intervals (symbolic 10) the domain and the range of this logarithmic function (proposition 10).

This method of analysis exemplified in S10 case at task 3d was the one employed in each task to each one of the 10 cases of study.

RESULTS

Once executed the analysis of the answers to each task of the 10 cases (in the methodology section we explain the case of S10 in task 3d as an example), the results are presented with particular detail in this section.

Task 1

This task is proposed for students to establish the part-whole mathematical connection to come across the regularity and connections among different representations of the exponential function $f(x) = 2^x$ (Figure 4). On the whole, the thematic analysis shows that both aspects were established by the 10 cases of study, and the onto-semiotic analysis evidences the

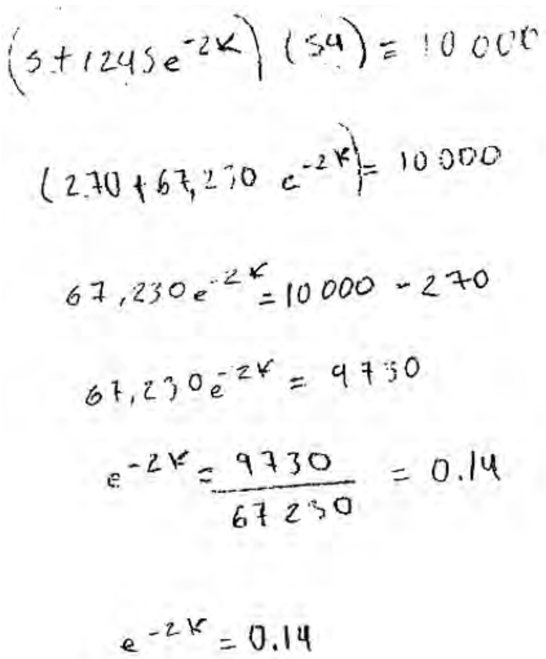


Figure 5. Answer provided by S3 to item b from task 2

semiotic functions' network conducted to achieve their establishment.

Task 2

The onto-semiotic analysis reveals in item a that S2, S3, S4, S5, S6, S8, and S10 recognize the conditions of the given task. They identify which is the variable to replace and evaluate the function in $t = 0$ correctly. Meanwhile, S1, S7, and S9 fail to establish the procedural connection, according to the thematic analysis, because they could not recognize the meaning of the word "initially" in the task.

At item b, the thematic analysis shows that this item is solved only by S3 who proved to have established the procedural and the reversibility connection (Figure 5); the latter is considered a needed connection ($\ln(e^t) = t$) for establishing the procedural connection.

Cases S1, S2, S9, and S10 failed to establish the procedural connection, hence the onto-semiotic analysis shows that those cases failed in both procedures, identifying and assessing the variables with the objective of going from a function to an equation. Meanwhile, S4 and S8 failed in clearing up (terms transposition) and in the reversibility process, while cases S5, S6, S7 failed in constructing the whole network of semiotic functions since they evidenced some lack of knowledge while solving such item.

Item c mainly demanded the establishment of the feature connection that simultaneously demanded to carry out the part-whole connection (between graph and extended graph) and the procedural connection in the graphic register (to visually find the domain, range, and intervals of growth). The thematic analysis shows that all students succeed to correctly answer this item

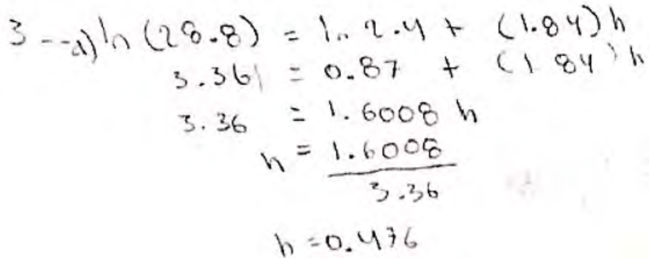


Figure 6. Answer provided by S9 to item b from task 3

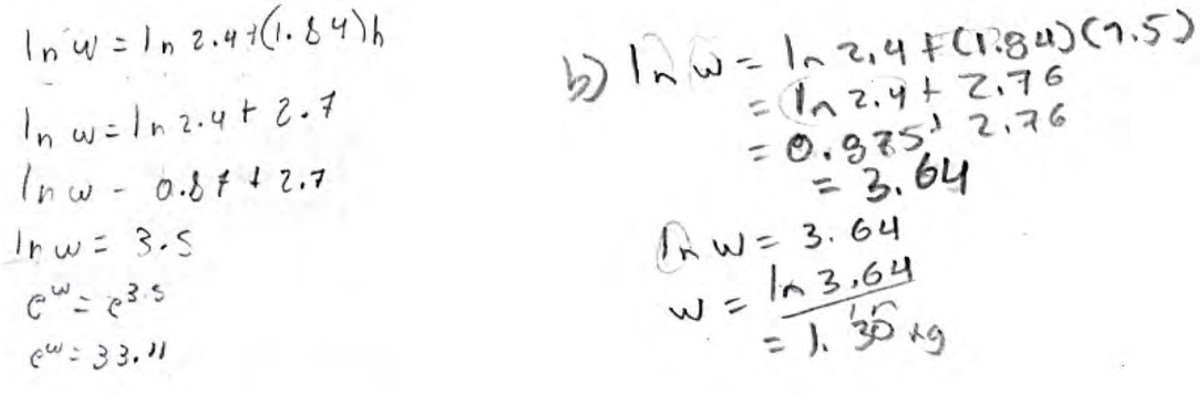
establishing these connections, and the onto-semiotic analysis shows that all students evidenced the semiotic functions for establishing these mathematical connections.

Task 3

In item a, the developed analysis indicates that only four students (S3, S5, S6 and S10) could establish the procedural connection. The onto-semiotic analysis indicates a possible cause for students' failure at establishing the mentioned connection. Actually, S1 and S4 fail to obtain the symbolic expression $3.360 = 0.875 + 1.84h$ (or equivalent expressions) because they make mistakes in the transposition of terms. S9 fails to correctly apply the hierarchy of operations in the symbolic expression $3.360 = 0.875 + 1.84h$ (Figure 6). S8 fails to correctly apply the hierarchy of operations in the symbolic expression $ln28.8 = ln2.4 + 1.84h$, and also the definition of logarithm. Whereas the case of S2 evidences a lack of the whole network of semiotic functions related to this item, because she makes some mistakes (for instance, converting ln 2.4 into 2.4). Finally, S7 only gets the symbolic expression $ln28.8 = ln2.4 + 1.84h$ replacing all the given values and does nothing else, that is why it is inferred that she lacks the whole correspondence of semiotic functions.

On the whole, in this item, the failure of the 6 students at establishing the procedural connection is due to diverse reasons going from the extreme case of being unable to assess the symbolic expression 1, to those making mistakes during the transposition of terms.

In item b, the analysis developed according to EMCM concludes that only 3 of the 10 students (S3, S4, and S10) established the procedural connection and the reversibility connection in the symbolic register ($e^{\ln w} = w$). The analysis based on the OSA showed that S1 and S8 lack part of the procedure for solving logarithmic equations (Figure 7). Though evidencing the use of the exponential as inverse process of the logarithm, S1 fails to correctly employ such condition to solve logarithmic equations (raises e to the terms of the equation, but obviates the ln, confusing the exponentiation with the power) and, therefore, fails to find the symbolic expression $w = e^{3.63}$ giving the wrong answer. Meanwhile, S8 fails to give a correct solution to the



Answer of S1 to item b

Answer of S8 to item b

Figure 7. Answer provided by S1 and S8 to item b from task 3

equation since he misuses the reversibility relation between power and logarithm.

Neither S6 nor S9 evidence signs of the reversibility connection since they only reach the symbolic expression $\ln w = 0.875 + 2.76$ (or equivalents). S9 just develops the four initial steps of the answer given by S1 and, when the interviewer asks him what should be done to clear up w, he answers that he does not know what to do. Whereas S6, who just develops the four initial steps of the answer given by S1, when asked the same question, replies that she has to use the inverse of the logarithm, but she does not know which one.

In the case of S5 and S7, the whole network of semiotic functions fails since they offer no answer to item *b* and they express that they do not know how to solve an equation with a variable located at the argument of a logarithm. The same happens to S2, but unlike the others, this student tries to solve item *b*, but fails to write the symbolic expression $\ln w = ln\, 2.4 + (1.84)1.5$.

On the whole, establishing the relation of reversibility is, in this item, a necessary condition to establish the procedural connection. The seven students who failed to correctly answer, could not do it because they cannot make the reversibility connection operative.

In item *c*, the analysis developed according to EMCM concludes that only 3 of the 10 students (S3, S4, and S10) established the procedural connection (finding the symbolic expression of the function, domain and range, and plotting a graph), the part-whole connection (by assuming that the function that models the weight is exponential), the reversibility connection ($e^{\ln w} = w$), the diverse feature connections (by establishing the domain and range) and the representation connection (relation among the symbolic expression, natural language and the graph of the function that models the weight).

Through the onto-semiotic analysis, the cases of S5, S6, S7 and S9 evidence a failure in the whole network of semiotic functions related to this item (they did not answer) because the process to be carried out in this item is similar to the one in item *b* (they could not solve it either). Moreover, when S2 tries to generalize the procedure 4 that she incorrectly used in item *b*, she evidences also a failure in the whole network of semiotic functions, since she writes another wrong expression.

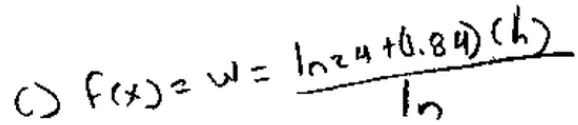


Figure 8. Answer provided by S8 at item c from task 3

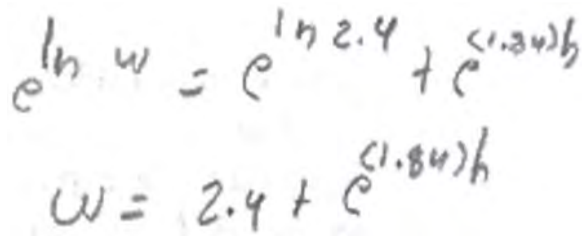


Figure 9. Answer provided by S1 at item c from task 3

S8 evidences a failure in the generalization of the procedure of solving logarithmic equations because she assumes that *ln* is multiplying *w* and therefore moves it to the other side as a divisor (Figure 8). This is another evidence (as in item b) that S8 does not establish the reversibility connection since the logarithm definition is unknown to her.

At the same time, S1 evidences a failure in generalizing the procedure of solving logarithmic equations (also incorrectly applied in item *b* by not using the reversibility connection). Particularly, she applies the property *-the image of the addition is the sum of the images-* (instead of using the product) (Figure 9), making impossible to find the symbolic expression $f(w) = e^{\ln 2.4 + 1.84h}$, though this time she uses the reversibility connection correctly to simplify the term in the left side of the equation and one term in the right side. Then, she assumes that the resultant expression is an exponential function and, to plot the graph, she uses the calculator to find some points of the function in order to link them later.

Figure 10. Answer provided by S1 in item d

In item *d*, the analysis developed according to EMCM concludes that cases S3, S4 and S10 answered correctly by establishing the procedural connections (finding the symbolic expression of the function, the domain, and range, and plotting a graph), the part-whole connection (by assuming that the function that allows to find the height as a function of the weight is logarithmic), the reversibility connection (among graphs, domains, and ranges), the diverse feature connections (by establishing the domain and range), the representation connection (relation among the symbolic expression, natural language and the graph of the function that models the average height) and the implication connection (relation of the condition of inverse with the domain and range, and the graphic representation).

S5 and S6 also solve this item correctly, but S5 plots the graph by using general features from the graph of a logarithmic function (there is no relation of reversibility since she did not plot the graphic representation in item *b*). The same happens with S6, though she uses a procedure of representing the graph through a chart of values found by substituting in the formula and using a calculator.

S1 represents the graph through a chart of values found by substituting in the formula and using a calculator. Surprisingly, he gets a very similar graph to the one from his answer in item *c* and he does not consider it as a correct answer since, implicitly, he applies the reversibility connection to conclude that the graph should be different from the previous one. Then, he plots a graph of the exponential function and gets the answer from item *d* making it symmetrical with respect to the ordinate axis (he mistakes inverse for symmetric) as it is shown in Figure 10. S2 and S8 failed to correctly answer because they (correctly) generalize the incorrect procedure from item *a*. S9 exchanges $\ln w$ for h at symbolic 1. Finally, S7 evidences a failure in the whole network of semiotic functions because she does not try to solve it.

This item is richer in terms of establishing connections. On the whole, it is shown that providing the correct answer in item *c* (which depended on the correct establishment of the reversibility connection) was necessary to establish mathematical connections in this item.

Task 4

Task 4 aims to determine if students know the relation of reversibility between exponential function and the logarithmic one, which demanded the establishment of the mathematical connection of meaning. The analysis related to the approaches shows that, according to the answers given by the students, they tend to define the exponential function correctly and they have a deeper knowledge of the relation of reversibility than of its competent use at solving tasks (see Table 6). An example of this is the case of S8 (Figure 11) showing that she knows the relation among the functions in general and knows how to plot a graph, but

Table 6. Answers to task 4

Students	Define the exponential function correctly	Define the logarithmic function correctly	Establish the reversibility relationship	Use the reversibility relationship in the previous tasks
S1	Yes	No	Yes	He uses it sporadically
S2	No	No	No	No
S3	Yes	Yes	Yes	Yes
S4	Yes	Yes	Yes	Yes
S5	Yes	No	Yes	No
S6	Yes	Yes	No	No
S7	No	No	No	No
S8	Yes	No	Yes	No
S9	No	No	No	No
S10	Yes	Yes	Yes	Yes

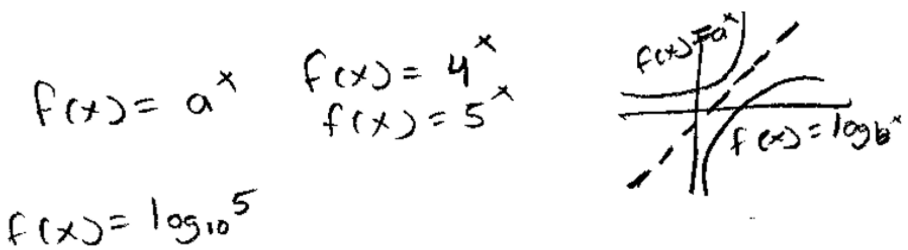


Figure 11. Answer provided by S8 from task 4

she is incapable of using this relation to solve the given tasks.

DISCUSSION AND CONCLUSIONS

This research aimed to study the mathematical connections activated in Mexican high school students' practice solving tasks regarding the exponential and logarithmic functions, establishing the study on two theoretical frameworks: the EMCM and the OSA. The resulting analysis shows that, in order to correctly answer the questionnaire (considered in this article as evidence of the student's understanding), a student must indispensably establish the seven categories of mathematical connections considered by the EMCM. Among them, the use of the reversibility connection stands out, due to its significance, which was expected if we consider that both functions, logarithmic and exponential, are inverse of each other.

Even though we conclude that the most significant mathematical connection for understanding the exponential and logarithmic functions is the reversibility connection, it must be highlighted that, to make the mentioned understanding possible, students must establish other mathematical connections, particularly the feature, part-whole and different representations connections (they are needed to solve the four tasks). In other words, reversibility is the mathematical connection specific to this topic, although the other types of connections are present at understanding almost every mathematical notion.

It must be highlighted that the mathematical connection of reversibility is an indispensable condition to correctly establish the connection of procedure (particularly while solving tasks 2 and 3), as well as the connection of implication (while solving task 3).

The mathematical connections activated by the students when they solve the tasks were expected in the theoretical framework. Five of the seven types of connections (except for implication and part-whole) were also reported by García-García and Dolores-Flores (2018) in their research regarding intra-mathematical connections associated with the derivative and the integral function. In addition, in one of the tasks, they also found that the most important mathematical connection (or main connection) between these objects in the symbolic register (developing procedures) is the connection of reversibility, since regarding the derivative and the integral as inverse processes is essential to establish other types of mathematical connections.

Now, once identified the mathematical connections (established by the students) through the thematic analysis that used the EMCM, the methodology of analysis of the mathematical activity proposed by the OSA was applied, which allowed for the establishment of each category of connections from EMCM.

Metaphorically, it can be understood as the visible part of an iceberg which underwater non-visible part is a vast network of semiotic functions that are at the basis. The non-establishment of some of those semiotic functions of the mentioned network shed light on the reasons why the students do not establish the wanted connection. This analysis with the onto-semiotic methodology allows for the observation of a wide spectrum of possible causes why the students fail to establish a determined connection. This is related to a coherent result with the networking research between the EMCM and the OSA developed in Rodríguez-Nieto et al. (2021).

The thorough analysis of the answers provided by the students to the questionnaire evidences that, although most of the students know that one function is the inverse of the other one, they do not show a competent use of this relation (in terms of developing practices that activate propositions and procedures that derive from the mentioned reversibility). This result, regarding properties of the functions in particular, is similar to the findings of Weber (2002) and Gruver (2018). The students poorly established the procedural connection in the symbolic register, due to the non-competent establishment of the mathematical connection of reversibility and the absence of procedure strategies to solve equations. Even though some students evidenced that they know how to define the functions, they were incapable of solving equations symbolically, being this a result that contrasts with the one from Escobar (2014), since he suggests that students are more likely to develop the procedural connection.

We can consider the results of this research as a departure point, so a question that demands further research can be: how can we study students' understanding of exponential and logarithmic functions based on the mathematical connections established by students (taking as reference the mathematical connection of reversibility)? In general, how can we study students' understanding regarding an object based on the most significant mathematical connections? These results may also be used as the basis to design a sequence of tasks for the study of exponential and logarithmic functions.

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