REFLECTIONS FROM THE LEARNING ENVIRONMENT DESIGNED ACCORDING TO THE VARIATION THEORY: CONGRUENT TRIANGLES[1]

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ABSTRACT

This article presents theoretical knowledge, sample activities, and reflections from the implementation process on how teachers can benefit from the Variation Theory to help students enhance their mathematical comprehension. In this context, "congruence of triangles" was conceptualized as an object of learning. Critical aspects were determined in order to help students discern dimensions of variations of congruence of triangles. In order to develop awareness in students about the critical aspects, learning tasks including patterns of variance and invariance in accordance with the Variation Theory were designed. The activity was implemented in 2020-2021 academic year by a volunteer mathematics teacher under the guidance of the researcher in distance education. Seven volunteer 8th grade students enrolled in a secondary school in Giresun participated in the study. Data analysis revealed that the activity designed based on the Variation Theory was effective in helping students discern the critical aspects of the congruence of triangles.
Keywords: Variation Theory, object of learning, congruence of triangles.

VARYASYON TEORİSİ'NE GÖRE TASARLANAN ÖĞRENME ORTAMINDAN YANSIMALAR: ÜÇGENLERİN EŞLİĞİ

ÖZ

Bu makale, öğretmenlerin öğrencilerin matematiksel anlayışlarını geliştirmeye yardımcı olabilmesi için Varyasyon Teorisi'den nasıl faydalanabileceklerine dair teorik bilgiler, örnek etkinlikler ve uygulama sürecinden yansımalar sunmaktadır. Bu kapsamda "üçgenlerin eşliği" öğrenme nesnesi olarak ele alınmıştır. Üçgenlerin eşliğine yönelik öğrencilerde farkındalık geliştirilebilmesi amacıyla kritik özellikler belirlenmiştir. Kritik özelliklere yönelik öğrencilerde farkındalık geliştirilebilmesi amacıyla Varyasyon Teorisi'ne uygun değişen ve değişmeyen örüntülerini içeren etkinlikler tasarlanmıştır. Etknlikler 2020-2021 eğitim öğretim yılında, gönüllü bir matematik öğretmeni tarafından, araştırmacı rehberliğinde uzaktan eğitim ile uygulanmıştır. Çalışmaya Giresun'da bulunan bir ortaokulun 8. sınıfında öğrenim gören 7 gönüllü öğrenci katılmıştır. Sonuç olarak Varyasyon Teorisi'ne uygun tasarlanan etkinliklerin, öğrencilerde üçgenlerin eşliğine yönelik kritik özelliklere dair farkındalık geliştirmede etkili olduğu görülmüştür.
Anahtar kelimeler: Varyasyon Teorisi, öğrenme nesnesi, üçgenlerin eşliği.

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INTRODUCTION

In general, learning processes are designed with the goal of students' constructing knowledge about the concepts to be taught. Teachers first determine the concepts that they want their students to learn. Then, they design the learning environments by developing activities for students to learn or discover these concepts. Learning environments usually focus on the concepts and their critical aspects. The curricula are also designed in this way. This also holds true for the mathematics curriculum currently in place in Turkey (Ministry of National Education [MoNE], 2018).

The concept of area can be examined as an example. When teaching the concept of area; rectangles, triangles, and parallelograms are treated as separate topics. Students are expected to learn to calculate the area of rectangles in the fifth grade. In the sixth grade, they practice drawing the height of different types of triangles. Students are expected to find the formula for calculating the area of a triangle by using the formula for the area of a rectangle. Then, students are engaged in tasks that require drawing the height of a parallelogram and finding the formula for the area of a parallelogram by using the formula for the area of a rectangle. When teaching the concept of height, the focus is on determining the heights of various triangles or parallelograms. There is not much discussion about the line segments that are not height for a chosen base. Lesson plans are designed in a similar manner in most textbooks, too. This approach represents an inductive approach that expects students to move from the specific to the general. The student studies the height of a particular polygon (e.g., a triangle) and tries to generalize the concept of height to all polygons. Nevertheless, it is a quite challenging process to move from a single special case to the general one.

In Marton's Variation Theory, different from an inductive approach, the *contrast* phase, varying one aspect (dimension) while keeping the other aspects fixed, precedes generalization (Marton, 2014). Thus, it is aimed for students to discern *the dimensions of variation* of an object of learning. In this way, students can distinguish the required aspects of the concept from the optional ones (Johnson, 2020). According to the Variation Theory, discernment of the features of a concept should be developed in order to learn the concept meaningfully (Marton & Booth, 1997). Studying with two different triangles does not ensure the discernment of height (Türker, 2020), nor does only examining the correctly drawn heights. In other words, learning occurs by distinguishing the critical aspects of the object of learning through focusing on the experience of differences rather than the experience of sameness (Marton & Pang, 2013; Pang et al., 2016). This is achieved by determining the critical aspects of the object of learning, and then by varying the focused aspect while keeping the other aspects invariant (Lo, 2012). Critical aspects are considered in terms of *variance* and *invariance*. Marton and Booth (1997) emphasize that there should be a certain pattern of variance and invariance in the learning environment in order for students to make sense of an object of learning. Variation should be used consciously and systematically. By this means, the critical aspects of the object of learning can be discovered.

When designing patterns of variance and invariance, teachers can use the following four principles (Marton & Pang, 2006, p.199). (1) *Contrast:* It is very important to experience mutually exclusive examples of a critical aspect. It is as important to know what an object of learning is as what it is not. (2) *Separation*: Experiencing a critical aspect can be accomplished by varying one dimension of variation while keeping the others invariant. (3) *Generalization*: The critical aspect should be experienced and generalized in different environments/situations. (4) *Fusion*: Two dimensions of variation should vary simultaneously and the student should experience this simultaneous variation. This will allow the student to make connections between the critical aspects. All these four principles aim to develop discernment for the dimensions of variation of an object of learning. While all four principles can be used to discern aspects of a particular object of learning, only one principle can be used as well (Lo, 2012).

There are a limited number of studies in which the learning environments are designed based on the Variation Theory. These studies emphasize that the Variation Theory is effective in helping students discern the critical aspects of an object of learning in topics such as algebraic expressions (Jing et al., 2017), fractions (Pang & Ling, 2011), slope (Pang, 2008), area (Türker, 2020), and multiple representations of graphs (Johnson, 2020). To achieve a level of discernment, it is not enough to just explain the critical aspects to students, rather these aspects should be experienced by the students (Marton & Tsui, 2004). It is the teachers who will provide these experiences to students. However, it is not that easy for teachers to determine the critical aspects of objects of learning (Olteanu & Olteanu, 2010) because these objects of learning are quite simple for them. Teachers may not be able to examine an object of learning from the viewpoint of a student. Thus, they may not be able to recognize the critical aspects. Based on the existing literature, it is important to research the critical aspects of different objects of learning, working with different groups of students. In addition, it is highly important to reveal how students experience the critical features to provide an example for teachers.

In this study, the object of learning is the concept of congruence of triangles. The concepts of congruence and similarity are used to prove a significant part of the properties of triangles (Güven, 2016). The concept of congruence has an important role in real life and in various disciplines. For example, it is predicted that Thales used the angle-side-angle congruence theorem in triangles to calculate the distance of a ship to land (Burton, 1985). In engineering designs, as scaffolding elements, congruent triangles are used to equally distribute the pressure arising from the weight to the carriers (Güven, 2016). Escher (1898-1972) used transformations of congruent and similar shapes in many of his work. These art works helped Escher attain a worldwide reputation among artists and mathematicians.

Previous research studies revealed that students have difficulties in solving questions related to similar triangles (Athanasopoulou, 2008; Biber, 2020; Gül, 2014; Parastuti et al., 2018). Even though understanding the concept of congruent triangles seems easier than understanding the similarity of two triangles, there is not much known in regards to what kind of knowledge students have about the congruence of triangles. In addition, it is very important to reveal which aspects of the concept of congruent triangles are critical for the students in order to help them develop concept awareness. In this context, the purpose of this study is to design an activity based on the Variation Theory for the concept of congruent triangles and to find out how the critical aspects of the concept are experienced by the students.

PLANNING THE ACTIVITY

Determining the Object of Learning

In the Variation Theory, first the object of learning should be determined. The object of learning refers to the concept or skill that we want to teach students. The object of learning in this study is the concept of "*congruence of triangles.*" The mathematics curriculum published by MoNE (2018) includes "congruence and similarity" as a sub-learning domain at the eighth grade level. According to the curriculum, students are expected to achieve the following learning outcomes: "Make connections between congruence and similarity; determine the side and angle properties of congruent and similar shapes." The curriculum also includes the following explanations to highlight some points to pay attention: a) The lessons should include activities that focus on comparing two-dimensional shapes to each other to decide whether they are congruent or not. b) It should be emphasized that the corresponding side lengths and angle measurements are equal in congruent polygons, and the corresponding angle measurements are equal in similar polygons, but the side lengths are proportional. It should also be emphasized that congruent polygons are similar, but similar polygons are not necessarily congruent to each other. The congruence and similarity rules in triangles such as Side-Side-Side, Angle-Side-Angle should not be explicitly taught at this grade level (MoNE, 2018). As evident in the expected learning outcomes given above, the curriculum also suggests comparing congruent and non-congruent shapes in teaching activities. This indicates that the selected object of learning is suitable for the Variation Theory.

Determining the Critical Aspects

Critical aspects are the key aspects in learning a concept. Polygons that have the same size and shape are called congruent polygons. Hence, the corresponding angles and sides of the two congruent polygons are equal in measure.

Similarly, two triangles are congruent if their corresponding sides and angles are equal in measure. The sufficient conditions for two triangles to be congruent can be shown by using the Side-Angle-Side (SAS) congruence axiom, Side-Side-Side (SSS) congruence theorem, and Angle-Side-Angle (ASA) congruence theorem. Based on these definitions, the critical aspects of the congruence of triangles can be listed as follows:

1. Position
2. Congruence of corresponding sides
3. Congruence of corresponding angles

Determining the Students' Prior Knowledge

When students interact with an object of learning, they focus more on features that align with their experiences. Accordingly, students' prior knowledge and perspectives about the object of learning should be determined and then the learning activities should be developed (Marton & Both, 1997). Students' perspectives about a concept can be determined by examining the related literature, administering a test, examining the curriculum, and using teachers' past experiences (Lo, 2012). In this study, a test developed by reviewing the related literature and curriculum was used in order to determine the students' prior knowledge. The test was designed based on the three critical aspects determined for the concept of congruence of triangles. The test consists of three questions that will reveal students' perspectives.

The first question aims to determine the students' prior knowledge about the *position*, the second question about the *congruence of corresponding angles*, and the third question about the *congruence of corresponding sides,* covering all critical aspects. The test is given in Appendix 1. The study was carried out in the fall semester of the 2020-2021 academic year. As the distance education has been in place due to the COVID-19 pandemic, the participants were limited to 7 eighth grade students who regularly participated in online lessons. The test was sent to the students using online tools and the students sent their answers back in the same way. When there was an ambiguous answer, the student was asked to clarify the answer. Students' achievement scores in the test are given in Table 1.

Table 1. Students' Discernment of Critical Aspects of before the Implementation

Question	Correct	%	Incorrect	%
1	5	%71	2	%29
2	4	%57	3	%43
3	4	%57	3	%43

The participating students did not have a formal experience related to "congruence of triangles." However, since the concept of *congruence* is frequently used in daily life, it is expected that the students had a preconception about this concept. Table 1 presents the total number of discernments of the critical aspects. The findings presented in the table do not mean that more than half of the students had discernment of all the critical aspects. There was only one student with knowledge of all the critical aspects. The other students had knowledge of one or two critical aspects while lacking knowledge of the other aspect(s).

According to Table 1, there is a considerable number of students who lacked discernment of critical aspects of the congruence of triangles. There were two students who did not distinguish the effect of *position* on the congruence of triangles. Regarding the *congruence of corresponding angles* and the *congruence of corresponding sides,* for each critical aspect, there were three students who did not discern these aspects. These findings showed that almost half of the students did not have discernment of these critical aspects.

Planning the Tasks

According to the Variation Theory, in order for students to discern a critical aspect, the focused critical aspect (dimension of variation) is varied while the other critical aspects are kept invariant. Based on this tenet of the theory, three tasks were designed towards the three critical aspects of the congruence of triangles (*position*, *congruence of corresponding angles*, and *congruence of corresponding sides*). In order to achieve the purpose of these tasks, students should know how to construct a triangle given the length of three sides, a triangle given the measures of two sides and one angle, and a triangle given the measures of two angles and one side. The participating students had experience of constructing all these triangles.

Task 1

Intended learning outcome: A triangle can be in any position. It remains the same triangle. (Side-Side-Side congruence theorem)

- The *position* of a triangle can be considered by students as a distinctive aspect for the congruence of triangles. Students can think of this dimension as a defining and required aspect.
- *Position* is not a required or defining dimension. It is a critical aspect that should be separated from the required critical aspects.
- In this task, the focused aspects do not vary. Side and angle are focused aspects for congruence. The unfocused aspect, *position*, varies.

<u>varying aspect</u>	<u>invariant aspects</u>
position	angle
	side

- This process may appear similar to induction. However, here, the student does not discover a new meaning (i.e., what a triangle is). Instead, the student is expected to generalize a particular meaning (triangle in any position). A change in the conceptualization of meaning in this way is called a generalization.

Instruction: Construct a triangle with side lengths of 3 cm, 6 cm, and 8 cm.

Principle: In this task, the *generalization* principle from the Variation Theory is employed. In the Variation Theory, *contrast* precedes *generalization*. However, the generalization in this task is not for a required and defining aspect. Here, the goal is to help students open up a dimension of variation.

Task 2

Intended learning outcome: In order for triangles to be congruent, two angles and the side between these angles must be congruent. (Angle-Side-Angle congruence theorem)

- The side and angles are the focused aspects for the congruence of triangles.
- The congruence of corresponding angles and sides is a required or defining dimension.
- In this task, the two angles (hence the third angle), one of the two focused critical aspects, and the position, the unfocused critical aspect, are *invariant*. The other focused critical aspect, side, *varies*.

<u>varying aspect</u>	<u>invariant aspects</u>
side	position
	three angles

Instruction: Construct a triangle with one angle 30˚, and the other angle 60˚.

Principle: In this task, the *contrast* and *separation* principles from the Variation Theory are used. New meanings are learned by simultaneously distinguishing properties and dimensions of variations. If the dimension of variation has been opened up, then the student can discern the critical aspect and construct new meanings. In the previous task, a dimension of variation of congruence, *position*, was made explicit. In the current task, there are other properties to be aware of (although the angles are congruent, the sides may not be congruent). While one critical aspect is invariant, the other critical aspect varies. Therefore, the goal is to develop discernment for the critical aspect that varies. In the process of *contrast,* it is important to know what the focused aspect is as well as to know what it is not. This task makes it possible to focus on non-congruent triangles as well as congruent triangles. The students will be able to discern the critical aspects by examining the examples and non-examples together. Hence, the principles of *contrast* and *separation* are employed.

Task 3

Intended learning outcome: In order for the triangles to be congruent, the two sides and the angle between these sides must be congruent. (Side-Angle-Side congruence theorem)

- The side and angles are the focused aspects for the congruence of triangles.
- The congruence of corresponding angles and sides is a required or defining dimension.
- In this task, lengths of two sides, a focused aspect, and position, the unfocused aspect, are *invariant.* The focused aspects angle and length of one side *vary*.

<u>varying aspect</u>	<u>invariant aspects</u>
angle	position
one side	two sides

Instruction: Construct a triangle with the lengths of two sides equal to 3 cm and 7 cm.

Principle: In this task, the *contrast* and *fusion* principles from the Variation Theory are used. The first task opened up *position* as a dimension of variation for students' discernment. The second task triggered an awareness for the critical aspect *side*. In this third task, the critical aspects "side" and "angle" vary simultaneously and the effect of this variance on congruence is experienced. Therefore, in this task, the principles of *contrast* and *fusion* are employed.

Materials

Since all tasks are designed to construct triangles, the required materials are pencil, paper, ruler, and protractor. Compass may also be used.

ACTIVITY IMPLEMENTATION

The activity was implemented by a volunteer mathematics teacher under the guidance of the researcher (author) in the fall semester of the 2020-2021 academic year. Ethics committee approval required for the study was obtained from Giresun University Ethics Committee (document number: 50288587-050.01.04-E.748, date: January 6, 2021). Prior to implementing each task, the researcher and the teacher held an online meeting. The researcher introduced the relevant task and its purpose to the teacher and shared important aspects to be taken into consideration during the implementation process. All the tasks were designed to be used in online education since at the time of the implementation, distance education was in place. The researcher participated in all lessons and was able to communicate with the teacher. When there was a problem or concern during the process, the researcher and the teacher communicated with each other and sorted out the problem. The research was carried out by having extra lessons with the participants in addition to regular mathematics lessons. Seven volunteer eighth grade students who were enrolled in a secondary school in Giresun province of Turkey participated in the study. Parental consent forms were obtained from the parents of the participating students. Pseudonyms were given to students to protect confidentiality.

The Implementation of Task 1

The class got together through an online application. The teacher explained to the students that they will construct various triangles in this lesson. Then, the students were given the instruction in Task 1: "Construct a triangle with side lengths of 3 cm, 6 cm, and 8 cm." The students were asked to construct the required triangle individually. Some students who were trying to construct a triangle according to this instruction asked with which side to start the construction. Since the goal of this task is to develop discernment of *position*, it is undesirable for all students to choose the same side as the base. For this reason, the teacher told the students that they could start from any side that they wanted.

Students completed their construction in 3 to 10 minutes and sent the photograph of their drawing to the teacher. The teacher examined the triangles and gave feedback about the drawings with incorrect or missing parts. As an example, Fuat's initial drawing is given in Figure 1.

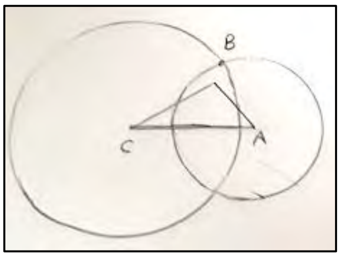


Figure 1. Fuat's Incorrect Drawing

As Figure 1 shows, the top vertex of the triangle that Fuat constructed initially is not at the intersection point of the circles. This shows that the constructed triangle does not meet the criteria given in the instruction. With the guidance of the researcher, the teacher had the following dialogue with Fuat:

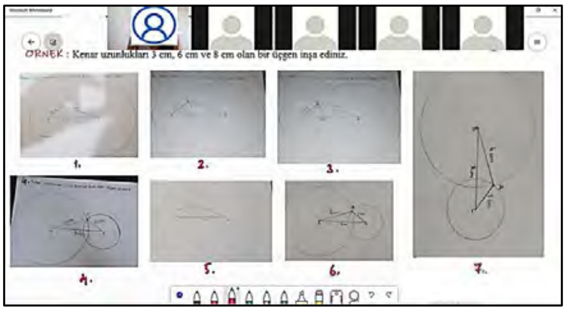
> Teacher: Fuat, why did you draw circles when constructing the triangle?
> Fuat: To determine the sides with the given lengths.
> Teacher: What is the relationship between the required lengths and the circles?
> Fuat: The required length is the radius of the circle.
> Teacher: What does the intersection point of the circles mean?
> Fuat: A vertex of the triangle.

Teacher: Well then, what is the reason that one vertex of the triangle you drew is not at the intersection of the circles?

Fuat: I drew it wrong, Miss.

Fuat noticed the incorrect part in his drawing and corrected it after the inquiry process with the teacher. He sent a photograph of the revised drawing to the teacher. Some students did not label the vertices and some students did not write the side lengths. The teacher warned these students as well and ensured that all the information was written. This is particularly important because only when this information is available, the students can have the opportunity to compare different drawings.

When all the drawings were completed, the teacher put all the drawings together to fit the screen, numbered them, and shared it with the students. In this part of the lesson, it is important to place the triangles in a way that they do not have the same *position* because in this task, the *position* is the *varying* aspect. The image of the screen shared by the teacher is given in Photograph 1.



Photograph 1. All Drawings in Task 1

An analysis of the students' drawings showed that they all constructed the longest side as the base. The reason for this finding might be that students usually have experiences with prototypical shapes and therefore construct a typical perception for those shapes (Clements & Sarama, 2000). In fact, this is an undesirable situation. However, the choices for the other sides of the constructed triangles varied. This provided an opportunity to raise awareness toward the concept of *position*. After the image of all triangles was shared with the students, the following discussion took place:

Teacher: You all drew a triangle according to the measurements that I gave to you. When you examine all the drawings, what can you say about the congruence of the triangles?

Buse: Miss, I think that they are congruent based on their measurements, but [triangles] 2 and 3 are similar in appearance.

Teacher: Any other ideas?

Ali: Miss, one is pointing towards the right and the other is pointing towards the left.

Teacher: What might this mean?

Ali: Because they are different triangles.

Buse: They were drawn in different positions but their measurements are the same.

Ali: How can they be the same, one is pointing that way, one is pointing this way.

This dialogue indicates that Buse was aware that different *position*s do not affect the congruence. On the other hand, Ali thought that the different *position*s affected the congruence of triangles. The goal of this task is to develop discernment for the concept of *position*. This discussion shows that the students started to open up the dimension of variation for the concept of *position*. After all students shared their opinions, the following discussion took place:

Teacher: If I cut these triangles out, give them to you, and ask you to put these triangles on top of each other, what do you think would happen?

Seda: There are triangles with one side short and one side long. In fact, they were all drawn with the same measurements, 6 cm and 3 cm, but one would be long and one would be short when put on top of each other.

Buse: How could they be different? When we rotate them in the same position, they will be the same triangle.

Teacher: Can we change the positions of the triangles as we want?

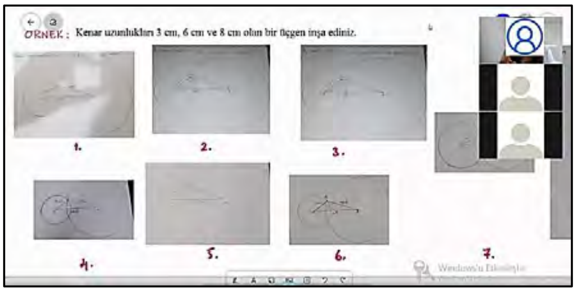
Buse: Of course. If we have a triangle cut out, it will be the same triangle when we rotate it as many times as we like.

Murat: They have the same lengths and also, we can rotate and put them on top of each other. I think it would look like a single triangle.

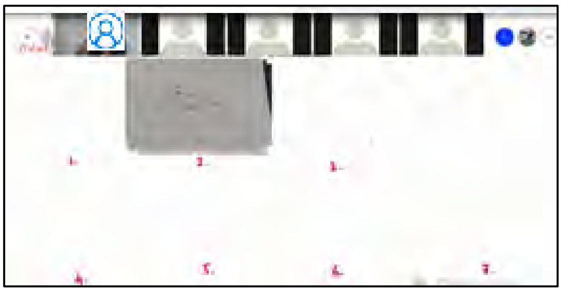
Teacher: Seda, what do you think about your peers' opinions?

Seda: I still hold my decision.

In this discussion, an awareness toward the concept of *position* was developed. The teacher rotated all the triangles in the same position (Photograph 2) and shared it with the students in order to provide a concrete observation about the dimension opened up in the discussion. The group discussed once again whether the triangles were congruent or not. Then, the teacher dragged all the triangles and put them on top of each other (Photograph 3).



Photograph 2. The Triangles Rotated in the Same Position



Photograph 3. The Triangles Put on Top of Each Other

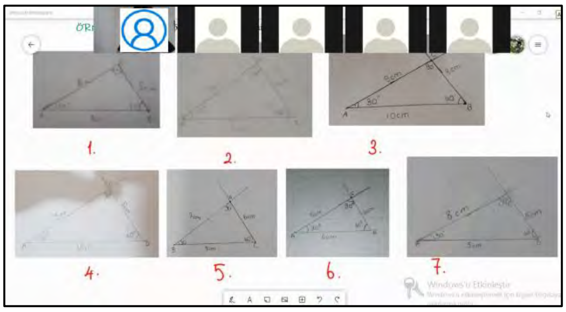
The final image was discussed again. The students were expected to reach a general rule. As a result, all students agreed that varying the position of the triangles does not change the triangle and that triangles are congruent to each other when their sides are congruent. The discussion lasted about 20 minutes and the whole task about 40 minutes.

The Implementation of Task 2

At the beginning of the lesson, the instruction of Task 2 was given to the students: "Construct a triangle with one angle 30˚, and the other angle 60˚." The students were asked to construct the triangle individually. Some students asked if they could determine the length of the first side. The teacher told the students that there was only a limitation on the angles and reminded them to write the necessary side, vertex, and angle information on their drawings. Students' work was observed through cameras. An image of Ali's work is given in Photograph 4.

Students constructed the triangles in about 5 to 12 minutes and sent the photograph of their drawing to the teacher. As in the first task, the students received feedback for the drawings with incorrect or missing parts. The teacher put all the drawings together to fit the screen, numbered them, and shared it with the students. In this part of the lesson, it is important to place the triangles in a way that they all have the same *position* because in this task, the *position* is the *invariant* aspect. The image of the screen shared by the teacher is given in Photograph 5.

Photograph 4. Ali's Drawing



Photograph 5. All Drawings in Task 2

An analysis of the students' drawings showed that there were different and congruent triangles. This is important in developing discernment for the congruence of the corresponding sides and angles. The *position*s of all the triangles drawn were the same due to the direction of the protractor. This does not pose a problem since *position* is considered as an *invariant* aspect in this task. After the image of all drawings was shared with the students, the following discussion took place:

Teacher: We talked about congruent triangles in our last lesson. Which triangles do you think could be congruent in this task?
Buse: I think 3 and 7, Miss.
Teacher: Why do you think so?
Buse: Because their length is equal.
Teacher: Can you examine the number 4 as well?
Buse: I think 4 is not congruent to 3 and 7. It does not have the same drawing.
Teacher: What can you say if you consider the properties that we focused on in the last lesson?

Buse: The angles are the same, and the lengths are the same. (She is examining the measurements). No, their length is not the same. I don't know.
Teacher: Any other ideas?
Ali: Some are not congruent because their side lengths are not the same; some are congruent because they [their side lengths] are the same.
Teacher: Which ones do you think are congruent?
Ali: 3 and 4.
Teacher: Which properties make you think that they are congruent?
Ali: The angles are in the same position; their lengths are the same.
Teacher: What other triangles can be congruent?

In this discussion, the students shared their opinions about which triangles were congruent and explained their reasoning. The lesson continued with the following discussion:

Teacher: I asked you to draw a triangle. Some of you drew the same and some of you drew different triangles. What might be the reason for this?
Murat: Miss, I have noticed something. If the lengths of the bases were the same, the other side lengths would all be the same as well. Because they all have the same angles, the other lengths would not change unless the length between the angles changed. There are 10 cm, 9 cm, 6 cm. That is why the other lengths change.
Seda: Well, Miss, what would happen if the two sides were congruent but the angle was different?
Teacher: Good question. We will examine this question in the next task.

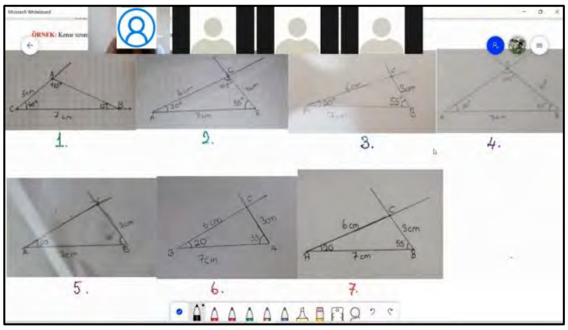
In this teaching episode, the teacher carefully facilitated the discussion, helping students notice how the *varying* aspects affect the congruence. During the class discussion, the students agreed that "in order for triangles to be congruent, two angles and the side between these angles must be congruent." The discussion took approximately 20 minutes. The whole task lasted about 50 minutes.

The Implementation of Task 3

The online lesson started with the instruction of Task 3: "Construct a triangle with the lengths of two sides equal to 3 cm and 7 cm." The students were asked to construct the triangle individually. Some students asked how to choose the angle measurements. Since the goal of this task is to develop awareness about the concept of angle, it is undesirable for all students to choose the same angle measurements. For this reason, the teacher told the students that there was only a limitation on the length of two sides and asked them to write the necessary side, vertex, and angle information on their drawings. However, the teacher did not remind about writing the necessary information again since any lacking information will contribute to the whole group discussion for Task 3. An image from Merve's work is given in Photograph 6.

Photograph 6. An Image from Merve's Work

Students constructed the triangles in about 5 to 15 minutes and sent the photograph of their drawing to the teacher. The teacher put all the drawings together to fit the screen, numbered them, and shared it with the students. In this part of the lesson, it is important to place the triangles in a way that they all have the same *position* because in this task, the *position* is the *invariant* aspect. The image of the screen shared by the teacher is given in Photograph 7.



Photograph 7. All Drawings in Task 3

An analysis of the students' drawings showed that except for one student, all other students chose 20° as one of the angle measurements. This is an undesirable situation in terms of having different answers. Nevertheless, the existence of congruent and non-congruent triangles provided an opportunity to develop discernment for the congruence of corresponding sides and angles. The *position*s of all the triangles drawn were the same. This does not pose a problem since *position* is considered as an *invariant* aspect in this task. After the image of all drawings was shared with the students, the following discussion took place:

> Teacher: Which triangles do you think might be congruent?
> Merve: I think 2 and 6 are congruent.
> Teacher: Why do you think so?
> Merve: Their angle and side measurements are equal.
> Teacher: Well, what do you think about 4? Might it be congruent to 2 and 6?
> Merve: One side was not written.
> Teacher: Can we predict it?
> Merve: All angles and two sides are the same. That side will be the same as well.
> Teacher: Can the person who drew the fourth triangle measure it and tell us its length?
> Seda: Miss, it is 6 cm.
> Teacher: What can we conclude?
> Seda: Miss, 2, 3, 4, 5, 6 and 7 all have 20˚ angle. The other angle is 55˚ in allof them, while it is 60˚ in 5. Therefore, this angle affects the congruence. For example, the side whose length is not written in 5 would not be 6 cm.

In this discussion, the students shared their opinions about which triangles were congruent and explained their reasoning. The teacher carefully facilitated the discussion, helping students notice how the *varying* aspects affect the congruence. By the end of the discussion, the students agreed that "in order for triangles to be congruent, two sides and the angle between these sides must be congruent". The discussion took about 15 minutes. The whole task lasted about 40 minutes.

CONCLUSION and SUGESTIONS

After all the tasks were completed, the students took the pre-test again. The students' achievement scores are given in Table 2. Table 2 shows that all students had awareness of the effect of *position* on the congruence of triangles at the end of the study. Also, all students developed awareness for the critical aspects of the *congruence of corresponding angles and sides*; only one student gave an incorrect answer to the third question, which tests the discernment of the *congruence of the sides*. This finding shows that the activity designed based on the Variation Theory is effective in developing discernment for the critical aspects of the congruence of triangles. This discernment was possibly developed due to conceptualizing the critical aspects in light of certain principles and using the patterns of variance and invariance in designing the learning tasks.

Table 2. Students' Pre-Test Scores After the Activity Implementation

Question	Correct	%	Incorrect	%
1	7	%100	0	%0
2	7	%100	0	%0
3	6	%86	1	%13

Students made measurement errors while constructing the triangles as part of the activity since they drew their triangles using a ruler and a protractor. This is an expected situation as perfect measurements are unlikely in paper and pencil drawings. This aspect of the activity will not affect developing discernment. The important point is to focus on the varying and unvarying aspects. Alternatively, these drawings can be made using a Dynamic Geometry Software to avoid measurement errors and to obtain perfect drawings. *Position* is considered as a varying aspect only in the first task. While implementing the tasks 2 and 3, positions of the triangles that are found to be congruent can also be discussed. In this way, students can make a new *generalization* by considering all critical aspects.

In the study, it was observed that some students constructed the triangles faster than the others. These students can be encouraged to think about different drawings of the same triangle, in order to manage the class effectively, to keep the faster students engaged in the task, and to help students develop awareness of the critical aspect. In this study, the activity was implemented in distance education due to the COVID-19 pandemic. A similar process can be used in the face-to-face education. In face-to-face teaching, the triangle constructions can be made in larger measurements by using colored cardboard. The constructed triangles can be cut out and posted on the board, taking into account the points to be considered in the task. A whole class discussion can be conducted based on the triangles posted on the board.

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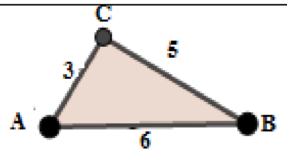
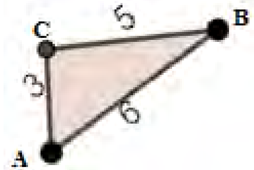
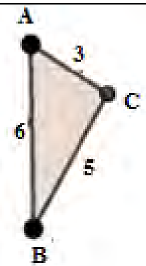
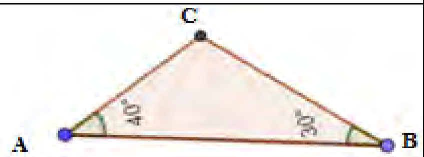
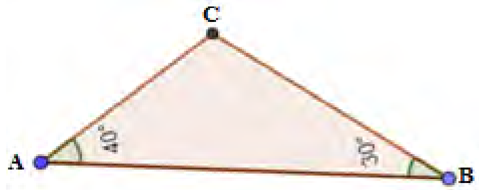
Appendix 1

A Test to Measure Students' Knowledge of Congruence of Triangles

Dear Students, There are three different questions below. For each question, examine the triangle in the first box. In the same row, mark the triangle(s) that could be congruent to this triangle. Write down why you think that it is a congruent triangle.					
1	 IACI= 3 br IABI=6 br IBCI=5 br	 IACI= 3 br IABI=6 br IBCI=5 br	 IACI= 3 br IABI=6 br IBCI=5 br	 IACI= 3 br IABI=6 br IBCI=5 br	 IACI= 3 br IABI=6 br IBCI=5 br
	Explain your answer. It is congruent because….. It is not congruent because….				
2					
	Explain your answer. It is congruent because….. It is not congruent because….				

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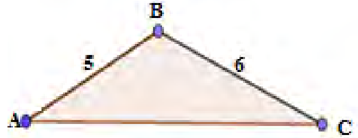
<td>IABI=5 br
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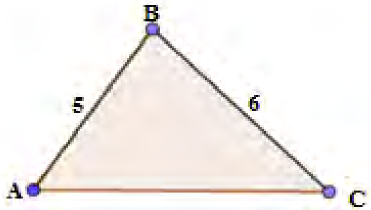
<td>IABI=5 br
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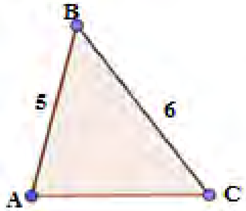
<td>IABI=5 br
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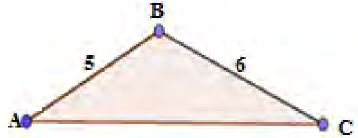
<td>IABI=5 br
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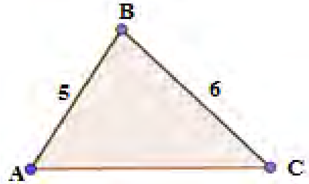
<td>IABI=5 br
IBCI=6 br</td>
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<td>Explain your answer.
It is congruent because…..
It is not congruent because….</td>
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