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Visual Reasoning in Mathematics Education: A Conceptual Framework Proposal

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Abstract: Reasoning is handled as a basic process skill in mathematics teaching. When the literature was examined, it was seen that many types of reasoning related to mathematics education were mentioned. In the present study, it was focused on visual reasoning, which is one of the types of reasoning and also used in different research areas. The purpose of the study was to propose a conceptual framework for what visual reasoning is and what its components are. The conceptual framework constructed consists of three components as visual representation using, visualization, and transition to mathematical thinking. In this framework, a clear distinction was made between the concepts of visual reasoning and visualization, which are thought to be intertwined with each other in the literature. At the same time, we tried to explain where visualization will take place in visual reasoning. Additionally, how visual reasoning will relate to mathematical thinking also distinguishes the framework from other frameworks.

Key words: mathematical thinking, visualization, visual reasoning, visual representation.

1. Introduction

Although the aim of teaching mathematics in primary education is not to teach a deep knowledge of mathematics, it can be said that creating a basis for people to gain mathematical thinking skills. According to Nunes (2014), in order to think mathematically, people must learn to represent concepts such as quantities, relationships, and areas, using numbers and other mathematical tools such as algebra, graphs, and calculators that are common in today's society. National Council of Teachers of Mathematics [NCTM] (2000) emphasized that the need to understand and use mathematics for the changing world conditions is very important and this need will increase gradually. It is also stated that mathematical thinking and problem solving are needed more in many fields of professions.

Stacey (2006) also emphasized that mathematical thinking, which is inevitably used in the solution of real life problems, has an important place in mathematics education and therefore is among the important aims of mathematics education. Similarly, Alkan and Bukova-Güzel (2005) stated that they use mathematical thinking in analyzing the events and phenomenon that people encounter at every stage of their lives, whether consciously or not. In short, learning mathematics is much more than learning some basic concepts and gaining operational skills. The important thing in mathematics education is to provide an individual to think mathematically in order to adapt to life.

Liu (2003, p. 417) defined mathematical thinking as "a combination of guessing, induction, deduction, specification, generalization, analogy, formal and informal reasoning, verification, and similar complex processes". Based on Liu's (2003) definition, reasoning can be said to be an important process for mathematical thinking. Artz and Yaloz-Femia (1999), who consider reasoning as a part of mathematical thinking, stated that reasoning involves attaining valid conclusions and generalizations. Lithner (2000) defines the term reasoning as a way of thinking that is adopted to make assumptions and reach conclusions. Since reasoning is important, NCTM (2000) also has considered reasoning as a basic process skill in mathematics teaching. To improve students' learning, it is necessary to understand the developmental state of their thinking and reasoning (Cai, 2003).

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Reasoning can be considered as the process of obtaining new information using the idiosyncratic tools of mathematics (symbols, definitions, relations, and so on.) and thinking techniques (induction, deduction, comparison, generalization, and so on.) based on the information available (Ministry of National Education [MoNE], 2013, p. 5). According to Ball and Bass (2003), reasoning is the process of thinking based on making claims using mathematical knowledge and patterns and constructing a new configuration. Reasoning is also expressed as a key component of doing mathematics (Stacey, 2006).

When the literature is analyzed, many types of reasoning (symbolic reasoning, deductive reasoning, inductive reasoning, algebraic reasoning, proportional reasoning, geometric reasoning, statistical reasoning, quantitative reasoning, analogical reasoning, adaptive reasoning, creative reasoning, and so on) related to mathematics education are mentioned (Battista, 2007; Ben-Zvi, 2014; Bergqvist & Lithner, 2012; Fonseca, 2018; Healy & Hoyles, 1999; Hoyles, Noss, & Pozzi, 2001; Smith & Thompson, 2007). The present study focused on visual reasoning, which is one of the types of reasoning and also used in different research areas. The purpose of the study was to propose a conceptual framework for what visual reasoning is and what its components are.

1.1. Literature review

Gülşen (2012) defined visual reasoning as the process of perceiving visual information in geometric or graphic representations representing the relationships between mathematical expressions, establishing the relationships between this visual information and making this information formal, and expressing visual information as mathematical relationships. According to Karrass (2012, p. 27), visual reasoning is as follows: “to engage in diagrammatic reasoning means to create or observe a graphically rendered cognitive construct; to perceive its components and inherent structures; to reflect upon these perceptions; to intuitively generate new hypotheses and verify them; to communicate ideas; and finally, to make connections”. At the same time, visual reasoning is about understanding and apprehending problems, concepts, objects, or processes in terms of visuals (Abd Hamid, 2017).

It is important for students in terms of developing communication skills to express their mathematical thoughts using different visual representation forms such as concrete model, shape, picture, graphic, table, and symbol (MoNE, 2013). Visual reasoning involving reasoning with visual representations; it is not about linguistic or algebraic tools but about understanding concepts and ideas visualized by the use of diagrams and images (Anderson & McCartney, 1997). Fischbein (1987) states that visual reasoning is an important factor not only in organizing data in meaningful structures but also in directing the analytical development of the solution. Hoffmann (2007), on the other hand, defines visual reasoning as the process of facilitating thinking and states that this kind of reasoning is about decision making and developing knowledge.

Another term in the literature that is said to be intertwined with visual reasoning is visualization (Hershkowitz, Arcavi, & Bruckheimer, 2001). Visualization is the process of creating or using a hand-drawn or computer-generated geometric or graphic representation of mathematical problems, principles, or concepts (Zimmermann & Cunningham, 1991). In recent years, interest in visualization and visual reasoning has increased in mathematics education studies (Abd Hamid, Idris, & Tapsir, 2019; Akkan, Akkan, Öztürk, & Demir, 2018; Alcock & Simpson, 2004; Duval, 2014; Gal & Linchevski, 2010; Gunčaga & Žilková, 2019; Natsheh & Karsenty, 2014; Presmeg, 2014; Yilmaz & Argun, 2018).

According to Arcavi (2003), “visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas, and advancing understandings.” (Arcavi, 2003, p. 217). It is expressed in the advanced dimension of this broad process and level of thinking that there is a concept of visual reasoning that can be defined as using visuals, images, or diagrams effectively to perform high level thinking tasks (Natsheh & Karsenty, 2014). In the framework of these opinions, it can be said that visualization is a process performed with a set of visual tools and visual reasoning is a way of thinking that covers this process.

2. Conceptual framework: Visual reasoning

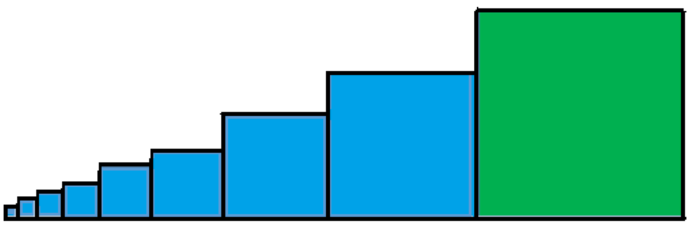
Visual reasoning plays a much more important role than is generally known in the work of today's mathematicians (Dreyfus, 1991). Visual reasoning is different from verbal reasoning because it is supported by diagrams. However, this does not mean that it is completely different from verbal reasoning rather visual reasoning complements verbal reasoning (Diezmann, 2005, as cited in Pantziara, M., Gagatsis, A., & Elia, 2009). Spatial and visual reasoning is thought to include mental skills related to visually understanding, manipulating, rearranging, or interpreting relationships (Booth & Thomas, 1999).

Visual reasoning is also associated with the mathematical understanding process (Trigueros & Martínez-Planell, 2010). Therefore, visual reasoning can be used to provide understanding in mathematics. Additionally, visual reasoning is associated with discovery (Zahner & Corter, 2010). This feature of visual reasoning can be used in exploratory mathematics activities. Abd Hamid (2017) stated that the studies investigating the effect of visual reasoning and visualization on academic achievement generally show positive results on various subjects and for many educational levels.

When the literature is examined, a limited number of conceptual frameworks for visual reasoning were found (Abd Hamid, 2017; Natsheh & Karsenty, 2014; Russell, 1997). Abd Hamid (2017) emphasized the coding and decoding processes in the process of studying visual reasoning skills. The coding process can be explained as students' using visual representation to explain verbal or written information. The decoding process can be expressed as students' reading, comprehending, and interpreting the visual information placed in visual representations.

The "visual inferential conceptual reasoning" framework constructed by Natsheh and Karsenty (2014) is also guiding for mathematics educators in terms of explanation of visual reasoning. This framework can be seen as a special type of visual reasoning in relation to the ability to use visual thoughts to achieve inferences that improve conceptualization. The framework consists of three components: visual displays, visual actions, and visual purposes. Visual displays are 2D or 3D objects on which certain visual actions can be performed. Pictures, graphics, tables, shapes, and so on tools can be said as an example to visual displays. Visual actions are different operations and actions that a person can perform on a visual display. It reflects the actions required to reveal information that is not explicitly provided in the visual display. For example, a person can look at a visual display and can also measure, read, make comparisons, add an auxiliary structure, cut a visual display into pieces, or create a new visual display. Visual purposes are related to the purposes of visual actions performed on visual displays. The student reflects the purpose of the action by referring it to a visual display created by her/his own or provided by others to explore the relationship between visual displays. Rephrasing, defining, explaining, and providing information can be given as examples of visual purposes.

Let's analyze the framework over an example problem given in Figure 1.



The figure shows squares with different edge lengths. The area of each square is equal to twice the area of the previous square.

a) In the direction of this informations, is the sum of the areas of the blue squares otherwise the area of the green square larger?
b) Please explain your answer with its reasons.

Figure 1. *A problem toward the visual inferential conceptual reasoning framework*

When we examine the problem in accordance with the process of visual reasoning, visual display is the form presented in the problem. The actions to be taken regarding the area of the squares on visual display are visual actions. These actions will be shaped according to the students' previous knowledge. In this process, the student may apply to other visual displays if needed. The visual purpose of the problem is that students can make explanations by interrelating relations between the squares.

Gülşen (2012) stated that there are many definitions and researches for visualization in the literature as a result of her examination but stated that visual reasoning is more limited. We think that these concepts that are intertwined with each other should have a clear distinction. Therefore, the framework constructed within the scope of this study explains where visualization, which is frequently included in the literature, will take place in visual reasoning. Additionally, how visual reasoning will relate to mathematical thinking also distinguishes the framework from other frameworks. The conceptual framework constructed as a result of the literature reviews is presented in Figure 2.

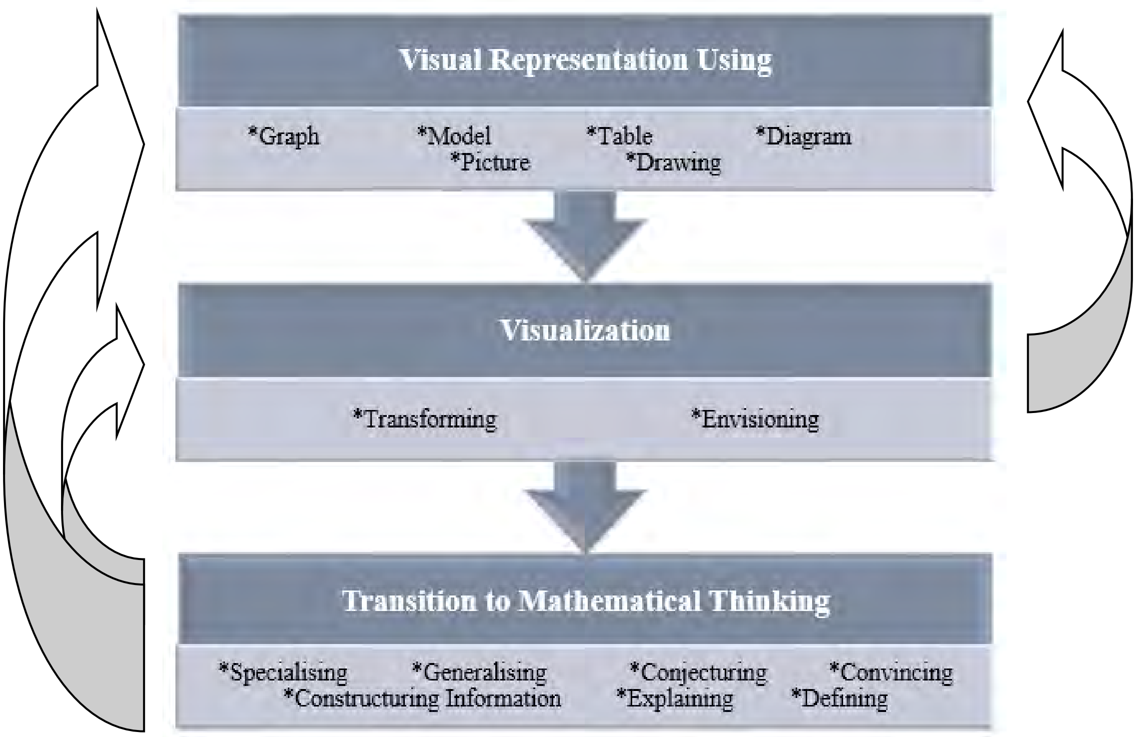


Figure 2. *The proposed conceptual framework toward visual reasoning*

When we look at Figure 2, it can be said that the process of visual reasoning takes place in three stages. For visual reasoning, the component that must be handled first is visual representation. It all starts with visual representation. Visual representations can be used in the solution of a problem or in the process of understanding a concept. After that, visualization takes over. An individual interacts with the visual representation that she/he has created or is presented to the visualization stage. Here, a some mental activities, such as transforming or envisioning, occur. If it seems necessary, it can interact with visual representation again. Finally, the individual who completes the visualization engages in mathematical thinking activities related to the situation that arises. In these activities, the individual can make explaining, conjecturing, or generalizing, and so on. In addition to these activities, the individual can return to previous stages and also be involved in other processes. Now we want to examine these components in more detail and then make explanations on a problem situation.

2.1. Visual representation using

Mathematical representations are visible or tangible productions such as diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, mathematical expressions, formulas, and equations, or depictions on the screen of a computer or calculator (Goldin, 2014, p. 409). When the literature is examined, it is seen that the concept of representation is classified in different ways (Gilbert, 2010; Goldin, 2014; Lesh, Post, & Behr, 1987). Gilbert (2010) divided the representations into two classes, symbolic and visual. In this context, abstract structures such as signs, discourses, or symbols are symbolic representations. However, structures that emphasize the visual form of information such as pictures, diagrams, or graphics are called visual representations.

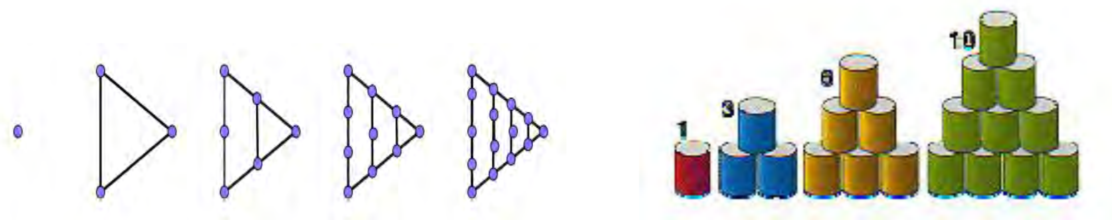
According to Russell (1997), a visual representation of a mathematical problem can be internal or external. Internal visual representation is an object of the mind created from our experiences. External visual representation includes paper and pencil drawings, models or other physical manipulators, and technological visuals of the problem. Additionally, it is expressed that there is an abstraction of internal visual representation for external visual representation. Goldin (1998) mentions that visual representations that play an auxiliary tool in the expression of mental representations are a language expressed with the help of visual images that allow the transfer of a concept and idea. According to Debrenti (2015), many studies show that the representation of mathematical objects and pictorial representations play an important role in the learning process. It is emphasized that the drawings facilitate a better understanding of the concepts or the problem and help mathematical reasoning (Debrenti, 2015).

Definition. A number is triangular if it is the sum of consecutive integers beginning with 1 (Donald, 1998, p. 2). For example,

$T_1 = 1$, $T_2 = 1+2 = 3$, $T_3 = 1+2+3 = 6$, $T_4 = 1+2+3+4 = 10$.

n^{th} the triangular number can be expressed as $T_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

When a student encounters a definition like above, a set of visual representations will facilitate one's work to make sense of the definition.



Sequence number	1	2	3	4	5
Value	1	3	6	10	15

Figure 3. *Examples of visual representation*

Kar and İpek (2009) stated that visual representations were frequently used in problem solving throughout the history of mathematics. They emphasized those visual representations are an important strategy for understanding and solving the problem and that visual representations should be used in mathematics teaching and problem solving process. Likewise, Bishop (1989) stated that emphasizing visual representations is an important factor in mathematics education.

Ergan (2018, p. 3) expresses the benefits of visual representations as follows:

- It helps the students determine the importance and operation sequence of the information given in the problem,
- It ensures that the problem parts are brought together and the steps to be taken for the solution are determined,

- It provides the systematic view of the concepts given in the problem in a concrete plane and together,
- It helps students construct their conceptual knowledge and connect with their existing knowledge.

Education researchers have long studied the role and importance of visual representation strategy in mathematics learning and problem solving (Arcavi, 2003; Debrenti, 2015; Goldin, 1998; Stylianou & Dubinsky, 1999). Creating visual representations should not be seen as an additional aid to problem solving but as a process developed by students and forming part of the problem solving process (Ergan, 2018). Stylianou (2011) emphasized that visual representations are the process itself rather than a final product. However, in order to a visual representation to be considered as a useful tool for problem solving, it must be fully understood and internalized by the student (Dufour-Janvier, Bednarz, & Belanger, 1987, as cited in Ergan, 2018).

2.2. Visualization

In order to approach the abstract relations of mathematics more effectively, the use of possible concrete representations of objects is called mathematical visualization (Guzman, 2002). Presmeg (2014), in a similar way, stated that signs such as symbols and diagrams are used to represent abstract concepts in mathematics and emphasized that visualization is involved in the representation of mathematics. Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect visual information (Hershkowitz, et al., 1989, p. 75, as cited in Hershkowitz, Arcavi, & Bruckheimer, 2001). For this reason, visualization is very important for learning mathematical concepts, especially geometric concepts (Hershkowitz, Arcavi, & Bruckheimer, 2001). According to Wheatley (1998), visualization is a complex process of transformation between structuring, representation, and mental images. Starting from pre-school education, visualization process can be utilized at every educational level in order to embody and better understand the abstract concepts and symbolic representations with visual representations (Presmeg, 2006).

We come across visualization in many fields such as engineering, art, medicine, economics, chemistry, automotive, and psychology besides mathematics (Gutiérrez, 1996). From a math point of view, the aim is not only to visualize but to understand mathematics with the help of visualization. In other words, instead of obtaining a product with visualization in mathematics, it can be said that it is aimed to use visualization as a tool to provide understanding (Gülşen-Turgut, 2019). For example, visualizing a diagram means creating an image of that diagram in the mind. Similarly, visualization of a problem refers to understanding the problem through a diagram or a visual image. In mathematical visualization, both images are created and with the help of these images, an effective mathematical discovery and understanding process is passed (Zimmermann & Cunningham, 1991).

Graphics, diagrams, and various geometric shapes or models are a tool for visualizing thoughts, ideas, and abstract concepts. Through them, human thought makes a relationship between the external world and abstract concepts. In other words, representing algebraic structures with geometric expressions helps to show students how logical theory was constructed based on a physical model (Konyalıoğlu, 2003). Visualization appears to be a useful approach in attracting students' attention, motivating the student, making it meaningful by concretizing learning, organizing the student's knowledge, and connecting with concrete and abstract expressions of concepts (Işık & Konyalıoğlu, 2005). Additionally, it is stated by many researchers (Presmeg, 2006; Rivera, 2013; Van Garderen, Scheuermann, & Poch, 2014) that visualization is important for teaching mathematics by going beyond understanding a subject in terms of visual representations. Besides, some researchers (Abdullah, Zakaria, & Halim, 2012; Alcock & Simpson, 2004; Stylianou & Dubinsky, 1999) have associated visual reasoning with an in depth understanding of mathematical concepts. Using the visualization approach in mathematics courses allows students to view abstract concepts and structures from different perspectives (Delice & Sevimli, 2010).

According to Tekin (2010), one of the important points in visualization is that the student gains the ability to think visually using the visual model and thus develops her/his understanding of the subject or concept. Visual thinking skills help individuals understand the events, processes, and objects they observe around them and create more detailed and rich abstract concept schemas in their minds. The network structure of the concept schemas in mind affects the permanence and meaningfulness of what is learned positively.

2.3. Transition to mathematical thinking

When looking at mathematics education research, there are researchers (Burton, 1984; Harel & Sowder, 2005; Mason, Burton, & Stacey, 2010; Schoenfeld, 1992) who examine mathematical thinking from the perspective of mathematical process such as problem solving, specializing, generalizing, conjecturing, and proving, and there are also researchers (Dreyfus, 2002; Tall, 1995) who go around mathematical thinking as the development of mathematical concepts. The point of view that deals with mathematical thinking from the perspective of the process focuses on "How does mathematical thinking take place?" but it can be said that the other perspective is focused on how the individual structures the concepts in her/his mind (Çelik, 2016).

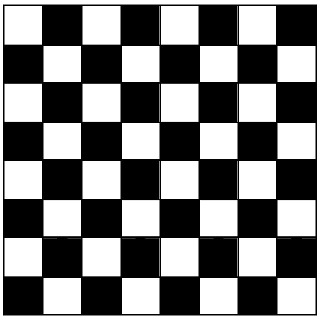
In the framework constructed, the person completing the visualization stage is expected to reveal some mathematical thinking components such as specializing, generalizing, conjecturing, proof, explaining, and so on. Therefore, in this framework, a perspective that takes mathematical thinking from a process perspective is adopted. If the person takes part in the process of transition to mathematical thinking, being aware of what to do, it can be said that she/he runs her/his visual reasoning skills to work even if she/he fails to achieve the targeted result.

The framework constructed for visual reasoning will be explained by the following problem taken from Mason, Burton, and Stacey (2010). The explanations on the problem were made in a way that a student at the middle school level can understand. The explanations should be evaluated only for this problem.

Problem: It is claimed that there are 204 squares on an ordinary chessboard. Can you show that this claim is true?

Visual representation using:

In the face of such a question, every student can think of a unique solution. Perhaps a person may want to draw a chessboard right away. In this way, a good start can be made to the problem by creating a visual representation.



A person who draws the side figure has created her/his visual representation to reach the solution of the problem. It can be said that it will be descriptive especially for those who cannot visualize the chessboard in their mind. Looking at the figure, it is seen that the chessboard consists of 8 rows, 8 columns, and 64 squares. However, after seeing that 204 squares are mentioned in the problem, the student will realize that there are other squares.

Visualization:

After this stage, not all students can reason properly. However, there will be students who think as described below.

Looking carefully at the figure, it can be understood by time that 64 squares found in the previous stage are 1x1 squares. It will also be understood that there are 2x2, 3x3, 4x4, 5x5, 6x6, 7x7, and 8x8 squares by envisioning or transforming the shape in the later stages.

It can be said that this stage is a very important step for the solution of the problem. However, it may take time to reach the correct result by counting other squares. Even in this process, wrong conclusions can be reached. Therefore, the student should try to make a generalization, which is one of the components of mathematical thinking, to reach the correct result of the problem.

Transition to mathematical thinking:

The students are expected to make specializing firstly in order to generalize. They can create a pattern after counting a certain square from the figure. Perhaps a new visual representation may be created as follows.

Edge lengths	8x8	7x7	6x6	5x5	4x4	3x3	2x2	1x1
Count	1							64

Students can find the result by counting 7x7 squares. Likewise, after seeing that the number of 6x6 squares is 9 and by discovering the pattern, they will show that

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = 204.$$

3. Discussion and conclusion

It is thought that the visual reasoning framework can be a useful framework that can be used in all areas of mathematics from primary to higher education. Apart from the field of geometry, where visual representations are used extensively, visual reasoning framework can be employed in areas such as numbers, algebra, and statistics. The framework will contribute to students' reasoning skills using in the solution processes of routine and non-routine problems. Additionally, a mathematics teacher's mathematics lessons designed based on the components of the framework will increase the quality of teaching.

The visual reasoning framework can be also applied for making proofs without words. According to Nelsen (1993), proofs without words, even if they are not accepted as real proofs, they express why a particular statement is true. The visualization (diagram or picture) during the proof without words helps to make a rigorous proof. Polat, Oflaz, and Akgün (2019) think that proofs without words using visual representations are important tools in mathematics education. Therefore, thanks to proofs without words, visual reasoning can be realized using visual representations, visualizing, and then proving.

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