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SPECIALISED CONTENT KNOWLEDGE: THE CONVENTION FOR NAMING ARRAYS AND DESCRIBING EQUAL GROUPS' PROBLEMS

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Abstract

Specialised Content Knowledge (SCK) is defined by Ball, Hoover-Thames, and Phelps (2008) as mathematical knowledge essential for effective teaching. It is knowledge of mathematics that is beyond knowledge which would be required outside of teaching; for instance, the capacity to determine what misconception(s) may lie behind an error in calculation. Such knowledge should be core business of teachers of mathematics, and any perceived shortfall in SCK viewed as problematic. The research reported on here is part of a large study about multiplicative thinking involving approximately two thousand children between nine and twelve years of age and their teachers. Data were generated from semi-structured interviews and a written diagnostic assessment quiz. As part of that large project, forty-four Australian and New Zealand primary and middle school teachers were asked to respond to student work related to multiplicative thinking, particularly to concepts of numbers of equal groups and commutativity. Participants' responses reflected confusion about a pivotal piece of SCK, the convention for naming arrays. As well, questionable assumptions about the children's work samples were made. Given that there is not unanimous agreement amongst mathematics educators about naming conventions, these observations may not be surprising. Due to the sample size, broad generalisations cannot be made, but the results suggest that many teachers may have limited SCK with regards to the important mathematical area of Multiplicative Thinking (MT).

Keywords: Conventions, arrays, multiplicative, teacher content knowledge.

INTRODUCTION

Background

Since the publication of Shulman's seminal work on Pedagogical Content Knowledge (PCK) in the 1980s, much has been written about PCK, as well as the variations of it. The attempt by Shulman to codify the elements which can be identified in effective teachers, has provided a platform from which the idea of Mathematical Knowledge for Teaching (MKT) (Ball, Hoover-Thames & Phelps, 2008) has evolved. This paper posits the question "To what extent can a sample of primary and middle school teachers articulate the Specialised Content Knowledge needed for teaching aspects of multiplicative thinking, in particular, with regard to equal groups' problems?" Further, it will make a case for the development of teachers' Multiplicative Thinking (MT), an essential 'big idea' (Siemon, Beswick, Brady, Clark, Faragher, & Warren, 2015) of mathematics which spans the domains of Common Content Knowledge (CCK) and Specialised Content Knowledge (SCK). It will assert that many of the teachers involved in this study did not display sufficient levels of SCK, and therefore Subject Matter Knowledge (SMK) regarding MT, that are necessary to effectively teach this vital concept. Further it will investigate a pedagogical tool, the multiplicative array, for the teaching and learning of MT, a tool which can be used to promote a conceptual understanding of this important mathematics.

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Literature review

Pedagogical Content Knowledge (PCK) and Mathematical Knowledge for Teaching (MKT)

Shulman (1986) argued the presence of Pedagogical Content Knowledge (PCK), a form of knowledge which could be described as the nexus between the content and pedagogy, and a third essential domain, context (Figure 1). PCK can be described as a practical knowledge of teaching and learning, through a contextualised lens of knowledge of a particular classroom setting (Shulman, 1986).

Figure 1. Shulman's (1986) domains of pedagogical content knowledge

Since the 1980s, Shulman's ideas and schema regarding PCK has been widely accepted and has been adapted by a number of authors. In elaborating on Shulman's construct of PCK, several teams of researchers (e.g. Hill, Ball & Schilling, 2008; Ball, Hoover-Thames & Phelps, 2008; Delaney, Ball, Hill, Schilling & Zopf, 2008) have used a construct (Figure 2) which aligns their stated domains of content knowledge for teaching onto two of Shulman's (1986) categories for PCK, Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge. This construct by Hill et al. (2008) to illustrate Mathematical Knowledge for Teaching (MKT) was described by Depaepe, Verschaffel and Kelchtermans in 2013, as being mathematics education's most influential reconceptualisation of teachers' Pedagogical Content Knowledge.

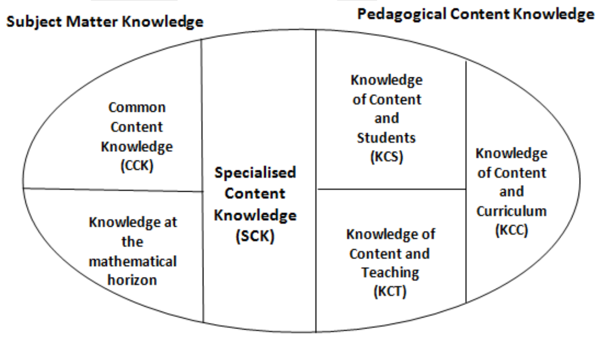

Figure 2. Mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008, p. 377)

Hill, Ball and Schilling (2008) assert that Pedagogical Content Knowledge can be broken into three domains: Knowledge of Content and Students (KCS); Knowledge of Content and Teaching (KCT); and Knowledge of Content and Curriculum (KCC). Subject Matter Knowledge (SMK) also contains three domains: Common Content Knowledge (CCK); Specialised Content Knowledge (SCK); and Knowledge at the mathematical horizon. This paper will primarily concern itself with only two of the domains CCK and SCK, while acknowledging that all domains are interrelated.

Common Content Knowledge (CCK) is mathematical knowledge and skill used in settings not necessarily unique to teaching (for example the ability to: calculate an answer correctly or recognise incorrect answers). Effective teaching requires more than CCK, it also requires Specialised Content Knowledge (SCK) which is the basis of quality instruction, instruction which is focussed and where decisions regarding both content and pedagogy are made (Delaney et al., 2008). SCK is the understanding of the mathematical knowledge and skills which are particular to teaching (for example the ability to respond to students' "why" questions or recognise what is involved by using a particular representation).

SCK of pedagogical concerns (for example, in recognising what is involved in using a particular representation or linking representations to underlying ideas and to other representations) is necessary to develop conceptual and/or procedural knowledge (Rittle-Johnson & Schneider, 2015). Bruner (1966) asserted that concepts are progressively developed, first in the 'enactive' stage through engagement with concrete experiences, then the 'iconic' stage, when pictorial and other graphic representations are used, and finally, the symbolic stage. Bruner's work underpins the contemporary practice of CRA (Concrete-Representational-Abstract), a graduated sequence of instruction proven to be an effective strategy for teaching mathematics concepts and skills (Agrawal & Morin, 2016; Goonen & Pittman-Shetler, 2012). An example of CRA is presented in Figure 3.

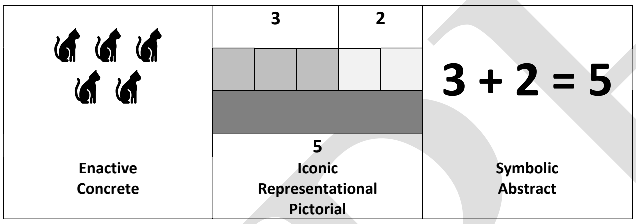

Figure 3. Example of CRA Model

CRA purports that it is developmentally advantageous for students to be afforded opportunities to move from the concrete, to the representational, and then to the abstract (Goonen & Pittman-Shetler, 2012), in order to be facilitated in gaining a deeper and lasting understandings of mathematical concepts (Mutodi & Ngirande, 2014). Research indicates that for learning benefits to occur, concrete material is appropriate to the topic (Askew, 2018; Swan & Marshall, 2010), in order to stimulate students' thinking. Merely providing students with materials to manipulate, will not provide the benefits of thoughtful selection and employment – that must be associated with particular SCK that is linked to the use of manipulatives. (Askew, 2018; Swan & Marshall, 2010).

More recently, various researchers and mathematics educators have indicated the need for teachers to hold a strong and connected knowledge of mathematical structure. Siemon, Warren, Beswick, Faragher, Miller, Horne, Jazby, Breed, Clark, and Brady (2021) noted its importance if teachers were to make learning meaningful for children. Similarly, Waller and Marzocchi (2020), Parker (2019), and McKenna (2019) all provided examples of how string teacher content knowledge could enhance teaching and learning of mathematics at a conceptual level. Also, Clements and Sarama (2019) clearly allude to the importance of strong teacher knowledge in understanding children's developmental learning paths.

Multiplicative Thinking (MT)

Multiplicative thinking is fundamental to the development of important mathematical concepts and understandings such as algebraic reasoning (Brown & Quinn, 2006), place value, proportional reasoning, rates and ratios, measurement, and statistical sampling (Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006). According to Siemon, Breed, Dole, Izard, and Virgona (2006), students who are not proficient with multiplicative thinking lack the foundational skills and knowledge to participate effectively in further school mathematics, or to avail themselves of some post-compulsory school opportunities. Further, Siegler, Duncan , Davis-Kean, Duckworth, Claessens, Enge, Susperreguy, & Chen (2012) advocate that knowledge of division and of fractions (another part of mathematics very much reliant on multiplicative thinking) are unique predictors of later mathematical achievement.

Multiplicative thinking is significantly more complex than addition so is consequently not easy to teach or to learn (Barmby, Harries, Higgins & Suggate, 2009). Although most students will enter school with an informal knowledge that underpins counting and initial additive thinking (Sophian & Madrid, 2003) re-conceptualisation of understanding about number is necessary to understand multiplicative relationships (Wright, 2011). Multiplicative thinking is markedly different from additive thinking (Jacob & Willis, 2001) and needs to be understood as being much more than the capacity to remember and use multiplication facts. What is needed is a developing ability to apply these facts to the many and varied situations which are built on multiplication. Jacob and Willis (2003) outlined five broad phases for the development of multiplicative thinking: One-to-One Counting; Additive Composition; Many-to-One Counting; Multiplicative Relations; and Operating on the Operator. It is at the third phase, the development of many-to-one counting when students can simultaneously hold two numbers in their head; the number of groups and the total in each group.

Traditionally the approach has been to use links with repeated addition to facilitate students' multiplicative thinking (Confrey & Smith, 1995), an approach which may not be helpful in addressing the variety of situations to which multiplication needs to be applied (Wright, 2011). Behr, Harel, Post and Lesh (1994) posit that students often have the inappropriate additive model presented to them by teachers, who themselves use an additive model. It may be useful for teachers to be given the opportunity to 'see' multiplication as standard and non-standard multiplicative partitioning and to create situations where they are mindful of the way in which questions and tasks are phrased and are aware of the power of particular representations. A multiplicative array is a powerful representation which may allow this to occur and teachers require a strong level of SCK to facilitate that.

Multiplicative Arrays as a powerful mediating artefact for MT

Research (Barmby et al., 2009; Young-Loveridge & Mills, 2009) suggests that the array is a potent way in which to represent multiplication. Multiplicative arrays have the potential to allow students to visualise commutativity, associativity and distributivity, (Nunes & Bryant, 1996; Young-Loveridge, 2005) and that if employed alongside other representations, can serve to "allow students to develop a deeper and more flexible understanding of multiplication/division" (Young-Loveridge, 2005, p. 38-39). Indeed, they enable students to understand commutativity through rotating the array, and, by seeing that although the multiplicand and multiplier can be exchanged without effecting the product, the two situations (e.g., 4×3 and 3×4) are different. Further, multiplicative arrays exemplify the dualistic character of multiplication (Barmby et al., 2009), and have value in as a representation, that they also connect to the mathematical ideas of area and volume and Cartesian products (Wright, 2011).

The representation of multiplicative thinking using the array is not a trivial decision, and one which allows students to develop from being solely additive thinkers (Askew, 2018; Downton, 2008). While representations such as ten-frames are adept at helping children to develop additive reasoning, arrays have been the preferred representation in the development of multiplicative thinking (Siemon et al., 2015; Vale, & Davies, 2007; Young-Loveridge, 2005). According to Askew, (2018) additive problems and additive thinking (AT) are essentially problems involving three numbers, where all three numbers relate to the same referent. For example, there are five sheep in a pen, three more sheep are placed in the pen, so there are now eight sheep in the pen. Multiplicative Thinking (MT) problems have numbers that do not all share the same referents; there are 4 pens, there are three sheep in each pen, there is a total of 12 sheep. The four tells of how many iterations, or the number of pens, and the three tells of how many sheep in each pen. This mental manipulation of the multiplier and the multiplicand can be facilitated through the use of arrays which allows students the opportunity to both visualise and manipulate the concrete representation, before moving to pictorial and then an abstract representation (Askew, 2018; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; Sriraman, 2006). A further advantage of the array is that it allows for collinearity (Jacob & Mulligan, 2014), the capacity to focus students' attention on the three quantities, the coordination of; the number of rows, the number in each row (both factors), and the whole amount (the multiple), simultaneously (Jacob &

Mulligan, 2014; Siemon et al., 2015). Importantly the multiplicative array also has agency in allowing the opportunity to see multiplication as a structure rather than a procedure (Askew, 2018). Again, teachers need to hold a strong level of SCK for this to occur.

Conventions for naming arrays and equal groups

There is an accepted convention for naming arrays and for describing numbers of equal groups, and this is an essential component of teachers' SCK. In a multiplication fact, the first number indicates the number of groups and the second number indicates the number in each group. In an array, the first number indicates the number of rows and the second number indicates the number in each row (Jacob & Mulligan, 2014). Hurst and Hurrell (2018, p. 25) noted that it must be understood that "four groups of seven (4 × 7) is conceptually different to seven groups of four (7 × 4) but that the order of factors can be changed and the product will remain the same". Hurst (2018) also pointed out the importance of a conceptual understanding of ideas like the commutative property and how it underpins the later use of mathematical algorithms and procedures. Indeed, Hurst (2017, p. 3) stated that, "being able to understand and articulate that four groups of seven is quite a different multiplicative situation to seven groups of four, demonstrates more powerful knowledge".

To ascertain the extent to which this convention is accepted, a content analysis of 24 reputable sources was conducted. Sources included research articles, conference papers, teacher education textbooks, and publisher websites. Twenty of those sources, a selection of which is provided here, (e.g., Benson, Wall, & Malm, 2013; Stott, 2016; Tipps, Johnson, & Kennedy, 2011; Van de Walle, Karp, & Bay-Williams, 2013; Way, 2011) clearly adhered to the convention. For example, Tipps et al. (2011, p. 236) stated that, "The first factor is called the multiplication operator or multiplier because it acts on the second factor, the multiplicand, which names the number of objects in each set". In four other sources, there was some confusion and/or contradiction about the convention (Barmby, Harries, & Higgins, 2010; Cotton, 2016; Haylock, 2010; Reys et al., 2020). As an example of this contradiction, Cotton (2016) discussed how 4 × 3 could be shown and promptly showed an array with three rows of four. Later, he showed a three by eight array of postage stamps and correctly described it as a 3 × 8 array. While over 80% of the analysed sources adhere to the convention, it would be preferable if they all did in order that teachers in all jurisdictions received the same message that the convention is important.

The problem here is that not all of the sources used the convention for naming arrays. In the context of the overall research described earlier, a sub-problem emerges – To what extent do teachers adhere to the convention when naming and describing arrays?

METHOD

Theoretical Framework

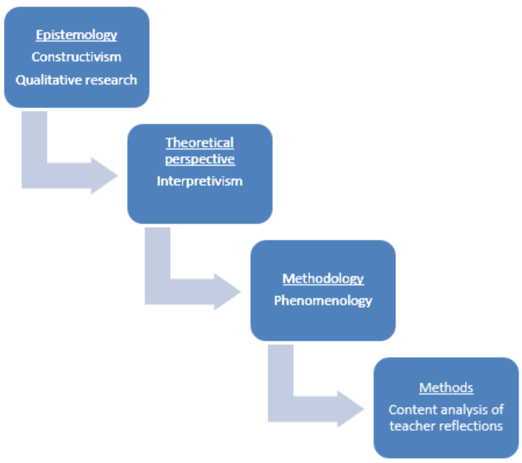

Figure 4. Theoretical framework

Research Design

The study utilised a qualitative method of investigation that follows a constructivist research paradigm and an interpretivist theoretical perspective, as outlined in Figure 4, which seeks to understand the primary importance of the meaning people attach to the world around them (Creswell, 2007). The qualitative methodology used content analysis of reflection sheets compiled by teacher participants.

Participants and Data Collection

Data were collected from 44 primary and middle school teachers from both Australian and New Zealand schools, and this was done through working with small focus groups. The participants were supplied with authentic work samples which had been collected through work with primary school aged students and were asked to make some determinations about those samples.

Participant teachers were presented with printed student work samples and asked to individually respond to two questions for each scenario:

a. What does each work sample tell you about the student's understanding of the mathematics involved?
b. What teaching strategies would you employ to help each student?

The first question to which teachers were asked to respond, and which is the focus of this paper, is shown in Figure 5.

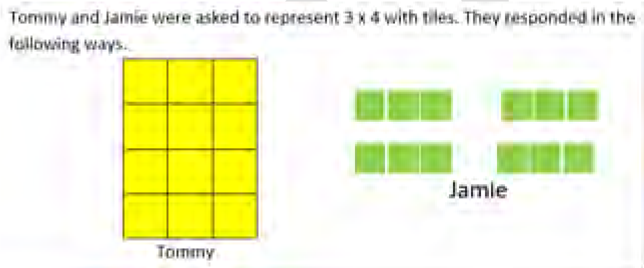

Figure 5. Question 1 from scenarios given to teachers

Fundamentally, the researchers were interested to determine which aspects of the identified SCK were evident in the participants' responses. Thus, there were two ways in which participants could demonstrate SCK – appropriately identifying the students' responses in terms of SCK, and identifying an appropriate intervention in terms of SCK.

After multiple readings of the scripts, the responses were categorised by one of the interviewers according to the aspects of SCK in the framework presented below. The responses were considered to be either:

- A valid or appropriate response indicating the presence of the specific SCK
- A vague or inappropriate response indicating partial presence of SCK, or lack of it

These categorisations were then given to the second interviewer and challenged until a final set of categories was established through consensus. This process required both researchers to agree on the intent of the written materials, both in what was written and what that writing was trying to convey. The process, although quite time consuming, proved to be instructive and challenging but remained systematic and thorough.

Framework for Data Analysis

The framework for this study is based on the premise of Specialised Content Knowledge (SCK) and the following five phases of development of multiplicative thinking proposed by Jacob & Willis (2003).

1. One-to-one counting – can count by one, but do not trust the count and do not count on

2. Additive composition – trust the count, do not count groups, and can count on.

3. Many-to-one counters – can hold two numbers (the number of groups and the number in each group) in their head and double count. They need to use arrays to move on. They do not understand the inverse relationship of the multiplicative situation, nor the commutative property. They understand the multiplicand but not the role of the multiplier.

4. Multiplicative relations – they know about the role of the multiplicand and multiplier and can coordinate the structure of grouping for both multiplication and division. They know which number tells them the number of groups and which one tells them the number in each group. They know whether to find five times the quantity or one fifth of the total. They are starting to use commutativity, inverse relationship, and part-whole understanding.

5. Operate on the operator – can work on variables in algebraic situations, operate on operators, and can flexibly use factors to make calculation of a multiplication sentence easier.

The first two phases related to counting and additive composition and were not considered relevant for this study and analysis. Also, the final phase relates to operating on variables and flexibly using factors, which are not appropriate for the specific mathematics involved in this sample. Essentially, Phases 3 and 4 focus on the transition from additive to multiplicative thinking on which the sample described above is based (see Figure 5).

Jacob and Willis stated that, "…unless teachers can actually recognise the difference between additive and multiplicative thinking, they would be unlikely to be able to help children develop the latter" (2003, p. 461). Clearly, they are alluding to the need for teachers to hold the necessary SCK in order for them to be able to assist their students to move beyond repeated addition. They need to recognise and use the array structure rather than continue to rely on the separate groups. Hence, the importance of teachers recognising the difference between the 'array response' and the 'group response' comes into focus and the need for teachers to hold strong SCK is emphasised. Jacob and Willis (2001) suggest that the power of the array lies in the fact that children can recognise the three aspects of the multiplicative situation – multiplier, multiplicand, and product – and that multiplicative thinking is about coordinating those three aspects in multiplication and division problems. Using two of the five phases outlined by Jacob and Willis (2001), the following framework was developed for the purpose of considering the data collected from the sample of teachers. The framework contains two aspects – first, the developmental phase of children, and second, the specialised content knowledge required of teachers to help children progress through each particular phase.

Table 1. Framework for data analysis

Children's developmental phase	Specialised content knowledge required by the teacher
Many-to-one counters – can hold two numbers (number of groups and number in each group) in their head and double count.	▪ 3 x 4 means 'three groups of four' because the convention is that the first factor indicates the 'number of groups' and the second factor indicates the 'number in each group'. ▪ The representation as separate groups likely indicates additive thinking and/or the use of repeated addition, and a lower level of thinking than is demonstrated by the use of the array.
Multiplicative relations – they know about the role of the multiplicand and multiplier and can coordinate the structure of grouping for both multiplication and division. They know which number tells them the number of groups and which one tells them the number in each group.	▪ The use of the array shows the two factors and the product as one entity. ▪ An array is read as the 'number of rows' multiplied by the 'number in each row'. Hence, a 3 x 4 array is read as '3 rows of 4'. ▪ The use of the array more likely indicates a level of multiplicative thinking. ▪ Rotating the array can be used to demonstrate the commutative property.

Developing Categories for Responses

The process of developing categories for participant responses took some time. Initially, it was thought important to identify what constituted a 'full response' to the questions about Tommy and Jamie and to see which participants were able to provide such a response. A 'full response' about Jamie's sample was considered to include the following points:

- Jamie should be asked to explain his thinking.
- Jamie needs to explain what 3 × 4 and 4 × 3 mean.
- Jamie needs to understand that 3 × 4 and 4 × 3 are different.
- Jamie needs to be exposed to arrays and the convention for naming them.
- Jamie needs to demonstrate an understanding of the convention for writing multiplication facts.
- Jamie needs assistance to understand the commutative property.

It soon became apparent that no participant gave such a complete response. It was then decided to classify responses as 'appropriate' or 'inappropriate', 'vague', or 'unclear/incoherent'. As the analysis continued, it was evident that participants frequently made assumptions about the work samples in terms of what they thought the samples showed about the knowledge of the two students. Even when challenged in conversation about whether their observations were assumptions or 'truths', few seemed to be able to concede that making the assumption could be erroneous. Through extensive interviews with children the researchers were abundantly aware that when such assumptions were made, they were almost always incorrect. For example, when students were questioned about the orientation of the array it was evident that it had been drawn with no understanding of the multiplicative situation or commutative property. Hence, responses where such assumptions could not be totally justified, were deemed to be inappropriate. Ultimately, it was decided that it would be better to focus on the percentage of appropriate responses based on the framework criteria (Table 1) and to include examples of inappropriate responses in the discussion.

RESULTS

The following results are presented in terms of the two phases of the framework – Table 1 shows results for identification of SCK about many to one counters and Table 2 shows results for identification of SCK about multiplicative relations. To provide some clarity about how the responses were categorised, sample responses which are typically appropriate, or inappropriate, are provided after Tables 2 and 3. Teacher respondents are identified with pseudonyms – R1to R44.

Table 2. Data summary relating to 'many to one counters'

Many-to-one counters	
(1) 3 x 4 means 'three groups of four' because the convention is that the first factor indicates the 'number of groups' and the second factor indicates the 'number in each group'.	
Part (a) Appropriate Identification 40.9%	Part (b) Appropriate Intervention 22.7%
(2) The representation as separate groups likely indicates additive thinking and/or the use of repeated addition, and a lower level of thinking than is demonstrated by the use of the array.	
Part (a) Appropriate Identification 11.4%	Part (b) Appropriate Intervention 2.3%

Typical appropriate responses about the SCK in Table 2 Criterion (1) included that 'Jamie has shown 4 x 3 and not 3 x 4 which he was asked to do' (R1) or that 'Jamie has made three, four times' (R2). Another participant responded with 'Tommy is linking to arrays but doesn't understand correct placement. He should have three rows and four in each' (R3). Inappropriate responses included that

'They have both created three groups of four' (R4) as well as making unjustified assumptions such as 'Tommy has shown the relationship between 3 × 4 and 4 × 3' (R5). Other inappropriate responses for Criterion (1) included 'They both understand grouping' (R6), or 'They both have an understanding of what 3 x 4 means' (R4). With regard to a possible intervention for Criterion (1), appropriate responses included 'I would get blocks and do hands on. [They need] visual representation of each one whilst writing the number sentences' (R7). Another response was 'Jamie needs careful reading of the question to adjust the groups of tiles. The first number is the number of groups and the second number is the number in each group' (R8). As well, another said that 'I would want to interview Tommy to find out why he arranged it that way' (R1). Finally, another said that 'Jamie needs to see 3 × 4 with tiles and 4 × 3 with tiles to see if he can see the difference' (R9). For Criterion (2), typical appropriate responses included that 'Jamie had laid the times out in a repeated addition format' (R10) or that 'Jamie is still thinking additively and possibly skip counting' (R11). Good suggested interventions included that Jamie needed to be introduced to arrays (R12) and that Tommy would benefit from using strips of three and four to show groups and overlaying them to form the arrays (R13).

Table 3. Results identifying aspects of multiplicative relations

Multiplicative relations	
(1) The use of the array shows the two factors and the product as one entity.	
Part (a)	Part (b)
Appropriate Identification	Appropriate Intervention
0%	0%
(2) An array is read as the 'number of rows' multiplied by the 'number in each row'. Hence, a 3 x 4 array is read as '3 rows of 4'.	
Part (a)	Part (b)
Appropriate Identification	Appropriate Intervention
6.8%	9.1%
(3) The use of the array more likely indicates a level of multiplicative thinking.	
Part (a)	Part (b)
Appropriate Identification	Appropriate Intervention
6.8%	2.3%
(4) Rotating the array can be used to demonstrate the commutative property.	
Part (a)	Part (b)
Appropriate Identification	Appropriate Intervention
2.3%	9.1%

There were few responses related to Table 3 Criterion (1) apart from 'Tommy has shown the total or product as one entity' (R14) (appropriate) and 'Tommy represents the answer, not the question' (R15) (inappropriate). For Table 3 Criterion (2), appropriate responses included 'Tommy has shown as an array but unsure if it shows 4 x 3 or 3 x 4' (R9) and 'Tommy is linking to arrays but doesn't understand correct placement' (R3). Inappropriate responses included that 'Tommy has laid out the tiles in a 3 x 4 pattern' (R10). With regard to a possible intervention, an appropriate response was 'Ask children to share what they have made. Discuss how many groups they have created. Discuss what 3 x 4 means' (R16), and 'Ask Tommy – 'You have the right amount but how many groups are there in 3 x 4?' (R15) For Table 3 Criterion 3, an appropriate response was that 'Tommy is thinking in a multiplicative way and has used an array' (R18), while an appropriate response for an intervention would be 'Ask Jamie if he can show it as an array' (R15).

Table 3 Criterion 4 produced by far the highest number of inappropriate responses. Only one appropriate response was recorded – 'It would be easy to assume that Tommy perhaps sees the relationship to 4 x 3 in his array, however, my experience of making this assumption in the past has shown that this is often not a connection necessarily made' (R16). Inappropriate responses generally assumed that the commutative property was understood because Tommy had not shown the array as a 3 x 4 but in showing it as a 4 x 3, it indicated that he understood commutativity. For example, 'Tommy shows perhaps greater awareness of commutative property of multiplication' (R17). However, there were a number of appropriate suggestions for interventions including, 'Is this (3 x 4) the same as 4 x 3 (i.e., 3 groups of 4 and 4 groups of 3)?' (R19), and, 'See if Tommy understands commutativity and is able to show as [rotated array drawn]' (R5).

Inappropriate responses for Table 3 included that 'Tommy has only put the total in and he needs separation to show groups' (R20) and 'Further investigation may be needed to see if they can be represented in other ways. If not, further teaching may be needed' (R21). The latter comment is not related to any of the specific SCK criteria and is therefore considered vague or inappropriate.

It is worth considering the range of individual results across the sample of 44 participant teachers as there was a great variation in the responses from person to person. The framework shown in Tables 1, 2, and 3 indicates six aspects of SCK that teachers could identify and about which they could respond. With regard to identifying the SCK contained in each criterion (i.e., the question asking 'What does each work sample tell you about the student's understanding of the mathematics involved?'), many participants gave appropriate responses alongside inappropriate, vague or unclear/incoherent responses. However, a large proportion (43.1%) provided no appropriate responses and 45.5% provided only one appropriate response to the six criteria. No participant recorded a majority of appropriate responses and 11.4% provided two appropriate responses. A total of 30 appropriate identifications/responses were given across the whole sample.

With regard to using the SCK for each criterion to suggest an appropriate intervention (i.e., the question asking 'What teaching strategies would you employ to help each student?'), there was a total of 20 appropriate interventions across the whole sample. Ten (10) of these related to the first of the 'many-to-one counters' criteria shown in Table 2, and there were four (4) appropriate interventions suggested for each of Criteria 2 and 4 about 'multiplicative relations' in Table 3. In all, 65.9% of participants were unable to suggest an appropriate intervention for any criteria, and 25.0% suggested one (1) appropriate intervention. A further 6.8% suggested two (2) appropriate interventions while 2.3% (one participant suggested three (3) appropriate interventions.

DISCUSSION and CONCLUSION

Numerous educators have continued to write about the importance of deep teacher knowledge of mathematical structure. In discussing pedagogical content knowledge, Siemon, Warren, Beswick, Faragher, Miller, Horne, Jazby, Breed, Clark, and Brady (2021), noted the importance of teachers having a strong understanding of structure in order for them to identify the best way/s of explaining a concept so that their students find it meaningful. Clements and Sarama (2019, p. 44) alluded to the importance of teacher knowledge, especially of mathematical connections, in discussing teacher awareness of appropriate learning trajectories, stating that, "As teachers come to understand children's probable developmental paths and become adept at anticipating children's strategies and misconceptions, their teaching practices may become more grounded and solidified". McKenna (2019) discussed children learning about factors through exploring, rather than simply being 'taught'. Clearly, if such an approach is to be successful, teachers need a deep and broad understanding of the structure about factors and multiples, a vital aspect of multiplicative thinking.

This is echoed by Waller & Marzocchi (2020) who assert that if children are to develop as deep and flexible thinkers, then the mathematical content knowledge held by their teachers needs to be strong and connected. Parker (2019) makes a similar point when discussing how children need to learn multiplication facts with conceptual understanding. For that to occur, teachers need a strong knowledge of multiplication and division. Hurst, Hurrell, & Huntley (2021) discussed children's fragmented understanding of factors and multiples. Their observations suggest that such fragmented knowledge is possibly due to the way in which factors are taught which is likely to have its origins in incomplete or unconnected teacher content knowledge.

The study clearly has limitations and it would not be appropriate, given the sample size (n = 44), to attempt to generalise widely from the results of this study. Nonetheless, it appears that there needs to be further research into teachers' SCK about aspects of multiplicative thinking. Even when given that the sample is small, some worrying indicators have emerged. First, the high proportion of participants who were unable to identify the specific mathematics nor suggest an appropriate intervention is of concern. Second, a number of participants made inappropriate assumptions about what the samples demonstrated. This was particularly evident in responses to the 'multiplicative relations' criterion about the commutative property where participants said that Tommy's sample indicated that he understood the property. It is just as likely that he does not understand commutativity nor the convention for writing number facts, as in 'many-to-one counters' criterion 1. Third, the high number of inappropriate responses that reflect a confused understanding of the mathematics needs further investigation. Finally, there seems to be no 'down-side' to adopting a consistent convention for naming arrays and describing equal group problems. In fact, research such as this would suggest that consistency is advisable and desirable.

Even considering the limitations in terms of the sample size and the relatively narrow focus of the study, there are some implications that are evident. First, the results suggest that further research should be undertaken with a larger sample and a broader content focus about multiplicative thinking. Second, with regard to teacher professional learning and pre-service teacher education, the results suggest that closer attention may need to be paid to Specialised Content Knowledge (SCK), specifically the fine points of mathematical structure about multiplicative thinking. In particular, the convention of describing number facts in terms of numbers of groups and the number in each group is important and forms the basis for understanding commutativity rather than considering it in a procedural way in terms of 'swaps and switches' (Anthony & Walshaw, 2002). Teachers need to better understand the structure if they are going to be in a position to assist their students to do so.

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