

Problem Solving With the Pythagorean Theorem: A Think Aloud Analysis of Secondary Students With Learning Disabilities

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While school performance suggests students with learning disabilities require intervention to demonstrate mathematics proficiency, little is known about how they approach problem solving in secondary geometry. Think aloud protocols highlight the higher order thinking skills underlying complex academic tasks. We conducted an exploratory descriptive analysis exploring the cognitive and metacognitive processes underlying problem solving with the Pythagorean Theorem in secondary students with and without learning disabilities (n = 8). Using an established cognitive-metacognitive theoretical framework, we coded and analyzed student verbalizations during a think aloud protocol. Results indicated that, compared to general education students, students with learning disabilities made shorter and fewer verbalizations advancing toward accurate problem solutions and longer and more frequent verbalizations hindering successful problem solving. Findings and implications for research and practice are discussed.

Keywords: **learning disability, geometry, secondary, problem solving, think aloud analysis**

Introduction

Since the 1975 implementation of the Education for All Handicapped Children Act (most recently reauthorized as the Individuals with Disabilities Education Act [IDEA] in 2004), high school graduation rates for students with disabilities have grown but remain lower than those for students without disabilities (Zaff et al., 2017). In 2017, 29% of the 430,000 students ages 14-21 receiving special education services, or 124,700 students, did not graduate high school with a regular diploma (National Center for Education Statistics [NCES], 2017). Life-long challenges are associated with a lack of a high school diploma, namely lower lifetime earnings, higher rates of incarceration, and poor health outcomes, thereby perpetuating cycles of poverty (Lansford et al., 2016). Addressing this low graduation rate for students with disabilities is incumbent on educators and researchers working with students with disabilities (Zaff et al., 2017). Given that the majority of students with disabilities (34%) receive services for a specific learning disability (SLD), it is clear that targeting academic proficiency for secondary students with SLD is an urgent matter for the field of special education and for society (NCES, 2017).

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Geometry and Students with Specific Learning Disabilities

Graduation requirements in all states include a minimum of two credits of mathematics, of which the first is typically algebra and the second is geometry (Common Core State Standards Initiative, 2010; NCES, 2017). Geometry is a mathematical science concerned with interpretation of space. Identified as a fundamental course in mathematics learning, geometry is also necessary in the preparation of students for post-secondary learning in science, technology, mathematics, and engineering fields (Common Core State Standards Initiative, 2010; National Mathematics Advisory Panel, 2008). Proficiency in geometry often hinges on successful word problem-solving, which, itself, depends on a range of cognitive and metacognitive components, such as visualization, problem representation, working memory, and self-regulation (Common Core State Standards Initiative, 2010; Peng et al., 2018). Furthermore, successful geometry problem solving reflects the cumulative nature of mathematical learning in that it is also dependent on algebraic proficiency (Common Core State Standards Initiative, 2010).

Given its complexity, successful problem solving in geometry poses a challenge for students with SLD. Scores from the 2017 administration of the National Assessment of Educational Progress (NAEP) indicate that only 8% of twelfth graders with disabilities scored at or above *Proficient* in mathematics (NCES, 2017). Students with SLD typically benefit from support in domain-specific and -general areas that can impede students' ability to integrate conceptual knowledge and skills to problem solve (Jitendra et al., 2016). In addition to coping with significant cognitive load, students with SLD contend with poor development of metacognition, further impeding successful problem solving (Montague, 2007; Montague & Applegate, 1993a). Situating current understandings of the impact of SLD on students' ability to successfully problem solve within the current rates of high school completion for students with disabilities highlights the need for research into the effectiveness of geometry instruction for high school students with SLD.

Existing Research on Geometry Instruction for Secondary Students with SLD

The existing literature on geometry instruction for secondary students with SLD comprises a small fraction of the larger body of literature on effective mathematics instruction for students with SLD. The current research base in secondary geometry instruction for this population of learners consists largely of investigations on the effectiveness of video modeling and manipulatives to improve outcomes on basic topics in secondary mathematics curricula, namely area and perimeter (Bergstrom & Zhang, 2016). This is clearly an area in need of expansion to more complex geometric concepts and their application in problem solving. Satsangi et al. (2019) established a functional relation between the implementation of video modeling and percent accuracy in geometry word problem-solving on perimeter and area in three secondary students with SLD. These findings expand on those obtained by Satsangi et al. (2018) in their investigation of explicit instruction and video modeling with three secondary students with SLD, in which both conditions enhanced students' problem-solving accuracy on area and perimeter word problems. Together, these two studies built on Cihak and Bowlin's (2009) study establishing a functional relation between video modeling and problem solving accuracy on area and perimeter word problems among three secondary students with SLD.

Two studies investigated the effectiveness of manipulatives in enhancing geometry problem solving on area and perimeter word problems. Satsangi and Bouck (2015) found that the use of virtual manipulatives, specifically polynomial tiles, improved problem-solving accuracy for three students with SLD in high school. Earlier findings by Cass and colleagues (2003) indicated that three secondary students with LD demonstrated gains in problem solving accuracy with the use of *GeoBoard* manipulatives. Thus, the existing literature on effective geometry instruction for secondary students with SLD focuses largely on area and perimeter, basic topics that are rarely assessed in isolation in standards-based geometry curricula at the secondary level. Little is known about how secondary students with SLD approach and can improve their performance in problem solving on advanced topics more commonly featured in standards-based geometry curricula. Clearly, the research base is underdeveloped as it is related to geometry problem solving for secondary students with SLD.

Think Aloud Analyses of Students' Problem-Solving Ability

Successful problem solving involves the integration of multiple mental processes that may not be fully evident by examining student solutions (Rosenzweig et al., 2011). The think aloud protocol provides one way to capture a more complete record of the multifaceted thinking processes in problem solving (Veenman & Spaans, 2005). In the think aloud protocol, problem solvers verbalize their thoughts as they occur during the problem-solving task, providing insight into the cognitive and metacognitive aspects of problem solving. As a result, students' problem-solving behavior is observed as it is executed. For students with SLD, for whom working memory is a challenge, the observational immediacy of the think aloud protocol bypasses the challenges presented by diminished working memory capacity (Young, 2000). Moreover, during a think aloud protocol, students' learning performance is not affected (Bannert & Mengelkamp, 2008). Students typically encounter the think aloud process in explicit instruction, which commonly features teacher think alouds in the descriptional component of modeling (Hughes et al., 2017; Rosenzweig et al., 2011).

Montague and Applegate (1993b) proposed the cognitive-metacognitive model of mathematical problem-solving to categorize think aloud verbalizations during problem solving as cognitive or metacognitive in nature (see Table 1). Cognitive verbalizations are further defined by the specific problem-solving strategy they encompass: reading, paraphrasing, visualizing, hypothesizing, estimating, computing, and checking. Metacognitive verbalizations refer to productive and nonproductive processes that either propel a student towards the problem solution by compelling them to use their prior experience and knowledge or reflect distractive thoughts about the problem-solving process and their success. There are four productive metacognitive (PM) verbalization subtypes: self-correction, self-instruction, self-monitoring, and self-questioning. In contrast, nonproductive metacognitive (NPM) verbalizations consist of two subtypes, comment and affect, that refer to negative self-talk about ability, knowledge, and emotional state. Thus, in the cognitive-metacognitive model of mathematical problem-solving, cognitive and PM verbalizations move students towards a problem solution, and NPM verbalizations stymie successful problem-solving efforts (Montague & Applegate, 1993b).

Table 1. ***Types of Verbalizations***

Type	Subtype	Description	Example
Cognitive	Reading	Reads the problem in full	[Student reads aloud problem.]
	Paraphrasing	Restates problem situation	"So, it's this triangle and I have to find this side." "It tells me the legs are 9 and the picture here shows that."
	Visualizing	Labels, points to, and creates diagrams	[Not captured by audio-recordings of think aloud task.]
	Hypothesizing	Identifies a goal, develops a plan	"I'm going to use the formula for the legs and hypotenuse." "I want to find this missing side, and I think I've got to use the other side to find it."
	Estimating	Predicts a numerical answer	"I think this leg is going to be close to the other leg." "I think I can label the side here the same as this other side because they look similar."
	Computing	Substitutes and evaluates	"I squared the 2 and then I added." "I multiplied the 2 and 4."
	Checking	Ensures answer is reasonable, all pertinent information has been used, computations are accurate	"I'm going to check it on the calculator." "I think I found the right side because the number is close to 4."

Table 1. *Types of Verbalizations (continued)*

Type		Subtype	Description	Example
Metacognitive	Productive	Self-Correction	Corrects errors	"Oh wait, I have to square this first." "I think I did this wrong, I want to do it over again."
		Self-Instruction	Gives directions to self	"First, I'm going to graph the points." "I should do the rounding, now."
		Self-Monitoring	Monitors pacing and progress	"Is that right?" "This is taking a long time, do I have time?"
		Self-Questioning	Clarifies strategy knowledge and use	"That doesn't make sense." "I'm trying to remember the steps from when I had to do this in my old class."
	Nonproductive	Comment	States perceptions of content, task, ability	"We were never taught this." "Oh, this seems really hard."
		Affect	States emotional disposition	"I'm just not good at math." "I usually have a hard time with these kinds of problems."

It is fair to conclude that think aloud analyses, during which students' verbalizations of their thoughts and actions are recorded, transcribed, and analyzed, are useful in understanding how students approach problem solving. While think aloud analyses comprise an important segment of the published literature in mathematics education research, to date, few have been published in the area of secondary mathematics for students with SLD. A review of the existing literature reveals that there exist two studies targeting this demographic and content area. Montague and Applegate (1993b) provided diverse groups of eighth grade learners (gifted [$n = 28$], average achieving [$n = 25$], students with SLD [$n = 28$]) with training in the think aloud protocol to study their strategy usage and self-regulation abilities. Findings indicated that, as problem complexity increased, students with SLD made fewer cognitive and metacognitive verbalizations than their average-achieving and gifted peers. Moreover, Montague and Applegate (1993b) found that students with SLD relied more heavily on cognitive verbalizations pertaining to reading and computing than other verbalization types and were more likely than their average achieving and gifted peers to perceive problems as being difficult. The researchers concluded that students with SLD were less equipped with metacognitive skills and cognitively "shut down" when presented with a problem they perceived as being challenging (Montague & Applegate, 1993b).

Rosenzweig and colleagues (2011) conducted a similar think aloud analysis with eighth grade learners of diverse abilities (average achieving [$n = 25$], low-achieving [$n = 34$], students with SLD [$n = 14$]). Participants received think-aloud training before solving three problems of varying complexity. Results of a 3 x 2 x 3 factorial ANOVA indicated that, as problem complexity increased, all participants made more metacognitive verbalizations. However, when faced with more challenging problems, students with SLD made more nonproductive metacognitive verbalizations, thereby stymieing successful problem solving efforts, than their low- or average-achieving peers. Rosenzweig and colleagues (2011) concluded that successful problem-solving is predicated on the activation of metacognitive strategies that are rooted in cognitive skills. For students with SLD, who typically have poorly developed cognitive skills, intervention targeting both metacognitive and cognitive skill sets is needed to positively impact their problem-solving abilities (Rosenzweig et al., 2011).

Data from think aloud analyses can be harnessed to differentiate and intensify instruction for individual students' needs (Rosenzweig et al., 2011). To date, think aloud analyses of the application of concepts in geometry with students with disabilities has not been explored at the secondary level. Such a think aloud analysis would constitute an important and instructionally useful first step in improving outcomes in mathematics for secondary students with disabilities.

Purpose and Research Questions

Given the complex nexus of cognitive and metacognitive processes underlying successful problem solving, the purpose of the present study is to contribute to the research base on effective mathematics instruction for secondary students with SLD by implementing a think aloud analysis with students as they engaged in problem solving in geometry. A think aloud analysis was chosen to obtain a deeper understanding of the specific cognitive and metacognitive thought processes students with SLD undergo through an analysis of their verbalizations during problem solving in important topics in geometry. One such topic is the Pythagorean Theorem, which involves the side lengths of right triangles (Common Core State Standards Initiative, 2010). In Common Core mathematics curricula, the Pythagorean Theorem is a fundamental topic upon which subsequent topics in secondary mathematics curricula are predicated and is first introduced to students in the eighth grade Common Core mathematics curriculum. Given the importance of the Pythagorean Theorem in students' ability to successfully navigate more advanced topics in geometry, deeper insight into students' problem-solving approaches within this topic are needed. As such, the present study was designed as an exploratory study to obtain a more detailed understanding of students' cognitive and metacognitive thought processes during problem solving with application of the Pythagorean Theorem.

The following research questions guided the present study:

1. Are there differences in cognitive and metacognitive verbalizations between students with and without learning disabilities as measured by (a) the duration of verbalizations; and (b) the frequency of types of verbalizations (cognitive vs. metacognitive, types of metacognitive)?

2. Do students' verbalization types vary as a function of (a) problem-solving accuracy or (b) problem-solving complexity?

We hypothesized that students with SLD would make fewer and shorter helpful verbalizations (cognitive and PM verbalizations) and more and longer verbalizations that do not (NPM verbalizations) than their general education peers. We also hypothesized that the frequency and duration of all types of verbalizations would increase with problem-solving accuracy and decrease with problem complexity. Our hypotheses were informed by the findings obtained by Montague and Applegate (1993b) pertaining to the cognitive "shut down" they observed in students with SLD when presented with challenging problems.

Method

The present study was a qualitative descriptive analysis of student think alouds as students with and without SLD solved geometry word problems of varying complexity involving the application of the Pythagorean Theorem. A qualitative approach was chosen to capture, to the best extent possible, the cognitive and metacognitive processes underlying students' problem-solving efforts. From a qualitative approach, the think aloud protocol was appropriate for the aims of the current study in that it was applied to a language-based activity of intermediate difficulty. As a result, students were required to provide more than an automatic response but were not presented with a cognitively overwhelming task (Charters, 2003).

Participants and Setting

Following university Institutional Review Board (IRB) approval, participants were recruited from a socio-linguistically diverse public high school just outside of a major northeastern city.

Table 2. ***Participant Characteristics***

Matched Pair	Participant	Sex	Age	Grade	Ethnicity	Disability Status	Scaled Score on NY Common Core Algebra Regents Examination	Free-Reduced Price Lunch
1	1	F	15	11	Latinx	SLD	75	Reduced
	2	M	15	10	Latinx	none	75	Free
2	3	M	15	10	Latinx	SLD	77	Free
	4	F	15	10	Latinx	none	77	Free
3	5	F	15	10	Latinx	SLD	68	Free
	6	F	15	10	Latinx	none	68	Free
4	7	M	17	11	White	SLD	71	None
	8	M	15	10	Latinx	none	71	Free

Note. SLD = Specific Learning Disability

Table 2 provides detailed descriptions of the participants. Informed consent forms were distributed to 176 students (n = 176) with and without disabilities and enrolled in geometry courses in the school. A total of 32 students (n = 32; 18%) with and without disabilities provided written consent to participate in the study. Of this pool of available students, eight participants in grades ten and eleven were selected to create four matched pairs of participants (n = 8). Each matched pair was created to include a student receiving special education services for a specific learning disability (as determined by the school district) and a general education student who had earned an identical scaled score on the New York State Common Core Algebra Regents Exam, a standardized assessment which all students who had provided consent had taken. Scores for all participants were passing and in the third performance level, defined as "partially" meeting Common Core expectations (New York State Education Department, 2015). All participants were enrolled in remedial Common Core geometry courses taught by approximately six different teachers. Although six different teachers taught this course, a common curriculum and scope and sequence were followed. At the time of the study, all teachers were within two days of the scope and sequence outlined in the curriculum. Information regarding participants characteristics were provided by the participating school district. Additional information regarding students' specific disability diagnoses and learning difficulties was not available.

All eight participants demonstrated fluency in English as determined by the school district, were enrolled in a cotaught inclusion geometry course with instruction in English, and had provided parental consent and student assent to participate in the study. Participants were excluded if they were receiving special education services for a disability other than a specific learning disability, did not provide parental consent or student assent to participate, or were receiving mathematics instruction in Spanish. Additional information regarding participants' SLD type was not available.

Procedure

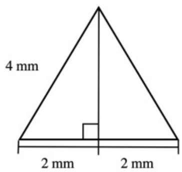
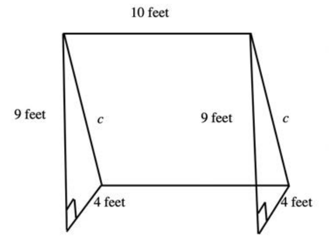
The study consisted of two parts: a video modeling session and a think aloud task. The first author, a certified special education teacher with a total of eight years of teaching experience, four of which were teaching secondary mathematics in inclusion settings, was the lead researcher for the current study and will be referred to as "the researcher" henceforth. Participants who provided parental consent and student assent met individually with the researcher to complete both parts of the study procedure. Study sessions took place in a small office in the school building during students' geometry class. The researcher sat next to students at a small round table throughout the study procedure.

First, participants viewed a brief video (3.5 minutes) on an iPad® during which the researcher modeled a think aloud task while solving a word problem on finding the area of a triangle, given a diagram with its altitude and base length. The researcher modeled the following typical steps in solving the problem: read the problem, identify pertinent information, label the diagram, write the correct formula for the area of a triangle, substitute, evaluate, and check the solution to ensure it answered the problem and was reasonable. The main purpose of the video think aloud was twofold, the first of which was to prime students for the upcoming think aloud task by highlighting the expectations for explaining their reasoning during problem solving. Secondly, we wanted to ensure a sufficient volume of meaningful data for our analysis and minimise the occurrence of unusable verbalizations (such as "I don't know"). The think aloud video was not designed nor intended to serve as an intervention.

Next, participants were presented with the think aloud task, which consisted of four word problems involving the application of the Pythagorean Theorem. Based on the type and progression of problems in the mathematics curriculum used in the EngageNY Mathematics Curriculum (EngageNY Module 7, Topic C, Lessons 15-18), two increasingly complex levels of word problems were developed by the researcher. The modified researcher-created problems were then submitted for review to the district's Secondary Mathematics Supervisor (who had 21 years of teaching experience) and to a dual-certified secondary mathematics special education teacher (who had seven years of teaching experience) in a neighboring district. Both individuals, neither of whom were not participating in the study, approved the problems as being appropriate for the think aloud task. Moderate level word problems required students to find the missing side or hypotenuse in a right triangle with an accompanying diagram. Advanced level word problems required students to find the missing side or hypotenuse in a right triangle without an accompanying diagram. Students were able to use a calculator with the radical function during the think aloud task. The task was presented on paper and untimed. Table 3 contains examples of the moderate and advanced problem types.

Before participants viewed the video modelling of the think aloud process, the researcher explained that they should pay attention to the general problem solving steps taken to solve the problem and how the speaker explained them verbally. After participants viewed the video, the researcher explained that they would be asked to do a similar think aloud as they solved problems on a different topic, application of the Pythagorean Theorem. The researcher emphasized that discussing and explaining the problem solving steps they were undertaking - rather than the accurate numerical solution - was the focus of the think aloud experiences. She explained that the expectation was that they do their best to "speak out loud" the steps they were taking, the strategies they were thinking of and using, and the questions they had as they solved the problems. Participants were presented with a note card with the following prompts to use during the think aloud task: "I know that . .", " I'm trying to figure out. . .", "One thing I can try is . . .", and "I want to try. . . because. . ." During the think aloud task, the researcher did not provide information or answers to participants' questions about the mathematical content in the word problems. Instead, she reminded participants to explain what they were thinking or writing if they were silent for more than 5 consecutive seconds and offered encouragement to continue as participants worked.

Table 3. ***Think Aloud Task***

Complexity Level	Word Problems	
Moderate	Determine the height, *h*, in mm, of the equilateral triangle below. Round your answer to the nearest tenths place. 	The diagram below represents a soccer goal. Determine the length of the bar, *c*. Round your answer to the nearest tenths place. 
Advanced	What is the distance between points (2, 9) and (-2, 1)? Round your answer to the nearest tenths place.	Find the side length of a square with a diagonal of 3.

Think Aloud Task Coding and Accuracy Scoring

Student verbalizations were defined and categorized based on earlier think aloud analyses of middle school students' problem-solving efforts (Montague & Applegate, 1993b; Rosenzweig et al., 2011) (see Table 1). The think aloud tasks were audio recorded and transcribed by the researcher. Recordings were collected using the Voice Memos application on a MacBook Pro® laptop. Transcriptions were completed by listening to the recordings and using a word processor. Each participant's think aloud recording was transcribed twice and included time stamps.

Transcriptions were coded twice according to verbalization type (cognitive, PM, and NPM) and duration. Frequencies of specific verbalization types were obtained by tallying the frequency with which they occurred in transcripts of individual participants' think aloud recordings. Data on the duration of verbalization subtypes was obtained by using a stopwatch to time the length of specific verbalization types in individual participants' think aloud recordings. A scoring sheet was developed and used to record frequency and duration of each verbalization subtype. Coding was completed by simultaneously reading the transcripts and listening to the audio recordings.

To obtain data on solution accuracy, students' work was scored. A rubric with four components was developed based on scoring guides for the New York State Common Core Geometry Regents Examination. The scoring rubric was then reviewed by the Secondary Mathematics Supervisor for review and approval. Each component was scored on a three point Likert scale: incorrect or blank response (0), partial accuracy (1), or complete accuracy (2). Three of the four components were identical for both moderate and advanced levels of problems: correct equation set up for the Pythagorean Theorem, correct substitution and evaluation, and correct numerical solution. The first component differed for each problem type: accurate labeling of the accompanying diagram (moderate level) and accurate drawing of diagram (advanced level). A maximum of eight points was available for each problem, for a total of 32 possible points in the think aloud task.

Interrater reliability (IRR) data were calculated for coding and problem accuracy. To obtain IRR on coding the verbalizations, a research assistant blind to the purpose of the study received a two hour training on identifying the types of verbalizations. Then, they independently coded 100% of the recordings and transcriptions. A nearly identical process was conducted to obtain IRR for problem accuracy. The same research assistant received a thirty minute training on using the researcher-developed rubric to score problems similar to those in the think aloud task. Then, they independently scored 100% of the participants' responses in the think aloud task. IRR was by calculated by dividing the total number of agreements by the total number of agreements and disagreements and then multiplying by 100. IRR was 100% for problem accuracy. Initial IRR was 93% for coding students' verbalizations. The researcher and research assistant discussed and resolved all disagreements. Subsequent IRR was 100% for coding student verbalizations.

Data Analysis

Using the cognitive-metacognitive model of mathematical problem-solving developed by Montague and Applegate (1993b), we coded the types of verbalizations made by both students with disabilities and those without disabilities. In order to answer the first research question regarding the difference in cognitive and metacognitive verbalizations between students with and without disabilities, we calculated frequency and duration of students' verbalizations. To answer the second research question, we examined types of verbalizations made by both groups of students as a function of problem complexity and solution accuracy.

Results

Results from the think aloud analysis are organized in relation to the duration and frequencies of verbalizations according to problem accuracy, type (complexity), and disability status. Table 1 provides specific student responses during the think aloud task. Tables 4, 5, and 6, provide detailed results, which are discussed in the following sections. Overall, results support our hypothesis that students with SLD made less frequent and shorter helpful verbalizations (cognitive and PM) and more frequent and longer NPM verbalizations.

Table 4. *Solution Accuracy*

Problem		Scores for Students with SLD				Scores for Students without SLD			
		S1	S3	S5	S7	S2	S4	S6	S8
Moderate	Problem #1	0	0	0	0	0	0	0	2
	Problem #2	0	0	0	0	0	0	0	2
Advanced	Problem #3	0	0	0	0	0	0	0	0
	Problem #4	0	0	0	0	0	0	0	0
Total		0	0	0	0	0	0	0	4

Note. Scoring scale: accurate (2), partially accurate (1), and an incorrect or blank response (0); SLD = Specific Learning Disability.

Table 5. ***Frequency and Duration of Student Verbalizations***

Problem Type	Verbalizations		Students with SLD					Students without SLD				
			F	% F	Total Time (s)	Average Time (*s*)	St. Dev.	F	% F	Total Time (s)	Average Time (*s*)	St. Dev.
Moderate	Cognitive		19	67.9	145	7.6	7.0	30	73.2	221	7.4	6.7
	Metacognitive	Productive	6	21.4	25	4.2	1.7	8	19.5	30	3.8	1.4
		Nonproductive	3	10.7	11	3.7	0.6	3	7.3	6	2.0	1.0
		Subtotal	9	32.1	36	4.0	1.4	11	26.8	36	3.3	1.2
	Total		28	100	181	6.5	6.0	41	100	257	6.3	5.9
Advanced	Cognitive		13	50.0	78	6.0	4.4	22	62.9	130	5.9	5.9
	Metacognitive	Productive	4	15.4	23	5.8	3.9	6	17.1	17	2.8	0.8
		Nonproductive	9	34.6	31	3.4	1.0	7	20.0	27	3.9	1.3
		Subtotal	13	50.0	54	4.2	2.4	13	37.1	44	3.4	1.2
	Total		26	100	132	5.1	3.6	35	100	174	5.0	4.6
Both	Cognitive		32	59.3	233	7.3	6.1	52	68.4	351	6.8	6.0
	Metacognitive	Productive	10	18.5	48	4.8	1.0	14	18.4	47	3.4	1.1
		Nonproductive	12	22.2	42	3.5	1.2	10	13.2	33	3.3	0.7
		Subtotal	22	40.7	90	4.1	7.0	24	31.6	80	3.3	0.9
	Total		54	100	313	5.8	7.5	76	100	431	5.7	7.4

Note. F = frequency; % F = (100%)(F of verbalizations/total F of verbalizations for given problem type); SLD = Specific Learning Disability; St. Dev. = standard deviation

Table 6. ***Frequency and Duration of Subtypes of Verbalizations***

Verbalization Type			Students with SLD		Students without SLD		Total	
			F	Time (s)	F	Time (s)	F	Time (s)
Cognitive		Reading	6	16	9	47	15	63
		Paraphrasing	2	6	2	4	4	10
		Visualizing	9	116	18	173	27	184
		Hypothesizing	0	0	4	15	4	15
		Estimating	1	2	2	5	3	7
		Computing	12	83	14	95	26	178
		Checking	2	10	3	12	5	22
		Subtotal	32	233	52	351	84	584
Metacognitive	Productive	Self-Correction	3	7	2	11	5	18
		Self-instruction	3	14	4	13	7	27
		Self-Monitoring	2	18	4	9	6	27
		Self-Questioning	2	9	4	14	6	23
	Nonproductive	Comment	10	33	7	23	17	56
		Affect	2	9	3	10	5	19
	Subtotal		22	90	24	80	46	170
Total			54	313	76	431	130	744

Note. F = frequency; SLD = Specific Learning Disability

Verbalizations Across Participant Groups

To answer the first research question related to differences in cognitive and metacognitive verbalizations between students with and without disabilities, we analyzed the frequency and duration of verbalizations for each participant group. Detailed results of the frequencies of verbalization are presented first, followed by results for duration of verbalizations.

Frequency

As shown in Tables 5 and 6, across both problem types, students with SLD made fewer verbalizations of any kind compared to students without SLD (54 verbalizations vs. 76 verbalizations). Comparing the frequency of verbalizations made by individual students within each matched pair, we found that, with the exception of one pair (Pair 3 with S5 and S6), the students with SLD made fewer verbalizations than their general education peers. Students in Pair 3 each made 17 total verbalizations.

Results of the frequency of different types of verbalizations made by students across both problem types indicate that students with SLD made fewer cognitive and PM verbalizations than students without SLD. Compared to their general education peers, students with SLD also made more NPM verbalizations across both problem types. These results are also evident in the proportions of verbalization types students made. Both students with SLD and without SLD spent the greatest percentage of their time in the think aloud task making cognitive verbalizations (59% and 68%, respectively). However, students with SLD spent a greater proportion of their time making NPM verbalizations, compared to their general education peers (22% vs. 13%). Both groups of students made similar proportions of PM verbalizations (18%).

When looking at the frequency of specific subtypes of verbalizations students made across both problem types, we found that the most common verbalization for both groups of students were visualizing (27) and computing (26) (see Table 6). The least and most common subtypes of verbalizations differed by student group. Students with SLD were most likely to make computing and comment verbalizations (12 and 10, respectively) and least likely to make hypothesizing verbalizations (0). In contrast, students without SLD were most likely to make visualizing and computing verbalizations (18 and 14, respectively) and least likely to make paraphrasing, estimating, and self-correction" verbalizations (2 of each subtype). No students with SLD made "Hypothesizing" verbalizations, whereas students without SLD made every subtype of verbalization.

Only two students (S4, a student without SLD, and S5, a student with SLD) made the "Reading" verbalization for all four problems in the think aloud task. Two students, both with SLD (S1 and S3), did not make this verbalization for any of the four problems. Three students, two without SLD (S6 and S8) and one with SLD (S7), read only two of the four problems. One student without SLD (S2) read only one problem, a moderate level problem.

Duration

Results of the duration of time students spent making verbalizations across both problem types indicate that both groups of students spent more time making cognitive verbalizations than metacognitive verbalizations (see Tables 5 and 6). Compared to their general education peers, students with SLD spent less time making cognitive verbalizations (351 seconds vs. 233 seconds) but more time making metacognitive verbalizations (80 seconds vs. 90 seconds). However, within the results for the duration of time spent making metacognitive verbalizations, we found that students with SLD spent more time than their general education peers making NPM verbalizations (42 seconds vs. 33 seconds). In fact, this difference accounts almost completely for the longer duration of time students with SLD spent making metacognitive verbalizations. Results for the average duration of verbalizations also indicate that, compared to their general education peers, students with SLD made, on average, slightly longer verbalizations of all three types.

Verbalizations Across Problem Accuracy and Complexity

To answer the second research question related to differences in cognitive and metacognitive verbalizations as a function of problem accuracy and complexity, we analyzed the frequency and duration of verbalizations across students' accuracy and problem types. We first present the results of students' accuracy on all problems in the think aloud task. Then, we present detailed results of the frequencies and duration of verbalizations for each problem type, moderate and advanced. Tables 4, 5, and 6 summarize these results.

Problem Accuracy

As seen in Table 4, results from scoring the think aloud task for accuracy show that students in both groups were largely unsuccessful in solving problems of both types as evidenced by accurate numerical solution. Students with SLD scored a total of seven points on the think aloud task, or 5.5%, and students without SLD scored a total of eight points, or 14%. Only one student correctly answered at least one problem in the task; S8, a student without SLD, correctly set up and solved both the moderate level problems in the task. Four of the total eight participants, two with SLD (S1 and S7) and two without SLD (S4 and S6), did not correctly solve or set up any components of the problems in the task. Among students with SLD, two participants, S3 and S5, demonstrated partial or full accuracy in labeling diagrams. S5 also set up the equation for one moderate level problem with partial accuracy. Among students without SLD, one participant, S2, accurately represented an advanced level problem with accurate graphing of a line with two given endpoints. These three participants, S2, S3, and S5, did not accurately complete any other components of the problems.

Results of the verbalizations made by the students who correctly responded to all or some components of the problems indicate a range of frequencies and durations of verbalizations. Total duration of verbalizations did not have a relationship with problem accuracy given the low scores for accuracy. The only participant to correctly solve a problem in completion, S8 (a student without SLD), made 10 cognitive and PM verbalizations of 45 seconds in duration and 1 NPM of 2 seconds in duration. The only other student without SLD who accurately completed one component of the problem, S2, made 8 cognitive and PM verbalizations of 38 seconds in duration and 0 NPM. Participant S3, the only student with SLD who accurately completed at least one component of the problems, made 5 cognitive and PM verbalizations of 34 seconds in duration and 1 NPM of 2 seconds in duration. Participant S5, a student with SLD who partially completed two components of the problems, made 9 cognitive and PM verbalizations of 69 seconds in duration and 1 NPM of 3 seconds in duration.

Comparing the frequencies and duration of verbalizations made in the fourth matched pair, consisting of participant S7 and S8, indicate differences. When working on the moderate level problems, S7, a student with SLD, made 5 cognitive and PM verbalizations of 14 total seconds and 2 NPM verbalizations of 8 total seconds. The results for this matched pair indicate that a higher frequency cognitive and PM verbalizations are related to increased problem accuracy.

Problem Complexity

The results summarized in Tables 5 and 6 indicate that students with SLD made fewer cognitive and PM verbalizations than their general education peers for both moderate and advanced level problems. In contrast, students with SLD made the same number of NPM verbalizations (3) as their general education peers did when working on moderate level problems. When working on advanced level problems, students with SLD made more frequent NPM verbalizations than their general education peers (9 vs. 7).

As problem complexity increased, both groups of students made more frequent NPM verbalizations and less frequent cognitive and PM verbalizations. The magnitude of the differences in these frequencies within each group of students was larger among students with SLD than students without SLD. As students with SLD progressed from moderate to advanced level problems, they made more metacognitive verbalizations (9 or 32% of all verbalizations for moderate level problems vs. 13 or 50% of all verbalizations for advanced level problems). This increase in metacognitive verbalizations is attributable to a slight decrease in PM verbalizations (6 or 21% of all verbalizations for moderate level problems vs. 4 or 15% of all verbalizations for advanced level problems) and a large increase in NPM verbalizations (3 or 10.7% of all verbalizations for moderate level problems vs. 9 or 34.6% of all verbalizations for advanced level problems).

The most and least common verbalization subtypes for each problem type differed for each group of students (see Table 6). When solving moderate level problems, students with SLD were most likely to make "visualizing" or "computing" verbalizations (7 of each subtype) and least likely to make "Hypothesizing", "Estimating", "Self-Monitoring", or "Affect" verbalizations (0 of each subtype). As they progressed to advanced level problems, students with SLD were most likely to make "Comment" or "Computing" verbalizations (7 and 5, respectively). Results of frequencies of verbalization subtypes indicate that students without SLD were most likely to make "Visualising" verbalizations (15) and least likely to make "Affect" verbalizations (0) when solving moderate problems. As this group of students worked on more challenging problems, they were most likely to make "Computing" verbalizations (9) and least likely to make "Self-Corrections" verbalizations (0).

Results for the duration of verbalizations for individual problem types mirror those reported for both problem types. At each problem level, students with SLD spent less total time making verbalizations than their general education peers (181 seconds vs. 257 seconds for moderate level problems and 132 seconds vs. 174 seconds for advanced level problems). However, results from the average time each group of students spent making verbalizations indicate that students with SLD made similar or slightly longer average verbalizations of all three types for both problem types.

As problem complexity increased, students in each group made shorter average verbalizations and spent less total time making verbalizations that were cognitive or productive in nature. The results for total and average time spent making metacognitive verbalizations are more nuanced (see Table 5). First, students with SLD were found to spend more time making metacognitive verbalizations as they progressed from moderate to advanced problems (36 seconds to 54 seconds), which is largely attributable to the increase in time spent making NPM verbalizations (11 seconds to 31 seconds) and a small reduction (2 seconds) in the time spent making PM verbalizations. As they worked on more challenging problems, students with LD made, on average, longer PM verbalizations (4.17 average seconds vs. 5.75 average seconds) and nearly identical NPM verbalizations (3.67 average seconds vs. 3.44 average seconds).

Results for students without SLD indicate a smaller increase in time spent making metacognitive verbalizations as problem complexity increased. Specifically, as they progressed to advanced problems, students without SLD spent less total and average time making PM verbalizations (30 total seconds with a 3.75 second average vs. 17 total seconds with a 2.83 second average) and more total and average time making NPM verbalizations (6 total seconds with a 2 second average vs. 27 total seconds with a 3.86 second average).

Discussion

The current study aimed to contribute to the knowledge base on how secondary students with SLD undergo cognitive and metacognitive problem-solving processes to arrive at solutions to word problems involving the application of the Pythagorean Theorem. Major results confirmed the hypothesis that students with SLD made fewer and shorter verbalizations. Other notable results involved specific subtypes of verbalizations students made. A discussion of these noteworthy findings from the present study follows. We conclude by discussing limitations and implications for practice and future research.

Noteworthy Findings

Primary findings from the present study suggest that, compared to their general education peers who had earned similar scaled scores on a standardized state assessment, students with SLD less frequently made helpful verbalizations (cognitive and PM) and more frequently made NPM verbalizations. Even more alarmingly, NPM verbalizations consisted of a third of the verbalizations students with SLD made when working on advanced level problems. In other words, students with SLD spent more time and mental effort experiencing negative thoughts about their ability and performance in mathematical problem solving than they did thinking about the mathematical content in the problems and strategies to solve the problems. These findings mirror those obtained by Rosenzweig and colleagues (2011), in which middle school students with SLD made nonproductive metacognitive verbalizations more often than their general education peers when approaching more challenging problems (Rosenzweig et al., 2011). The authors of the present study recently tested an intervention for problem solving with the Pythagorean Theorem for secondary students with SLD and other disability types (manuscript under preparation) and obtained findings which provide additional context for the findings in the present analysis. Following intervention, which led to increased problem solving accuracy for all participants, students with SLD demonstrated no change in frequency or duration of NPM verbalizations while students with non-SLD disability types made no NPM verbalizations (Deshpande et al., 2021). A possible explanation that may account for the persistence of NPM verbalizations among students with SLD is that this population of learners have poorer perceptions of their ability and knowledge when approaching challenging problem-solving tasks. Research indicates that, in addition to poor conceptual understanding and procedural fluency, students with LD struggle with perceptions of problem difficulty (Bryant et al., 2000; Montague & Applegate, 2000). Montague and Applegate (2000) found that middle school students with SLD, compared to average achieving peers, more often perceived word problems to be difficult and demonstrated less persistence in attempting to solve them. In light of this research, it follows that, when being asked to verbalize their thinking during problem solving, students with SLD made more NPM verbalizations than helpful verbalization types.

A second noteworthy finding in the present analysis involves the specific subtypes of verbalizations that students in each group made. First, the limited frequency with which students, especially those with SLD, made verbalizations suggesting that they read the problem reflects a core obstacle in their ability to successfully problem-solve. This finding can be situated in existing research on how students with SLD approach problem solving. Bryant and colleagues (2000) identified impulsivity in problem solving, "jumping into arithmetic procedures," as a characteristic behavior in students with SLD. In a broader sense, the infrequency with which students with SLD read the problems in the think aloud task reflects the poor sequencing of problem-solving steps identified in this population of learners (Montague, 2007). Second, our findings that students with SLD were most likely to make "computing" verbalizations reflect the nature of a majority of instruction and interventions that focus heavily on skill acquisition and procedural knowledge (Myers et al., 2015). In contrast, students without SLD were most likely to make "visualizing" verbalizations, reflecting a distinguishing feature of SLD, poor spatial visualisation (Peng et al., 2018; Jitendra et al., 2016).

Limitations and Implications for Future Research

Four major limitations exist in the present study. The first limitation is the small sample size, which constrains the generalizability of the present findings to the broader population of students with SLD or other disability types. Our data collection method, the think aloud protocol, constitutes the second limitation. Given that think alouds rely on students' expressive language abilities, participants with limited abilities in this area may have been at a disadvantage in fully communicating their cognitive and metacognitive processes. Relatedly, although English proficiency was a component of inclusion criteria, it is possible that one or more participants may not have met the school district's threshold for English Language Learner (ELL) status yet may have had limited English proficiency. The participating school had a large population of students who were new to the school and to the country. Detailed demographic data on students' time in the school district was not available. As a result, participants who were identified as being proficient in English nonetheless may have been at a disadvantage in adequately expressing their thinking during the think aloud protocol. The third limitation we identify is the selection of the Pythagorean Theorem as the topic of focus in the think aloud task. Although a foundational topic in secondary Common Core mathematics curricula, application of the Pythagorean Theorem relies heavily on proficiency in algebraic topics, thereby limiting the generalizability of our findings to other topics in geometry curricula. The fourth limitation we identify is the role the video modeling component may have played in inadvertently priming participants for the incorrect problem type, namely finding area of a triangle rather than application of the Pythagorean Theorem. Although the researcher explicitly stated to participants that the purpose of the video modeling component was to model the think aloud process, it is possible that participants with learning difficulties may have incorrectly generalized the content of the video modeling component to the the think aloud task.

In considering implications of the present study on future research, the most immediate is the need for further research on how secondary students with SLD approach problem solving with the Pythagorean Theorem. Specific research questions involve identification of strategies students use, common errors they make, and determining if each varies with problem complexity. Error analysis paves the way to more effective intervention by pinpointing specific junctures in the problem-solving process at which students have misconceptions and require support (Kingsdorf & Krawec, 2014). Given the complex interplay of cognitive skills needed to successfully problem solve and the unique domain-specific and -general deficits seen in SLD, such insights are important. Such an error analysis is especially salient when considering that proficiency in application of the Pythagorean Theorem, and other topics in geometry curricula, is predicated on algebraic proficiency. Obtaining more generalizable findings on students' strategy usage and errors will enable researchers and practitioners to develop and test interventions targeting the cognitive and metacognitive challenges students face when problem solving.

Using the think aloud protocol to obtain deeper understandings of students' geometric problem solving generates new research questions as well. Post hoc analysis of specific language students used in verbalizations would provide deeper understandings on more individualized levels. Exploring how to effectively scaffold students' responses during the protocol has potential to more effectively capture students' cognitive and metacognitive processes. Considering the challenges commonly facing students with SLD in language processing, research on effective scaffolds of think alouds is a worthy endeavor. Similarly, the present study raises questions regarding the interaction between verbalizations and written responses in mathematics. In other words, are there links between students' abilities to talk about problem solving and to explain and justify their reasoning in written form? The relationship between written expression and conceptual understanding and reasoning has been identified by the NCTM (2010) as important in students' ability to use mathematical language to draw logical conclusions (Hughes et al., 2019). It is also evident in standards-based curricula starting as early as third grade (Common Core State Standards Initiative, 2010). Research is therefore needed to contribute to the field's understanding of the role verbalizations play in building proficiency in mathematical writing.

More broadly, there is a need for research on how students with SLD approach problem solving in foundational topics commonly featured in geometry curricula. Such high-yield topics include but are not limited to properties of quadrilaterals and triangles, unknown angles, and measurement and dimension of three-dimensional figures (Common Core State Standards Initiative, 2010). Developing a more robust knowledge base on how students with SLD tackle the daunting conceptual and procedural tasks of problem solving in geometry will pave the way for the development of effective interventions to improve their overall mathematical abilities.

Implications for Practice

Despite its limitations, the present study leads to four instructional implications for students with SLD in geometry. First, findings on the least and most common verbalizations students made provide a rationale for inclusion of specific components in designing instructional interventions. The infrequency with which students made the "Reading" verbalization suggests that interventions targeting problem solving should begin with reminders or prompts for students to read problems. A number of stepwise strategies for problem solving in younger grades include this feature; emphasis on this step for older students can enhance their problem-solving ability. Similarly, findings that students with SLD made more frequent NPM verbalizations of longer duration indicate that fostering more positive perceptions of themselves and mathematics can improve overall problem solving ability. Including components that address metacognition relating to these perceptions should be included in problem solving interventions for students in secondary grades.

Because valuable insights on students' reasoning can be gleaned from engaging students in the think aloud protocol, instructional decisions that improve student outcomes on think aloud tasks hold promise. Specifically, instructional practices and scaffolds that tighten the focus on students' mathematical reasoning, instead of problem accuracy, hold potential for teachers to identify their students' specific strengths and areas of support during problem solving. In this vein, a second instructional implication of the present study stems from our findings that problem solving accuracy was low for all participants. It follows that students then struggled to verbalize the thinking underlying the steps they took to attempt to solve the problems during the think aloud task. To eliminate the hurdle of obtaining the correct problem solution and isolate the task to reasoning, teachers can scaffold students' responses in think aloud tasks by providing them with accurate problem solutions, such as those in worked solutions (Star et al., 2015). Students study the solutions before using them as the basis for their verbalizations during the think aloud protocol. The use of worked solutions reduces cognitive load and promotes self-explanation, thereby addressing typically challenging components of problem solving for students with SLD and promoting metacognition (Riccomini & Morano, 2019).

A third instructional recommendation, also designed around the goal of scaffolding students' ability to respond during the think aloud protocol, is to provide them with language supports. Scaffolds such as word banks have been used extensively to support learners' language output by infusing learners' current levels of understanding with specialized, academic vocabulary they are being taught (Lucero, 2013). For students with SLD facing challenges in language processing, word banks are a helpful tool to enable them to focus on the reasoning underlying their problem solving, the goal of the think aloud protocol in mathematics. It is worth noting that the scaffolds discussed here - and the think aloud protocol itself - are components of explicit instruction, an evidence-based practice for students with disabilities (Hughes et al., 2017).

A fourth, broader instructional implication of the present study is the use of explicit instruction of the think aloud protocol via video modeling. A low cost evidence-based practice for enhancing academic skills in students with disabilities, video modeling is an instructional method that is easily accessible for learners and facilitates independence, which is especially important when considering age-appropriate interventions for secondary students (Kellems & Edwards, 2016; Satsangi et al., 2019). Point-of-view modeling, in which the learner observes the desired performance from their perspective, holds instructional potential for teaching students how to demonstrate their mathematical reasoning during a think aloud. In this way, video modeling is an instructional method that enables students to learn how to respond during a think aloud protocol, thereby strengthening the value of the protocol in improving student outcomes during problem solving.

Conclusion

While the promise of special education has materialized for many students with disabilities, this population of learners requires continued intervention to meet the demands of a society that places a premium on high school completion. We believe the findings from the present exploratory study contribute to the knowledge base on improving academic outcomes for secondary students with SLD and establish the need for further research in effective geometry instruction for students of all abilities.

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