



## ERRORS IN USING CONVERGENCE TESTS IN INFINITE SERIES

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**Abstract:** The present study aimed to identify the errors made by pre-service elementary mathematics teachers while investigating the convergence of infinite series. A qualitative exploratory case study design was used with a total of 43 undergraduate students. Data were obtained from a test administered in a paper-and-pencil form consisting of seven open-ended questions. The data analysis was done using descriptive and content analysis techniques. Findings were presented as follows: inappropriate test selections; failure to check convergence criteria; incorrect use of a comparison test; limit comparison test error; re-test convergence test results; considering  $\sum$  as a multiplicative function; misunderstanding of special series; considering that series has no character when the convergence test is inconclusive; confusing sequences with series; misunderstanding of the  $n$ th-term test; misinterpretation of convergence test results. Findings showed that students with insufficient procedural knowledge had difficulty in solving the given problem even if they understood it, whereas those with insufficient conceptual knowledge could not literally understand what they did even if they solved the problem. Therefore, the establishment of a moderate balance between procedural and conceptual knowledge in the learning of the convergence of series is essential in reducing the errors or learning difficulties for developing deep mathematical understanding.

**Keywords:** infinite series, convergence tests, errors, procedural knowledge, conceptual knowledge

### 1. Introduction

In today's developing world, mathematics in many areas is becoming increasingly important for individuals, communities, and nations to function across the globe. Because of the importance of mathematics in such a wide range of fields, the difficulties and misunderstandings of students studying mathematics, science, and engineering, especially in calculus courses, have been investigated and reported by several researchers and mathematics educators (Aspinwall & Miller, 2001; Davis & Vinner, 1986; Martin, 2009; Martínez-Planell, Gonzalez, DiCristina & Acevedo, 2012; Monaghan, 2001; Orton, 1983; Tall & Schwarzenberger, 1978). Some of the studies have shown that students taking calculus courses have difficulty in conceptual understanding because they have a very superficial and insufficient understanding of the basic concepts of mathematical analysis due to the rote-based learning instruction (Steen, 1998; White & Mitchelmore, 1996). The introduction to calculus begins with the concept of limit, but this concept is inherently difficult, and no matter how it is taught many students have several difficulties in understanding the definition of limit intuitively (Barnes, 1995). These difficulties mostly stem from the inability to understand the notion of infinity or a fairly complex definition of the limit conceptually (Cornu, 1991, Tall, 1992; Tall & Vinner, 1981). Many students also experience similar difficulties in understanding the convergence of an infinite series which is one of the important concepts of analysis (Akgün & Duru, 2007). These difficulties have been said to increase further along with the inadequacy in algebra (Brown, 1996).

Sequences and series are the most fundamental topics in any analysis course and can serve as a basis for the other topics in calculus including limits, continuity, derivatives, and integrals. For example, the topic of series, which is one of the important subjects in the process of calculating the area under a curve, can also offer alternative ways to model some mathematical relations such as the distribution of drugs and populations. (González-Martín, Nardi, & Biza, 2011). On the other hand, Tall and Schwarzenberger (1978) argued that the subject of the series is one of the significant anomalies of the mathematics curriculum. Many students encountering infinite series for the first time cannot comprehend its

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mathematical essence, cannot complete the simple procedures and solutions related to the subject, and consider it as one of the hardest topics in mathematics (Monaghan, 2001; Nardi, Biza and González-Martín, 2008). The research showed that the most common learning difficulties in the series are the difficulties arising from the inability to distinguish the fundamental difference between sequences and series (Brown, 1996; Davis, 1982; Yazgan-Sağ & Argün, 2012). Lee (1993) mentioned that the meaning of partial sums could not be understood exactly and the sequence of partial sums could not be obtained explicitly from the given series. Moreover, the fact that an infinite series is an infinite sum of the terms of a sequence of numbers, or the limit of an infinite series is equal to the limit of a sequence of partial sums of a series has led to the idea that the concept of sequences is more central than the concept of series. This has caused the concept of series to be automatically pushed aside in the curriculum and hence in the textbooks. Even the most popular and widely used calculus textbooks do not explain the series much in relation to other mathematical concepts. They generally contain routine practices of convergence test formulae requiring only operational procedures. Studying the series only at the procedural level may result in the conceptual dimension being neglected as a result (Akgün, Işık, Tatar, İşleyen & Soylu, 2012).

In a study by Alcock and Simpson (2009), after the definition of convergence of an infinite series was made in the classroom, students were asked to explain it with their own sentences. However, it was seen that students differed in their explanations when defining the convergence of a series and that most of them could not achieve a sufficient level of understanding about the definition. Similarly, in other studies, it was found that students were unable to understand what the convergence of a series means and showed an inadequate understanding of identifying whether or not a series was convergent (Brown, 1996; Martínez-Planell, Gonzalez, DiCristina & Acevedo, 2012). The study conducted by Akbayır (2004) examined the errors made by the students in meeting some convergence criteria about the series. The errors encountered were stated as a lack of understanding of the notion of a general term of a series, a confusion of convergence criteria of series with one another, a preference of the D'Alembert ratio test in particular for checking the convergence of any given series, and an insufficient understanding of comparison test criteria. Furthermore, it has been revealed that some students have difficulty with the concept of infinity as well as accepting that a convergence test may be inconclusive (Champney, 2013; Nardi & Iannone, 2001; Sierpińska, 1987). Given the importance of infinite series in calculus, exploring various difficulties, especially ones not seen in previous research, faced by the students in determining the convergence of the series still remains an important issue for the effective teaching and learning of series. Accordingly, the present study attempts to explore and understand the errors that occurred when studying the character of the series. For this purpose, the following research question has guided this study: What are the errors that undergraduate students of an elementary mathematics teacher education program make while identifying the convergence or divergence of the infinite series?

## 2. Theoretical Framework

Conceptual and procedural knowledge are known to be two types of knowledge required for mathematics learning (Hiebert & Lefevre, 1986). Conceptual knowledge can be thought of not only as a knowledge of concepts but also as an understanding of concept knowledge along with mathematical processes, deep relations, and rich connections. (Kilpatrick, Swafford, & Findell, 2001; Star, 2005). Since conceptual knowledge involves interpreting the concepts and understanding the relationships between them, it is sometimes also called conceptual understanding. On the other hand, procedural knowledge is defined by two separate parts that make it up. The first part of the procedural knowledge involves symbols and language of mathematics, while the second part involves rules, operations, visual shapes, and algorithms used to solve mathematical problems (Hiebert & Lefevre, 1986). Star (2005) states that this definition includes only superficial relations, whereas procedural knowledge can in fact contain deeper and more comprehensive relations. In other words, there can be a wide variety of processes other than algorithms or rules, and the processes within each procedure can have different levels of complexity or relations. Thus, procedural understanding involves having an idea about the formal language or symbolic representations of mathematics and being able to use related processes effectively and accurately by making conscious choices. Thus, procedural knowledge is characterized not only as a knowledge of procedures but also as the ability to execute procedures and algorithms used

in problem-solving (Star, 2005, 2007). Moreover, Skemp (1978, 1987) refers to two forms of understanding in mathematics that he called relational understanding and instrumental understanding, which are basically similar to conceptual understanding and procedural understanding respectively. An individual with an instrumental understanding knows and effectively uses rules in mathematics but does not understand why these rules work. Relational understanding is explained as knowing what is done and why. Although Skemp (1978) recognizes that relational understanding is more important than instrumental understanding, it has been asserted that as there can be procedural knowledge within conceptual knowledge or conceptual knowledge within procedural knowledge, they cannot be separated by precise boundaries and that it is also not possible to talk about the superiority of one over the other (Carpenter, 1986). Baki (1998) emphasized the importance of using conceptual knowledge and procedural knowledge in a balanced way in mathematics education. Students cannot demonstrate the required mathematical competence if they are inadequate in any of the knowledge, or if they perform these two types of knowledge separately without relating them with one another (Hiebert and Lefevre, 1986). Therefore, conceptual and procedural knowledge about series can play an important role in understanding the errors that emerged when identifying the characters of infinite series that is one of the challenging issues many students often struggle with.

### 3. Method

#### 3.1. Research Design

Qualitative research in an exploratory vein is usually preferred for characterizing, evaluating, and interpreting the specific case when ‘what’ questions are being posed as it allows for rich descriptions and understanding of the phenomena under investigation (Merriam, 2009). This research study lends itself well to the use of qualitative exploratory case study design (Yin, 2014), where the case of interest is the errors made by undergraduate students of teaching mathematics while studying the convergence or divergence of infinite series.

#### 3.2. Participants

The study was carried out with a total of 43 undergraduate students (23 females and 20 males) enrolled in the third year of the elementary mathematics teacher education program at a public university in Turkey. The participants of the study were selected via the criteria sampling method that is one of the purposeful sampling methods (Patton, 2015). Having taken Analysis-I and Analysis-II courses beforehand and being registered for Analysis-III course had been taken as criteria to select the study group of this research. It was also assumed that the prerequisite knowledge needed for infinite series is available since the participants had already seen the fundamental issues of the calculus such as summation signs, trigonometric functions, limits, derivatives, and integrals in General Mathematics, Analysis-I, and Analysis-II courses.

#### 3.3. Data Collection

The data were obtained from a test administered in a paper-and-pencil form involving open-ended questions prepared to identify the character of the infinite series as convergent or divergent. The evaluation in this way not only provided the researcher with flexibility in time but also provided more systematic and comparable data from the participants. While preparing open-ended questions, common errors made by students were taken into consideration when determining the convergence of infinite series, such as the difficulties with the concept of infinity, a lack of understanding of the meaning of a general term of a series, misunderstanding of the criteria for the convergence tests of the series or the confusion of these criteria with each other.

The test consisted of eight open-ended questions at the beginning. Two experts in Mathematics Education examined both whether the questions are valid in measuring the learning outcomes related to the convergence of infinite series and whether the questions are understood when they are read. After the necessary arrangements were made, all the details of each question aimed to detect different types of student errors were explained to these two experts and the final version of the test was formed by taking their opinions on the validity of each question in achieving this. There were seven open-ended

questions in the last form of the test (See Annex 1 for more details). One question that was thought to measure the same learning outcome and three questions aimed at identifying similar student errors were excluded from the test. For example, the ratio test is not appropriate for investigating the convergence of both series  $\sum 1/\sqrt{n^2 + n}$  and  $\sum 1/(1 + \sqrt{n})\sqrt{n}$ . Their convergence can be found by using either the integral test or limit comparison test. Hence, they were thought to measure the same learning outcome and one of them was excluded from the test. On the other hand, at the request of the experts, three questions related to the convergence of series with negative terms were also added to the test. The purpose of the study was explained to pre-service teachers before the test was applied. It was emphasized that answering the questions in a way that explicitly and clearly reveals their solutions is important for the purpose of the research. In addition, the researchers were present with the participants from the beginning to the end of the implementation of the test in order to intervene in any problems that may arise.

### 3.4. Data Analysis

The data analysis was performed using descriptive and content analysis techniques. The data summarized and interpreted in a descriptive way were analyzed in more detail through content analysis, enabling the emergence of the themes that were not noticeable with a descriptive approach (Corbin and Strauss, 2015). Accordingly, with the in-depth analysis made through the idea that the mental structure of each individual is important, it was examined whether there were any answers that point to the familiar errors as well as non-familiar errors in the solutions of the students. The participants were coded from P1 to P43 for easy reference. Their responses to open-ended questions were analyzed by the researchers systematically, and a code list was created by assigning a different code for each distinct error made by the prospective teachers.

After the coding phase was completed, similar codes were combined under meaningful categories in accordance with the research purpose. For example, the category of ‘failure to check convergence criteria of tests’ was created by combining the codes obtained from the errors made by the students such as ‘direct application of convergence tests used in series with positive terms to series with negative terms’, ‘application of an integral test to alternate series’. Similarly, the category of ‘misunderstanding of special series’ was formed by merging the codes such as ‘thinking of a non-telescopic series as a telescopic series’, ‘thinking of a series that is not a p-series as p-series’.

While presenting the findings, the most striking ones were tried to be quoted directly from the errors made by the participants in determining the character of the series, and the findings were interpreted based on these errors. At all these stages, the content of the data was constantly examined by considering the purpose of the research, and care was taken to ensure that the errors placed under the same category were as consistent and meaningful as possible (Merriam, 2009). Besides, in order to ensure that the findings of the research accurately reflect the errors made by prospective teachers, the codes and categories created were checked by an expert in the field of pure mathematics, and the consensus was achieved. Furthermore, one mathematics education researcher was asked to act as an external rater. This mathematics education researcher and the researchers double-checked codes and re-examined emerged categories. The comparison of the two coding outcomes showed a 93% agreement. The researchers resolved all disagreements and revised the code definitions until a full-agreement on the categories was reached and conferred the dependability and credibility of the data analysis by ensuring inter-coder reliability (Miles and Huberman, 1994).

## 4. Findings

A cross-case analysis of the data from 43 participants revealed eleven emergent categories for the errors pre-service mathematics teachers made while identifying the convergence or divergence of series. These categories were as following: (i) Inappropriate test selections for the convergence of series; (ii) Failure to check convergence criteria of tests; (iii) Incorrect use of a comparison test; (iv) Limit comparison test error; (v) Re-test convergence test results; (vi) Considering  $\sum$  as a multiplicative function; (vii) Misunderstanding of special series; (viii) Considering that a series has no character when the convergence test is inconclusive; (ix) Confusing sequences with series; (x) Misunderstanding of the  $n$ th-

term test; (xi) Misinterpretation of the convergence test results. The following sections provide the findings with respect to these emergent categories.

### 4. 1. Inappropriate Test Selections for the Convergence of Series

One of the errors made was to test the convergence of the series by using the test which was not appropriate for the given series. For example, one of the participants came up with a solution to the second question in the test as follows:

$$\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n} \quad \text{Kök testi uygulanarak}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{3^n \cdot n!}{n^n} \right|} = \lim_{n \rightarrow \infty} \frac{3 \cdot (n!)^{1/n}}{n} \Rightarrow \lim_{n \rightarrow \infty} \frac{3}{n} \cdot \lim_{n \rightarrow \infty} (n!)^{1/n}$$

$$= 0 \cdot 1 = 0$$

Kök testine göre  $\lim_{n \rightarrow \infty} a_n < 1$  şartından dolayı seri yakınsaktır.

Figure 1. Participant-18’s Solution to Question 2

In Figure 1, the participant’s insistence on the Cauchy root test instead of the D’Alembert ratio test led the participant to the inaccurate inferences and caused him to find the divergent series to be mistakenly convergent.

### 4. 2 Failure to Check Convergence Criteria of Tests

Another error was that participants used convergence tests regardless of the characteristics that the series should have in the criteria of the convergence tests. For instance, the fourth question was answered by one of the participants as follows:

$$\sum_{n=1}^{\infty} \frac{1-n}{n \cdot 2^n} \quad \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1-n}{n \cdot 2^n} \right|} = \frac{1}{2} < 1 \quad \text{Yakınsak bir seridir.}$$

Figure 2. Participant-5’s Solution to Question 4

As seen in Figure 2, Participant-5 applied the Cauchy root test used in series with only positive terms to a series involving negative terms due to the expression  $(1-n)$  in the general term of the series and casually stated that the series is convergent by interpreting the limit value of  $\lim_{n \rightarrow \infty} (1-n)^{1/n}$  as 1.

In Figure 3, another participant applied the integral test to the first question in the test and identified the series to be divergent as follows:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n \cdot (n+1)} \quad \text{Alternan seri}$$

$$\text{Integral testi: } \lim_{n \rightarrow \infty} \frac{2n+1}{n^2+n} = \lim_{n \rightarrow \infty} \int_1^{\infty} \frac{2x+1}{x^2+x} dx \quad \begin{aligned} x^2+x &= u \\ (2x+1)dx &= du \\ &= \ln(x^2+x) \Big|_1^{\infty} \\ &= \ln(\infty) - \ln 2 = \infty \end{aligned}$$

Figure 3. Participant-43’s Solution to Question 1

The participant’s application of the integral test to the given alternating series, which is indeed convergent, led to the misinterpretation of the character of the series.

### 4.3. Incorrect Use of a Comparison Test

The other error was to consider either the series whose  $n$ th term was smaller than the  $n$ th term of the divergent series as divergent or the series whose  $n$ th term was greater than the  $n$ th term of the convergent series as convergent. One of the participants, for example, responded to the sixth question as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\frac{1}{n(n+1)} < \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{n^{1/2}} \quad p = \frac{1}{2} < 1 \text{ iraksal.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ serisi iraksal ise ondan küçük olan } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ serisi de iraksal //}$$

Figure 4. Participant-28's Solution to Question 6

In Figure 4, the participant compared the terms of the given series with those of another series whose divergence is known and erroneously interpreted the series  $\sum 1/\sqrt{n^2+n}$ , which is smaller than the divergent series  $\sum 1/\sqrt{n}$ , as divergent.

### 4.4. Limit Comparison Test Error

This type of error was usually caused by the fact that the series selected to perform the limit comparison test did not become an appropriate series for performing the limit comparison test. For example, one of the participants answered the sixth question as following:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} = \text{Karşılaştırma yapalım } \frac{1}{\sqrt{n(n+1)}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0 \text{ a.k.t.}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0 \text{ a.k.t.} \text{ için } \sum b_n \text{ yakınsak ise } \sum a_n \text{ de yakınsaktır.}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ p serisinde yakınsak gelmez.}$$

Figure 5. Participant-12's Solution to Question 6

As shown in Figure 5, the participant compared the series  $\sum 1/\sqrt{n(n+1)}$  with the divergent series  $\sum 1/\sqrt{n}$ , and after finding the limit value as zero, he should have seen that the test was inconclusive, but he identified the series  $\sum 1/\sqrt{n(n+1)}$  as convergent.

### 4.5. Re-test Convergence Test Results

Another error made by the participants was that once applying one of the convergence tests, another convergence test was applied to the result obtained. For example, one of the participants responded to the second question as follows:

$$\sum_{n=1}^{\infty} \frac{3^n n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = 3 \frac{3^{n+1} (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} = 3 \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = 3 \lim_{n \rightarrow \infty} \frac{n}{n+1} = 3/4 < 1 \text{ iraksal}$$

Figure 6. Participant-21's Solution to Question 2

In Figure 6, the participant first applied the D'Alembert ratio test to the series  $\sum 3^n n! / n^n$ , and then he should have found the limit of the expression  $3(n/n+1)^n$  as  $3/e$  and should have indicated that the series was divergent. However, the participant concluded that the series was divergent by applying the Cauchy root test, another convergence test, to the expression  $3(n/n+1)^n$  he obtained after applying the ratio test to the series  $\sum 3^n n! / n^n$ .

**4. 6. Considering  $\Sigma$  as a Multiplicative Function**

The other error was that the sigma function was thought to be a multiplicative function. For instance, in Figure 7, the sixth question in the test was answered by one of the participants as follows:

Figure 7. Participant-36's Solution to Question 6

After noticing that the ratio test did not respond to the series  $\sum 1/\sqrt{n(n+1)}$ , the participant incorrectly wrote the series  $\sum 1/\sqrt{n(n+1)}$  in the multiplicative form  $\sum 1/\sqrt{n} \cdot \sum 1/\sqrt{(n+1)}$  in order to examine the convergence of both series  $\sum 1/\sqrt{n}$  and  $\sum 1/\sqrt{(n+1)}$  separately and expressed the character of the series  $\sum 1/\sqrt{n(n+1)}$  as divergent.

**4. 7. Misunderstanding of Special Series**

In this type of error, some special series (*p*-series, telescopic series, etc.) were perceived as a special series even under any functional operation. For example, one participant responded to the sixth question as following:

Figure 8. Participant-38's Solution to Question 6

As seen in Figure 8, the participant considered the divergent series  $\sum 1/\sqrt{n(n+1)}$  as a telescopic series and determined its character. However, she made her calculations using a method similar to the method performed in finding the convergence of the telescopic series  $\sum 1/\sqrt{n(n+1)}$  and stated that the series was convergent.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \tan \frac{1}{n\sqrt{n}} \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \tan \left( \frac{1}{n^{3/2}} \right) \rightarrow p \text{ serisi}$$

$\frac{3}{2} > 1$  old.  
seri yakınsor.

Figure 9. Participant-34's Solution to Question 7

Similarly, in Figure 9, another participant erroneously described the absolute convergent series  $\sum (-1)^{n-1} \tan 1/n\sqrt{n}$  as convergent by regarding the convergent  $p$ -series of  $\sum 1/n\sqrt{n} = \sum 1/n^{3/2}$ .

#### 4. 8. Considering that a Series has no Character When the Convergence Test is Inconclusive

Another important error was that if the convergence test for a series is inconclusive, it is assumed that the series does not have any character. For instance, one of the participants answered the fifth question as follows:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{10}} = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2}}{(n+1)\sqrt{10}}}{\frac{(-1)^{n+1}}{n\sqrt{10}}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n\sqrt{10}}{(n+1)\sqrt{10}} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

Test sonuçu yetersiz.

Figure 10. Participant-10's Solution to Question 5.

In Figure 10, the D'Alembert ratio test was applied to the divergent series  $\sum (-1)^{n-1}/n\sqrt{10}$  and it was stated that the test was inconclusive after seeing that the result was one, but it was then concluded that the given series did not have any character without the need to apply another convergence test.

#### 4. 9. Confusing Sequences with Series

Many participants also made some serious mistakes in confusing the convergence of the sequence of partial sums with the convergence of the sequence of numbers and tried to use the facts concerning one another. For example, as shown in Figure 11, one of the participants expressed that the series  $\sum (n!)^2/(2n)!$  was convergent after finding the limit of its general term, as in determining the convergence of sequences, and thought that the given series converged to zero.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \quad n=1 \Rightarrow \frac{1}{2} \quad n=2 \Rightarrow \frac{1}{6} \quad n=3 \Rightarrow \frac{1}{24}$$

0'a yakınsıyor.

Figure 11. Participant-7's Solution to Question 3

Likewise, as seen in Figure 12, the same participant characterized the series  $\sum 3^n n!/n^n$  as divergent by finding the limit of its general term infinite.

$$\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n} \quad n=1 \Rightarrow 3 \quad n=2 \Rightarrow \frac{9}{2} \quad n=3 \Rightarrow 6 \quad n=4 \Rightarrow \dots$$

$\infty$ 'a yakınsıyor.

Figure 12. Participant-7's Solution to Question 2



### 4. 10. Misunderstanding of the *n*th-term Test

The other error was that some participants considered the *n*th-term test as sufficient for the convergence of a series. They had the tendency of using the *n*th term test to check the convergence of the series instead of using it to determine the divergence of the given series. For example, in Figure 13, one of the participants responded to the sixth question as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \Rightarrow \frac{1}{n^2+n}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}$$
*n. terim testinden sifira yaklasir. Bu da yakinsaktir*  

$$\lim_{n \rightarrow \infty} a_n = 0 \text{ ise}$$

Figure 13. Participant-21’s Solution to Question 6

By applying the *n*th term test to the divergent series  $\sum 1/\sqrt{n(n+1)}$ , the participant found the result to be zero, and erroneously stated the character of the series as convergent. However, when  $a_n \rightarrow 0$ , the series  $\sum_{n=1}^{\infty} a_n$  cannot always be said to converge. It is possible for a series to diverge when  $a_n \rightarrow 0$ .

### 4. 11. Misinterpretation of the Convergence Test Results

Another error was the misinterpretation of the convergence test results as the sum of the given series. For example, as seen in Figure 14, one of the participants applied the Cauchy root test to the series  $\sum (1-n)/n2^n$  and found the result  $e/2$  in her solution for the fourth question. However, as seen in Figure 14, the participant considered this value as the sum of the given series, and mistakenly characterized the series as convergent.

$$\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$$

$$\lim_{n \rightarrow \infty} \frac{1-n}{n2^n} = 0 \quad n < 2^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1-n}{n2^n}} = \lim_{n \rightarrow \infty} \frac{1}{2} \sqrt[n]{\frac{1-n}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1-n}{n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left( 1 - \frac{1}{n} \right)^{n-1}$$

$$= \frac{1}{2} e^{-1} = \frac{1}{2} e \text{ yakinsaktir}$$

Figure 14. Participant-23’s Solution to Question 4

## 5. Discussion and Conclusion

In this study, it was attempted to explore the types of errors that elementary mathematics pre-service teachers make when examining the characters of the series. First of all, as Earls (2017) argued, the study revealed that students’ conceptual understanding of the concepts of function, limit, infinity, and sequence is essential for accurately determining the convergence of the series in all aspects. On the other hand, the study also yielded that even if conceptual learning is assumed to occur, it is not possible to accurately investigate the convergence of the series. As McCombs (2014) found, this study likewise showed that various errors were encountered in cases where students were unable to use symbolic representations, related rules, and algorithms effectively in some subjects such as trigonometry, summations, derivative, and algebra. Therefore, in addition to the level of students’ readiness or their lack of previous learning that prevent them from having a rich conceptual understanding (Hiebert and

Lefevre, 1986), some students' failure to use procedural knowledge effectively can lead them to make several errors while investigating the convergence of the series.

Another error made in this study was to confuse the convergence of the series with the convergence of the sequences. As done in the sequences, the students calculated the limit of the general term of the series when identifying the convergence of the series. As mentioned in previous studies (Earls, 2017; Lindaman, 2007; Tall and Schwarzenberger, 1978), this type of error can be due to the lack of ability of students to recognize the subtle difference between the convergence of a sequence of partial sums and the convergence of a sequence of numbers. The other error encountered was that some students assumed that the series had no character when the test applied to examine the convergence of the series did not respond. In other words, the students focused on the convergence test results they obtained and failed to understand that the series defined as the limit of the infinite sequence of partial sums should have actually been either convergent or divergent. That is, a series converges if its sequence of partial sums converges (i.e. its limit exists and is finite). Likewise, a series diverges if its sequence of partial sums diverges (i.e. its limit does not exist or is plus or minus infinity) (Thomas, Weir, Hass and Giordano, 2016). Students' failure to pay attention to this absolute distinction between a sequence and a series can be attributed to their inability to develop sufficient conceptual understanding of the definition of a series.

Moreover, as illustrated by Martin (2009), in this study, some of the students considered the  $n$ th term test as a sufficient condition for the convergence of a series. However, the  $n$ th term test is a necessary condition but not a sufficient condition for a series to converge (i.e. if  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ ). In other words, the  $n$ th term test is a test for checking the divergence of a series (i.e. if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , the series diverges). Therefore, some students failed to understand that the necessary condition of the convergence of a series is used to demonstrate that a series diverges. As Laudien (1999) pointed out, the proposition that if a series converges, the limit of the general term of a series is zero was mistakenly interpreted as if the limit of the general term of a series tends to zero, the series is convergent. This misinterpretation led some students of this study to the thought that the series was convergent when they found the limit of the general term of the series to be zero. This kind of error can be explained by the lack of conceptual understanding as a result of the students' inability to differentiate the nuanced difference between the conditional statement  $p \Rightarrow q$  and its contrapositive  $q' \Rightarrow p'$ .

Additionally, as seen in previous research (Burcu-Dereli, 2017; Earls, 2017; Nardi and Iannone, 2001), this study showed that while determining the convergence of the series, some students preferred inappropriate series for applying the comparison test or limit comparison test. This error led students to label a convergent series as divergent or a divergent series as convergent. In addition to the selection of series not suitable for the convergence tests applied, it was also seen that some students made inappropriate test selections for investigating the convergence of the series. These inappropriate selections can result from students' desire to apply the schemes and algorithms they have created in the learning of the convergence tests to the series they analyze. For example, if the algebraic expression that generates the general term of a series includes the  $n$ th power of any mathematical expression, it was seen that many of the students tended to apply the Cauchy root test. Students thought that this test was the most appropriate test to draw a direct conclusion about the convergence of the given series. The preference of examples or in-class practices, where each convergence test can be easily applied, can pave the way for the formation of these schemes and algorithms that lead students to create didactic barriers (Arslan and Kanbolat, 2016).

The study also demonstrated that some students expressed the results received from convergence tests as the converged value of the series. Therefore, as in the study of Kung and Speer (2010), this study also showed that students could not accurately interpret the test results obtained. Even though procedures for applying the convergence tests were realized, students could not conceptually understand what they were testing in using convergence tests. The application of the convergence test chosen to determine the character of a series indeed depends on the students' conceptual understanding as well as their procedural understanding. It is not possible to separate these two understandings from each other with definite boundaries, and there may be procedural knowledge in conceptual understanding and

conceptual knowledge in procedural understanding (Carpenter, 1986). Therefore, the failure to fully understand or correctly interpret the values received after applying the selected convergence tests to the given problems as a result of procedural understanding can reflect an indication of the students' lack of conceptual understanding of convergence tests.

In addition, the study indicated that while investigating the convergence of the series, some students thought that special series, which have specific features, can also be available under the image of any function. They made a generalization that telescopic series and p-series keep their features under even the functional structure. That is to say, the series  $\sum f\left(\frac{1}{n^p}\right)$  that accepts the image of the general term

of the p-series  $\sum \frac{1}{n^p}$  as its general term is also considered to be a p-series. The fact that some of the students' attempts to determine the convergence of the series without fully understanding the telescopic series and p-series can make it impossible for them to establish the relationship between conceptual understanding and procedural understanding. Another error preventing the establishment of this relationship was that some students tried to use the series convergence tests without checking under which conditions they could be applied. This shows that rather than conceptually understanding the convergence tests of the series, as discussed in some studies, students generally focus on the procedural steps used in the application of the tests (Earls, 2017; Morrel, 1992). However, learning occurs when procedural knowledge and conceptual knowledge are dealt with alternately. Thus, based on the principle assumption that the development of the procedural or conceptual understanding affects the development of the other (Hiebert and Wearne, 1986), it is very important to relate both understandings with one another to internalize the related learning (Rittle-Johnson and Schneider, 2015).

In sum, considering that conceptual knowledge refers to a full understanding of mathematical concepts, it is not possible for students who cannot realize the conceptual understanding to see the comprehensive relationships in different problem situations while determining the convergence of infinite series. The failure to realize this understanding also brings with it learning difficulties. Since conceptual understanding strengthens the relational structure in learning, it can be thought that it creates effective and deep learning about the series by allowing students to establish relationships between their mathematical knowledge (Gelman and Williams, 1998). Hence, the role of conceptual understanding in preventing errors or overcoming learning difficulties experienced while investigating the convergence of the infinite series cannot be underestimated. Another important point here is to make sure that conceptual and procedural understanding are not independent of each other but rather supporting each other in order to overcome the learning difficulties. In other words, students with insufficient procedural knowledge have difficulty in solving the given problem even if they understand it, whereas those with insufficient conceptual knowledge cannot fully understand what they do even if they solve the problem. Even though conceptual understanding is a goal that students must strive for, it cannot be guaranteed. We must sometimes allow students to make progress without fully understanding the concepts. Therefore, the establishment of a moderate balance between the procedural and conceptual knowledge in the learning of the convergence of series is also essential in reducing the errors or learning difficulties in this subject. Otherwise, the students cannot achieve robust and deep mathematical understanding in the related learning area if they are inadequate in either of these two types of knowledge or if they perform the two without relating to each other (Hiebert and Lefevre, 1986). Even though it is essential for students to realize conceptual and procedural understanding without preferring one another in identifying the characters of the infinite series, the question of how to deal with them in harmony still remains one of the controversial issues of mathematics education.

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**Annex 1**

Aşağıda verilen serilerin yakınsak mı yoksa ıraksak mı olduklarını belirleyiniz. Lütfen cevaplarınızı detaylı bir şekilde açıklayınız.

(Determine if the series given below is convergent or divergent. Please explain your answers in detail.)

1.  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)}$

2.  $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

3.  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

4.  $\sum_{n=1}^{\infty} \frac{1-n}{n 2^n}$

5.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1\sqrt{10}}$

6.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$

7.  $\sum_{n=1}^{\infty} (-1)^{n-1} \operatorname{tg} \left( \frac{1}{n\sqrt{n}} \right)$