Leveraging Tutorial Instructional Software to Enhance Classroom Mathematical Discussions: An Exploratory Mixed-Methods Study

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Abstract: *Using a convergent parallel mixed methods design, this exploratory study examined mathematics lessons in two third-grade classrooms to analyze the mathematical classroom discourse when using tutorial instructional software. A chi-square test for independence revealed statistically significant correlations between teachers use of the focusing and revoicing talk moves and student discourse levels of generalization and justification. The qualitative analysis found two major themes: (1) responsive teaching with technology, and (2) using digital representations to question students. From this analysis, the Facilitating Mathematical Discourse with Technology framework emerges connecting three key elements of instruction with technology: the students, the teacher, and the instructional software. The relationships among these elements illustrate the classroom interactions when using technology to enhance classroom mathematical discourse.*

Key Words: mathematics education, virtual manipulatives, classroom discourse, tutorial instructional software

Purpose of Study

The purpose of this exploratory study was to build theory and knowledge about how teachers leverage tutorial instructional software to support elementary students' discussion and learning of mathematics concepts. In the mathematics classroom, several tutorial computer software applications have been developed to assist teachers in addressing individual learning needs. For example, tutorial virtual manipulatives, either online or tablet-based, guide students step-by-step through mathematical procedures and concepts. Other commercially produced applications, such as ST Math (MIND Research Institute), and ALEKS (McGraw-Hill), provide teachers with detailed reports of individual student progress in addition to the tutorial software. However, the individual nature of such math tutorials does not—on its own—promote a rich classroom discourse community. As a tool for instruction, it becomes the role of the teacher to expand students' experiences with such tutorials to create a community of mathematics learners (National Council of Teachers of Mathematics [NCTM], 2014). This study examined the influence

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of tutorial instruction software on two teachers' instructional practices when teaching mathematics concepts.

LITERATURE REVIEW

VIRTUAL MANIPULATIVES

Beginning in the 1990s, virtual manipulatives (VMs) emerged as an alternative to physical manipulatives, expanding the possibilities for student use in the classroom. In 2002, Moyer et al. proposed a definition for these virtual manipulatives as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for construction mathematical knowledge" (p. 373). While this definition was beneficial at the time for identifying programs containing dynamic vs. non-dynamic virtual math objects, online platforms and devices have vastly changed in the past 18 years. Recently, Moyer-Packenham and Bolyard proposed an updated definition of VMs: "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features at allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (2016, p. 3). The extension and elaboration of the definition provides more structure for what qualifies as a VM. One main difference between a static object on a screen and a VM is the programmable features which allow objects to be manipulated while also representing mathematical concepts, relationships, and/or procedures. Virtual manipulatives now exist on platforms such as laptops, tablets, phones, and other touch-screen devices. This updated definition also allows for future technology expansion such as projected 3D or even holographic images (Moyer-Packenham & Bolyard, 2016).

GAMIFIED VIRTUAL MANIPULATIVES

While some virtual manipulatives exist in a more basic environment (i.e. single-representation), there are representations which have been "gamified", creating a more engaging learning experience (Moyer-Packenham & Bolyard, 2016). These VMs tend to have various representations of mathematical concepts as well as features of games such as goals, challenges, time constraints, and levels through which the student must progress (Deterding et al., 2011). Furthermore, these VMs have been found to actively engage students with disabilities in learning mathematics (Shin et al., 2017).

TUTORIAL VIRTUAL MANIPULATIVES

With the advancement of computer capabilities, Tutorial Instructional Software (TIS) packages and applications have been developed by mathematics education researchers as well as by publishers of mathematics curriculum resources. The tutorials dictate specific solution strategies and give direct feedback to users based on their adherence to those strategies in the form of pictures, numbers, and/or text. Research suggests that this direct feedback is valuable and effective in helping students to learn (e.g., Reimer & Moyer, 2005). However, TIS seems to be ineffective in promoting discourse among students without facilitation from a teacher (Anderson-Pence & Moyer-Packenham, 2016). Some TIS programs focus on procedural fluency and recall of math facts (e.g., ALEKS). Other programs and applications focus on conceptual understanding of mathematics by presenting students with interactive visual representations of the mathematical concepts (e.g., ST Math, NLVM Fractions–Adding). Overall, research indicates that interactive visual representations of mathematics (i.e., virtual manipulatives) positively contribute to students' learning of mathematics concepts (Moyer-Packenham & Westenskow, 2013).

Spatial-Temporal (ST) Math is a tutorial instructional software created by the MIND Research Institute which incorporates virtual manipulatives in a gamified environment. ST Math is an intuitive, online mathematics program designed for students in pre-K through grade 8. The software relies on non-linguistic, spatial-temporal representations of mathematics. This approach enables students to work through the program without language barriers (MIND Research Institute, n.d.). The tasks in ST Math offer no words or directions as all the concepts are displayed visually with symbols, numbers, and pictures. Students must solve puzzles to complete levels. These levels exist within different games with objectives tailored to master the state standards upon 100% completion. Students are guided through the levels by JiJi, a penguin who physically progresses to the next puzzle when students complete puzzles correctly or is blocked in cases of an incorrect answer. Feedback is given immediately within each puzzle to show students how their answer is correct or incorrect, allowing reinforcement of the concept or redirection. ST Math can be played on a desktop, tablet, laptop, or any supported mobile device. Curriculum is presented through objectives which can be modified by the teacher. Students can also play at home to work towards completion of the program. Pre-tests and post-tests are taken at the beginning and end of each objective to show growth and assess student understanding. All assessments and struggles can be closely monitored by the teacher. Representations in the game include number lines, place value charts, visual patterns, area models, fraction circular models, algebra tiles, and more (Wright, n.d.).

Virtual manipulatives have seen an uptick in use in classrooms around the country. However, their benefits seem to be contradicted across studies as some researchers have found positive results from using VMs (Steen et al., 2006; Watts et al., 2016) while others have found that VMs, such as ST Math, did not promote significant growth in students mathematical understanding (Rutherford et al., 2014). In an analysis of how ST Math use impacted student scores on state standardized tests, it was found that students in the third grade who used ST Math outperformed those in the control group, which was found to be statistically significant (Guise, 2017). These studies help to support the fact that there is still some contention about the advantages of the use of this specific virtual manipulative in classrooms, especially compared to the use of physical manipulatives. However, its potential cannot be dismissed and must be researched further as developing conceptual understanding and actively engaging students through the VM could have great impact on student learning (Bouck & Flanagan, 2010).

CLASSROOM DISCOURSE

Classroom discourse also plays a major role in students' learning. Students develop understanding as they interact with others through verbal or nonverbal communications and written word (Vygotsky, 1978). The teacher is responsible for setting the tone and culture of the classroom and for orchestrating discussions throughout the lesson. Appropriate verbal cues throughout the lesson made by the teacher serve to lead the students to a deeper understanding and reflection of their own knowledge (Chapin, O'Connor, & Anderson, 2013). During a lesson, teachers also make decisions based on informal assessments and observations. By paying close attention to what students are thinking and saying, the teacher can know which students to call on to provoke meaningful discourse (Mendez et al., 2007). He or she can also sequence students' solution strategies or ideas to build collective class knowledge or to assist individual students in making connections. This idea of scaffolding student discourse benefits students and their productive struggle in mathematics (Dale & Sherrer, 2015). Smith and Stein (2018) present five practices for teachers to promote student discourse and encourage this productive struggle while supporting students in a way that requires planning and reflection. These practices direct teachers to *anticipate* how students will solve problems, *monitor* and observe students as they work on the problems, *select* students to share their work with the class, *sequence* the specific order in which strategies will be shared, and explicitly *connect* students' strategies to each other through questioning techniques. To increase student understanding and ownership of essential mathematical practices (Hattie et al., 2017), students must be allowed to engage in discourse in a structured but student-centered environment (Brooks & Dixon, 2013).

With the addition of technology to the classroom environment, the responsibility of the teacher in facilitating mathematical discussion increases, as the teacher becomes responsible for orchestrating students' interactions with any technology being used as well as interactions with each other. Extensive research has been conducted on the teachers' role in facilitating classroom mathematical discourse (e.g., Gee, 2005; Herbel-Eisenmann & Wagner, 2010; Iiskala et al., 2011; Imm & Stylianou, 2012; Kotsopoulos, 2010; Nathan & Knuth, 2003; Wood & Kalinec, 2012, Mendez et al., 2007). However, few studies exist on how teachers facilitate discussions while using technology tools in their instruction (e.g., Anderson-Pence, 2014; Ares, Stroup, & Schademan, 2008; Evans et al., 2011; Sinclair, 2005).

Successful learners are actively and consistently engaged in their learning and have the capacity to connect their knowledge to real-world situations. The development of conceptual understanding requires students to make sense of math by giving them the "opportunity to develop their own rich and deep understanding of our number system" (Flynn, 2017, p. 8). While procedural fluency is an end goal, students should be able to make connections among multiple representations (visual, symbolic, verbal, contextual, physical) in or order to truly grasp a mathematical concept (NCTM, 2014). Hattie et al. (2017) outlines classroom practices and strategies that work best for students and reports effect sizes for these practices, claiming that those with an effect size of 0.40 or higher create desired effects in students. Discourse, student-centered teaching, and feedback were found to have an effect size of 0.82, 0.54, and 0.75 respectively, indicating these practices to be necessary for growth of conceptual understanding. Baroody et al. (2016), analyzed different approaches to subtraction to determine which approach fostered more conceptual understanding and led to transfer of knowledge. Results indicated that direct instruction was not as effective in developing concepts as allowing students to experiment with the numbers and discuss their findings.

Virtual Manipulatives and Classroom Discourse

Few studies have examined the effects of virtual manipulatives used in conjunction with purposeful classroom discourse on student conceptual understanding. Different types of VMs have been shown to have an impact on student-student discourse (Anderson-Pence, 2014), providing evidence that technology tools used in combination with worthwhile mathematical tasks can elicit higher-level thinking and discussion among students. However, discourse was examined solely in student pairs, and did not examine how technology was used within the whole classroom to allow the teacher to elicit further thinking. The Techno-Mathematical Discourse framework describes the relationships among technology tools, classroom discourse, and worthwhile mathematical tasks (Anderson-Pence, 2017). This framework provides a way for researchers and educators to think about the potential synergy between virtual manipulatives and student discussions. Reiten (2018) explored using a VM in whole group instruction, small groups, and individually with instructional guides to encourage different levels of thinking and to provide feedback in case of misconceptions or confusion. Reiten found that students who frequently got frustrated in math showed more perseverance with a VM and that the immediate feedback from the VM helped to quell those frustrations. These positive aspects of VMs support the idea that they can be used to promote discourse between students and between teachers and students, and this study will fill the gap in the literature by exploring this connection. Furthermore, this study extends the current research in examining the impact of a TIS with embedded virtual manipulatives on teachers' instructional practices and on students' mathematical discourse.

METHODOLOGY

RESEARCH QUESTIONS

The individual nature of TIS does not—on its own—promote a rich classroom discourse community. As a tool for instruction, it becomes the role of the teacher to expand students' experiences with such tutorials to create a community of mathematics learners. By focusing on the changing role of teachers to facilitate learning using technology and discourse, this study helps to build theory and knowledge about how teachers leverage TIS to support elementary students' discussion and learning of mathematics concepts. Using a convergent parallel mixed methods design, the researchers collected quantitative and qualitative data concurrently, analyzed the two data sets separately, and then merged the results to interpret the findings (Creswell & Plano Clark, 2018). The benefit of using a convergent parallel mixed methods design is that it provides greater insight into the impact of TIS in students' discussion and learning. The current study addresses the following research questions: 1) In what ways do teachers leverage tutorial instructional software to facilitate whole-class discussions of mathematics? What criteria do teachers use to make decisions on the direction and focus of a mathematical discussion? 2) What is the level of rigor of students' whole-class mathematical discussions when supported by tutorial instructional software?

PARTICIPANTS

This study included two third-grade classrooms of students ages 8–9 years in two different schools. The classrooms were purposively selected based on the teachers' frequency of assigning students to work with ST Math (an example of TIS) as part of their regular mathematics instruction. For the purposes of this paper, the teachers will be referred to using the pseudonyms—Ms. Martin and Ms. Whittaker.

DATA COLLECTION

Data was collected using video-recordings of whole-class discussions, observation protocols, and teacher interviews. Over the course of two months, a researcher observed and video-recorded mathematical discussions in the two third-grade classrooms for six class sessions each. In both classrooms, the discussions served to prepare the students for working individually on a specific section of ST Math. The video-recording equipment was set up in an unobtrusive location in the classroom, as to minimize any distraction for teachers and/or students. Each of the 12 observed discussions lasted approximately 15 minutes. Following each video-recorded discussion, the teacher participated in an audio-recorded semi-structured interview to reflect on the lesson.

A series of questions were created as the interview protocol to promote teacher reflection and analysis following the mathematics lesson. Not all questions were asked in every interview session, as the researcher asked appropriate questions related to the lesson observation. The interview protocol questions were as follows: What was your main learning objective for this lesson? How did you decide on the questions to ask students in the discussion? How did you decide on what features of the tutorial VM to focus on during the whole-class discussion? How well do you think the students responded when you asked, "…?"? What were some interesting/meaningful teaching moments during the discussion? What concepts do you feel your students are still struggling with?

The video and audio recordings were then transcribed and analyzed qualitatively and quantitatively as described in the following sections.

Qualitative Analysis

Data was analyzed qualitatively to examine patterns and trends identified in the video-recordings and teacher interviews, and the data was coded in cycles (Miles et al., 2014). The first cycle of analysis consisted of an initial read-through of the video and audio transcriptions. During this cycle, the researchers wrote memoing notes to identify preliminary codes and emerging patterns based on their first impressions of the data, focusing on teacher talk moves and student responses. In the second cycle of analysis, researchers read through the transcripts again, this time focusing on refining and relabeling the preliminary codes and solidifying patterns. A cross-case analysis was conducted to determine similarities and differences among the teacher talk moves and student responses. Finally, in the third cycle of analysis, overall themes were identified, and the articulation of a framework began to emerge.

Quantitative Analysis

Data was analyzed quantitatively to identify facilitations techniques used by the teachers and to determine the rigor of students' mathematical discussion during these lessons. Transcripts of the lessons were coded based on the teachers' talk moves and the students' responses to those talk moves.

Teacher talk moves were identified, defined, coded, and counted using a framework modified from Chapin et al. (2013). Each instance of the teacher spoke to the students was counted as one talk turn. Each talk turn was analyzed to determine which talk moves were present. Two researchers independently coded multiple lessons. After the initial lessons were coded, inter-rater reliability fell between 19 and 60 percent for some codes. It became clear that more focused definitions were needed. Subsequently, codes were combined and refined for clarity. After refinement, final inter-rater reliability was greater than 80% for all codes. Any discrepancy in coding was reconciled by discussion among the researchers.

Final codes for teacher talk moves were information, focusing, revoicing, funneling, restating, and waiting. Information was defined as instructional statements such as stating a fact of piece of information relevant to the lesson. Focusing was defined as a guiding question to elicit justification of student thinking, deepen responses, or further their initial response. This was viewed as questions into how or why something works conceptually. Revoicing was defined as restating a student's response. Funneling was defined as asked a student a specific question to elicit a certain answer. This type of question generally resulted in one-word answers and gave attention to procedures. Restating was defined as asking a student to restate another student's response. Waiting was defined as allowing adequate wait time for student response. Talk turns during which the teacher was engaging in classroom management or had to repeat a question due to management were not coded.

Student discourse was coded using the Robust Mathematical Discussion framework developed by Mendez et al. (2007). Based on this framework, students' speaking turns were coded to determine the intensity, justification, and generalizability of student answers as well as the level of student collaboration. Intensity of talk turns was coded as either an elicited answer or a volunteered answer. If a student joined the discussed without a prompt from the teacher, it was coded as volunteer but if the teacher called on the student, it was coded as elicited. Justification was coded as either statement, explanation, or proof. A student turn was coded as statement if the answer was stated but no explanation was given. Explanation was chosen if the student explained why the answer was correct or how they went about finding it. The student turn was coded as proof if the student gave a logical argument for their answer or a counterexample. Generalization of answers was coded as either concrete, comparison, or application. Concrete was the chosen code if the statement did not expand beyond the context of the current math problem. Comparison was coded if the student compared to one other context. Application was chosen if the student showed pattern recognition, applied their answer to multiple contexts, or generalized their answer for all situations. Student collaboration was labeled as Building and was coded as either an unrelated idea, response, or elaboration. A student turn was coded as unrelated idea if the student directly responded to the teacher or responded to another student in a way unconnected to the lesson. Response was coded if a student agreed or disagreed with a peer but did not elaborate. Elaboration was coded if the student continued the thinking of another student and added onto their previous response. Student talk turns were not coded if they were making a comment unrelated to lesson or speaking to a peer about an unrelated topic.

Once teacher talk moves and student discourse were coded, individual code counts for each lesson were combined for an overall total of each code. Percentages were then calculated to determine proportions of teacher talk moves and levels of discourse from student responses.

Three separate chi-square tests for independence were used to analyze the relationships between the categorical variables of teacher talk moves and student responses. The chi-square test was chosen as a non-parametric tool designed to analyze nominal group differences, comparing observed phenomenon with expected results (McHugh, 2013). As the talk moves and student discourse responses were not ranked but were merely categorical in nature, this test was the most appropriate. The coded teacher talk moves and student responses were used as the observed values. Expected values were calculated to find the counts that would be expected if there were no relationship between the variables. Three separate tests were conducted to focus on justification, generalization, and building. The differences between the observed and expected values were calculated with the chi-square statistic and p values were found to determine statistical significance. Instances for combinations of variables were counted by looking at the transcripts of the lessons, locating the instances of each talk move, and then finding the student response that immediately followed the teacher question or statement. For example, a teacher talk move of focusing followed by a student response of comparison was one instance of the combination of those two variables. Any combination that did not have more than five instances were removed from the chi-square calculations.

RESULTS

The following sections report the results of the quantitative and qualitative analyses for each research question of the study. The quantitative results show the frequency of each type of teacher talk move, the frequency of student discourse levels, and the chi-square tests of independence for the teacher talk moves and student response levels. The qualitative results provide a description, categorization, and interpretation of the teacher talk moves and student discourse in these lessons.

RESEARCH QUESTION 1: TEACHER TALK MOVES TO LEVERAGE TUTORIAL INSTRUCTIONAL SOFTWARE

Research questions 1a and 1b were: In what ways do teacher leverage tutorial instructional software to facilitate whole-class discussions of mathematics? What criteria do teachers use to make decisions on the direction and focus of the discussion? To answer these research questions, a quantitative analysis of the teacher talk moves in different mathematics lessons, and a qualitative analysis of the instructional session observations and the teacher interviews and were conducted.

QUANTITATIVE ANALYSIS

The coding of the transcripts from the 12 instructional sessions revealed 1,176 distinct instances of teacher talk moves. Figure 1 shows the frequency of talk moves throughout the lessons with funneling (39.29%) and revoicing (29.25%) occurring the most often. The restating and waiting teacher talk moves occurred only one and three times, respectively, resulting in miniscule frequencies and impact.

Table 1

Frequency of teacher talk moves

Talk Move	Frequency (N = 1,176)	%
Information	231	19.64
Focusing	135	11.48
Revoicing	344	29.25
Funneling	462	39.29
Restating	1	0.09
Waiting	3	0.26

QUALITATIVE ANALYSIS

After coding and analyzing the interviews and observations with the teachers who utilized TIS in their classrooms, two major themes emerged from the data. To leverage TIS and facilitate whole-class mathematic discussions, the teachers engaged in *responsive teaching* and *questioning of students*. The teachers were intentional in the ways they planned and crafted their lessons, using TIS as the tool for facilitating instruction and mathematical discussions. They wove *responsive teaching* and *questioning of students* into every aspect of their lessons. Within the two themes, there were several strategies and purposes that the teachers implemented. Teachers engaged in responsive teaching by connecting to students' prior knowledge and targeting areas of struggle. Teachers used questioning throughout the lessons to engage in mathematical discussions that expanded on student answers and problem-solving strategies, connected prior knowledge, redirected misconceptions, and demonstrated multiple solution paths.

RESPONSIVE TEACHING WITH TECHNOLOGY. The two teachers, Ms. Whitaker and Ms. Martin, utilized TIS regularly in their classrooms, and they had specific patterns in place when using the software to engage in whole group mathematics discussions. Even though the teachers were at different schools, they had similar routines when implementing TIS in their classrooms. To begin each lesson, both teachers reviewed what the students had been working on with their math curriculum or on the TIS to *engage prior knowledge.* Ms. Martin reviewed math vocabulary words as well as part of the prior knowledge review. The students were actively engaged in the review of the material by answering questions that connected to their prior knowledge or participating in the review of the material. The teachers then displayed a specific activity or problem from the TIS on a Smart Board and reviewed it with the students as a whole group. The teachers guided the students through the game or activity on the TIS, asking students to describe how to complete the activity and giving individual students the opportunity to "teach" the rest of the class as well. According to the interviews, the teachers chose the TIS activity to discuss as a class, based on the *areas of struggle or need* that the teachers noticed with their students. As Ms. Martin described, "We are looking at what types of problems kids have gotten hung up on, so if there are multiple students who got stuck on a certain level or certain type of game, that's what we are pulling out to do a lesson on to show how to solve, to give them more strategies than what they have already."

The teachers were attentive to the areas in the TIS where students were having difficulties and designed their lessons around the areas of struggle in order strengthen student understanding. In addition to areas of struggle on TIS, they also connected areas of struggle with the math curriculum to problem solving on the TIS. For example, Ms. Whitaker's students were struggling with skip counting by twos and threes, which was a part of the day-to-day curriculum as well as the work on the TIS. She reviewed the strategies for skip counting that they had learned in their math curriculum and showed students how to apply those strategies in the TIS activities. This kind of responsive teaching was evident in every interview and observation with both teachers.

The teachers also reviewed activities or strategies in the TIS with the purpose of *ensuring student understanding.* The teachers were responsive to areas of struggle and purposefully engaged in direct instruction of specific games or activities on the TIS to strengthen student understanding of a particular activity or strategy. The teachers displayed a solid understanding of their students' abilities in each observation and interview. During the interviews, the teachers identified what went well in each lesson, what students understood or grasped, and areas where students needed more support. While both teachers had a plan prior to each lesson, they would respond to students' questions, answers, and problem-solving strategies to build on the lesson and cover the material that students needed to ensure their understanding. As Ms. Whitaker described, she started with a plan based on student need and then "if they have questions [during the lesson], I build off of that." In this way, the students' questions and learning needs drove the teachers' instructional decisions.

By engaging in responsive teaching to connect prior knowledge and clarify areas of struggle, the teachers created mathematical classroom discussions that met the students' specific needs. Students responded to the teachers by answering questions, demonstrating their knowledge on the TIS, and solving problems.

USING DIGITAL REPRESENTATIONS TO QUESTION STUDENT. Throughout each lesson both teachers actively engaged in questioning of the students for several purposes. The use of questioning was used throughout each lesson and smoothly transitioned from one purpose to another, creating more opportunities for student responses and dialogue while interacting with the TIS tool. One use of questioning was for students to *expand on the answers and explain the problem-solving strategies* they used. The teacher would prompt the student who provided an answer with a question in order for the student to provide more details regarding his or her answer, provide a reason for the answer, or describe how he or she solved the problem. The following is an example of Ms. Martin using questioning so a student would expand on his answer. The students were working through an activity on the TIS together, which was displayed on the Smart Board.

Ms. Martin	What operation do we need to use?
Student 1	You need to add some
Ms. Martin	Ok, why do you say I need to add some?
Student 1	One of the monsters wants two and one [monster] wants one, and there's only one pie, so one monster would eat that, and you need two more pies, so you'd have to add two more pies so there's enough.
Ms. Martin	Very good. Alright, what operation do I need to do to solve this problem?
Student 2	Subtraction.
Ms. Martin	Subtraction. Ok tell me how you figured that out.

By asking the students to explain how they figured out the solution to a problem or activity on the TIS, they engaged in mathematical dialogue that expanded student answers and provided concrete examples for students to understand the problem-solving strategies. Ms. Whitaker also used questioning to prompt students to explain how they arrived at their answers and solved the problem. The TIS activity was displayed on the Smart Board so all students in the class were able see the activity as they worked through the problem together.

Ms. Whitaker	Ok, now am I done?
Students	No!
Ms. Whitaker	No? Why not?
Student 1	Because there's like . . .
Ms. Whitaker	You can come up and point it out to us.
Student 1	There's a corner right here so that means there's another cube behind this one.
Ms. Whitaker	Ok. So, do you see that little corner that she's pointing to? That's the top of another row of cubes.
Student 1	There's two more.
Ms. Whitaker	It looks like the corner of this one is about the same height as these. So, if it's the same height, how many [cubes] are going to be back there?

By asking students "why?" and how they figured out the solution to a problem, the teachers allowed the students to serve as key components of the mathematical discussions. The teacher provided the prompts, but the students explained their problem-solving strategies and answers in their own words.

In addition, the teachers used questioning to *connect students to prior knowledge*. In one TIS activity, students had to be able to read thermometers, identify temperature, and solve equations using thermometers. Ms. Martin used questioning to engage students' prior knowledge of thermometers so that they would be prepared to learn how to do the new TIS thermometer activity.

Ms. Martin	First, what is a thermometer?
Student 1	It tells us the temperature.
Ms. Martin	It tells us the temperature. So, what kind of temperature could we be checking with a thermometer?
Student 2	Hot or cold.

Ms. Martin	We are going to see how hot or cold something is. What kind of things do we use a thermometer on?
Student 3	There's one with a stab on it that you can tell how hot food is.
Ms. Martin	Ok, so we can put it in our food like if you're cooking a great big turkey and you need to know if it's cooked enough. You put a thermometer in it to see how hot it is and make sure it has cooked long enough. What do you think?
Student 4	A refrigerator.

The conversation continued with students identifying different kinds of thermometers. Ms. Martin connected to their prior knowledge by asking students to identify thermometers, and then after each student answered, she expanded on their answers to connect it to the topic they were discussing.

Ms. Whitaker also used questioning to connect to students' prior knowledge. In one activity that involved skip counting, she asked students to provide examples of skip counting by twos and threes, and even and odd numbers. Students answered by providing examples and then she introduced the skip counting activity on the TIS. Practicing skip counting before engaging in the activity on the TIS prepared students to be more successful with the activity as they had reviewed the concept together before completing the TIS activity.

Questioning was also used throughout the lessons to *redirect and clarify student misconceptions.* As discussed previously, the teachers chose the TIS activities to review based on student need or areas of struggle. The use of questioning techniques while reviewing the TIS activity allowed the teachers to identify the specific areas where students misunderstood or struggled with the activity. Since the TIS activity was displayed on the Smart Board, the teachers could go step-by-step through each activity with the class and review any area where students were struggling. In one interaction, a student told Ms. Whitaker that he was stuck on an activity and did not know what step to take next.

Ms. Whitaker	Why are you stuck on this? What do you think is going wrong here?
Student 1	The cubic units.
Ms. Whitaker	The cubic units . . .
Student 1	And how they are.
Ms. Whitaker	Ok, so how they're set up and how to count them? Or you can't see . . .
Student 1	Like how they're set up.
Ms. Whitaker	Ok, because you can't really see all of them. So, if you can't see all of them, does that make it difficult? How could we do this to make this a little bit easier? How many of you think you know this one...?
Student 2	I just counted them all. When I was stuck, I just started counting them.
Ms. Whitaker	You just started counting them. So, let's try that.

The students worked through the problem together by counting the cubic units. Through questioning, the teacher was able to identify why the student was stuck on the specific level, and she was able to engage other students in the discussion by asking how they solved the problem. Instead of providing the answer to the students, the teacher worked through the area of difficulty, asked students to participate in solving the problem, and engaged the class in mathematical discussions to redirect and clarify misunderstandings. When students answered incorrectly during the review of the TIS activity, the teachers used questions to redirect the students to the correct answer.

The teachers also used questioning to discover *multiple solution paths* that students used to complete the TIS activity. As Ms. Martin described, “we want [the students] to be explaining their thought processes to show that there are different ways to look at every problem, most problems, to find the solution.” The teachers were intentional in asking students to provide other ways to solve the problems so the students would understand that there is often more than one way to find the solution. Asking students to provide examples of how they solved the problems differently on the TIS provided the teachers with an assessment of the ways in which students were approaching the problems on the TIS and their thought processes for solving them. In one example, Ms. Martin specifically asked her students for other ways to solve the problem that they were working on as a class.

Ms. Martin	Is there another way we could solve this problem? What’s a different operation we could use to solve this? What do you think?
Student 1	Multiplication.
Ms. Martin	Multiplication. How would I use multiplication to solve this problem?
Student 1	You could multiply 5 until you get to 220 and still know you need 5 more to get up to 50.
Ms. Martin	Ok, that’s one way of doing it. What if I just use the numbers that are here in the question? What if I wanted to use multiplication?
Student 2	You could do 50 times 5.
Ms. Martin	If I do 50 times 5, what’s that going to tell me?
Student 2	That you still don’t have enough because it’s more than 220. Because 50 times 5 is 250.
Ms. Martin	So [multiplication] shows me that I need 250?
Student 3	Chocolate chips.
Ms. Martin	Chocolate chips to make 50 cookies. This problem we found two different operations, two different ways to do it.

By continuing to prompt the students with questions to find multiple solution paths, the teachers gave students the opportunity to share their problem-solving strategies and remain engaged in the discussion and dialogue. In another class example, Ms. Whitaker also specifically asked her students for another way to solve the problem on the TIS.

Ms. Whitaker	Nine. We just counted them one by one. How else can I count them besides just counting each individual cube?
Student 1	Since there is four on the bottom and four on the top, you can do four, eight, and another one up there, you can add that one.
Ms. Whitaker	Ok, so I can skip count and I can group them. He said this is four and this is four. So how much is that altogether?
Students	Eight!
Ms. Whitaker	Eight, plus one more makes?
Students	Nine!

The teachers engaged in questioning throughout each lesson so that their students remained active in the mathematical discussion and strengthened their understanding of the TIS activity, problem solving strategies, and the multiple ways to solve problems. The teachers were responsive to their students' needs throughout each lesson by building the lesson based on areas of struggle, responding to the answers their students provided to guide the direction of the discussion and lesson, and using questions throughout to prompt students to connect to prior knowledge, explain their answers, provide multiple solution paths, and clarify or redirect incorrect answers. The TIS was the tool in the interactions between the students and teachers, but it was the teachers' responsive teaching strategies that created the discussions and dialogue that occurred in each lesson.

Research Question 2: Level of Rigor of Student Discourse

Research question 2 was: What is the level of rigor of students' whole-class mathematical discussions when supported by tutorial instructional software? To answer this research question, transcripts from the 12 instructional sessions were coded for levels of generalization (concrete, comparison, application), justification (statement, explanation, proof), and building (unrelated idea, response, collaboration). Overall, most students' speaking turns were coded as concrete, statement, and unrelated idea in their respective categories (see Table 2). Among student speaking turns coded for generalization, concrete speaking turns occurred 597 times (96.6%). Comparison followed with 20 occurrences (3.24%), and application with one occurrence (0.16%). Among student speaking turns coded or justification, students answered with statements 573 times (92.87%) and explanations only 44 times (7.13%). No instances of proof were coded in these lessons. Among student speaking turns coded for building, unrelated ideas occurred 626 times (96.9%), response codes occurred 19 times (2.94%), and elaboration had one occurrence (0.15%).

Table 2

Frequency and Percentages of Students' Discourse Levels of Rigor

Level of Rigor	Frequency	%
Generalization		*N* = 618
Concrete	597	96.60
Comparison	20	3.24
Application	1	0.16
Justification		*N* = 617
Statement	573	92.87
Explanation	44	7.13
Proof	0	0.00
Building		*N* = 646
Unrelated Idea	626	96.90
Response	19	2.94
Elaboration	1	0.15

Relationship Between Teacher Talk Moves and Students' Speaking Turns

Chi-square tests for independence were used to compare types of teacher talk moves and rigor of student discourse in the generalization, justification, and building categories. For each category of student discourse (generalization, justification, building), every teacher talk move was tabulated with the level of rigor of the student speaking turn immediately following it. For example, the teacher talk move, "12. Okay. Why did you choose to count down instead of across by twos?" was coded as focusing. The student speaking turn immediately following this teacher talk move, "Because that's just...oh! Because it's like this (pointing to the vertical squares) and if we did 2, 4, 6, 8, 10 (horizontally), it would be like that." was coded as comparison for generalization. Therefore, this teacher talk move/student speaking turn pairing was marked as focusing/comparison. Results are displayed in Table 2.

The results of the chi-square tests examining teacher talk moves and levels of generalization, χ^2 (3, N = 1031) = 14.32, p = 0.002, and levels of justification χ^2 (3, N = 1039) = 116.16, p < .001, were significant at a p < 0.005 level. These results indicate that when teachers use focusing and revoicing talk moves, they have significant impact on student levels of generalization and justification. Chi-square test results for building were not significant, χ^2 (3, N = 1074) = 2.09, p = 0.552. These results show minimal connection between any of the teacher talk moves and students' building on each other's ideas. Table 3 includes the frequency data for teacher talk moves and subsequent student speaking turns. The frequencies of student speaking turns coded as the highest level of each category (application, proof, and elaboration, respectively) each had less than five instances, and were therefore removed from the chi-square analysis.

Table 3
Crosstabulation of Teacher Talk Moves and Student Discourse Level of Rigor

Level of Rigor	Information	Focusing	Revoicing	Funneling	χ^2
Generalization					χ^2 (3) = 14.32
Concrete	187 (14.2)	104 (-83.8)	281 (-61.8)	427 (98.27)	p = 0.002*
Comparison	5 (-14.2)	8 (83.8)	14 (61.58)	5 (-98.27)	N = 1031
Justification					χ^2 (3) = 116.16
Statement	187 (90.77)	78 (-336.53)	272 (-9.63)	425 (150.44)	p < .001*
Explanation	5 (-90.77)	36 (336.53)	23 (9.63)	13 (-150.44)	N = 1039
Building					χ^2 (3) = 2.09
Unrelated Idea	190 (-26.79)	120 (-31.26)	303 (5.41)	432 (36.47)	p = 0.552
Response	7 (26.79)	5 (31.26)	8 (-5.41)	9 (-36.47)	N = 1074

Note. Adjusted standardized residuals appear in parentheses next to group frequencies.
*statistically significant, p < .005

Summary of Findings

In summary, the results of this study identified the factors of student struggles and understanding, prior knowledge, and student engagement in the discussion as influencing teachers' decision-making when using tutorial instructional software in their lessons. These aspects of their lessons influenced how the teachers responded to students and how they posed questions to facilitate discussions. The teachers in this study primarily used funneling and revoicing talk moves to facilitate the whole-class discussion. Results of this study also indicated that teacher talk moves have a significant influence on students' generalization and justification discourse in a large group setting. However, these same talk moves did not significantly impact the students' level of building on each other's ideas.

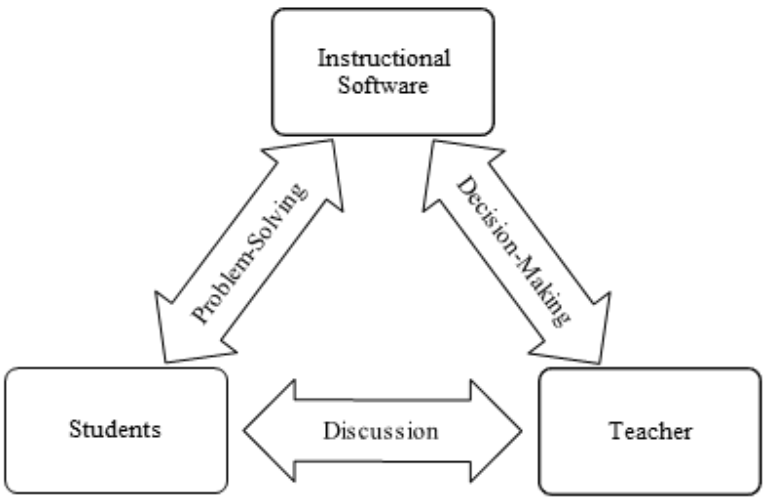
DISCUSSION

RESPONSIVE TEACHING WITH TECHNOLOGY FRAMEWORK

From the themes identified in the qualitative analysis, a framework emerges that connects three key elements of instruction with technology: the students, the teacher, and the instructional software. The relationships among these elements illustrate the classroom interactions when using technology to enhance classroom mathematical discourse (see Figure 1). The teacher interacts with the instructional software (TIS) when making decisions regarding the level of support to give to students and what features of the TIS to focus on. The students interact with the mathematical representations in the TIS as they problem solve tasks and experiment with mathematical concepts. Finally, the teacher and students interact with each other as they both use the TIS to represent and communicate their mathematical thinking. Without this purposeful interaction between teacher and students, the instructional software is not as beneficial. The use of focusing questions to have students explain their thinking strengthens their problem-solving skills and builds conceptual understanding.

Figure 1

Facilitating Mathematical Discourse with Technology Framework



This study highlights the importance of a teacher's role in providing students access to mathematical concepts via technology. In and of itself, technology can obstruct learning if a student is not familiar with it. However, through teachers' facilitation of discussion, problem solving, and building of community, students can become empowered to navigate instructional software and utilize its value as a learning tool. This study found that teachers use different types of talk moves to facilitate different levels of discourse among their students and to maximize the value of using a TIS in the classroom. The results showed that the teachers used more funneling moves than any other to guide student answers. Revoicing closely followed, allowing the teachers to affirm the right path to the answer while including others in the conversation.

However, the use of focusing questions to have students dig deeper into the "why" as the levels get harder required the students to master the concept instead of just answering teacher questions. In one of the lessons, this evolved into peer-to-peer discussions and small group discussions about the particular level in the TIS. By modeling the work and asking focusing questions, the students were able to talk about their understanding and share with others. On the other hand, most of the lessons involved the teacher engaging students one-on-one for answers or involving the whole class at once to collectively give answers. There was minimal student-to-student collaboration and discussion which is consistent with the findings from Anderson-Pence & Moyer-Packenham (2016). This shows the importance of teacher decision making when approaching TIS. When the goal of learning mathematics is to understand concepts and apply them to other contexts, teachers must choose their questioning strategies wisely.

The lessons in this study focused on mathematical ideas already familiar to the students and reinforced prior learning. This reinforcement is seen in the abundance of funneling talk moves used as compared to focusing talk moves used. The teachers in these lessons began their work with the students by activating background knowledge whether it was about thermometers, scales, or vocabulary surrounding equivalent fractions. The TIS acted as more of another opportunity for the students to be exposed to the concept which focused the discussion on answering step-by-step rather than conceptually working through a problem using different strategies. Even Ms. Martin noted in an interview that "The discussion about comparison, was where their gap was, which it wasn't the math part of it, but *their explanation of what was happening was a little bit lacking.*" She went on to explain how she decided to use the explanation of another student who was looking at the thermometer in the TIS in a different way than the others.

During the lessons, the teachers noticed that students were stuck and that they needed to slow down regarding certain areas of need. According to Ms. Whittaker, "They get stuck and they don't know what to do. Then they don't progress. So, having those little mini-lessons *makes them feel a little bit more confident and have that familiarity*." This kind of responsive teaching with TIS shows how teachers can identify an issue and use the TIS to model learning with whole-class discussions. However, this breaking down of the problems into more manageable parts also leads to using mostly funneling questions. As another example of responsive teaching, teachers use students' previous explanations to help other students understand certain concepts.
According to the results of this study, the teacher talk moves do not have an impact on student collaboration or building discourse with respect to TIS.

With the use of TIS, the results of this study showed two different teachers using funneling for almost 40 percent of the total talk moves. This speaks to the type of learning that occurs with TIS, showing step-by-step how to reach an answer. When modeling this for students, funneling is an obvious choice to actively point students to the next right answer. However, the results of this study show that focusing questions lead to more rigorous levels of student discourse. By guiding students to an answer and requiring a higher level of thought, students are more apt to apply their knowledge and compare the problem they are working on to different contexts. Teachers seem to easily fall into the trap of funneling to get the next right answer from their students. When this occurs, it seems as though students understand because the teacher is hearing the expected responses. However, students are only answering the one specific question being asked of them.

Purposeful questioning to obtain contextual understanding from students by using focusing talk moves increases the level of rigor according to the results. However, these levels were low when using the TIS. This shows that when using a TIS, teachers should be more deliberate about guiding students to think more deeply instead of pointing them to the next right answer. Focusing and revoicing talk moves are vital to support a deep and rigorous mathematics discussion. Funneling talk moves do have a place within mathematics instruction. However, they must be used intentionally to make the most impact on student discussion.

CONCLUSION

The patterns and trends identified in this study contribute to the existing literature on the complex issues that surround mathematical discourse and the use of technology in the classroom. Specifically, the results a) extend the existing literature on how teachers leverage technology in mathematics instruction, and b) suggest effective practices for engaging students in meaningful mathematics discussion while using technology. This study also provides a framework for effectively facilitating mathematical discourse with technology that illustrates the importance of relationships and meaningful discourse when using technology in the classroom. Furthermore, it emphasizes a teacher's role as the instructional decision-maker in the classroom and the need for teachers to make effective use of technology to further students' mathematical understanding.

The main limitation in this study was sample characteristic. This study was designed as an exploratory study of how teachers leverage tutorial instructional software to support elementary students' discussion and learning of mathematics concepts. The sample size was small and limited to two third-grade classrooms. With so few classrooms participating in the study, the variations in students' and classrooms' characteristics can profoundly influence the results. The students in these classrooms were accustomed to working with the TIS and attended schools serving a diverse lower-middle class population. It is possible that findings may differ with other populations such as students of different ages, ethnic groups, or socio-economic statuses. Other factors that may have influenced the results of the study include the students' familiarity with the TIS, perception of the TIS, and overall academic achievement. However, these factors were beyond the scope of this study.

The Facilitating Mathematical Discourse with Technology Framework, which emerged from this study, points to potential areas of further research. For example, future research could examine more deeply how teachers use the representations presented in the TIS to encourage student discourse and mathematical connections. While, this study did not focus on achievement factors, a future study could be conducted to determine how students of varying achievement interact with TIS and how they participate in class discussions. Likewise, research could be conducted on how the teachers' experience and familiarity with the TIS impacts the classroom discussion. Examination of these factors could deepen understanding of how students and teachers interact with TIS during mathematics instruction.

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