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# DETERMINATION OF STUDENTS' PERFORMANCE IN TESSELLATIONS

**Funda AYDIN-GÜÇ, Mihriban HACISALIHOĞLU-KARADENIZ**

**Abstract:** The purpose of this study was to investigate the tessellation performance of 8th-grade students and factors affecting their performance. Student performances have been examined in two dimensions: identification and completion of tessellation. In this context, a diagnostic test containing two main questions with the content of determining whether the motifs are tessellations and completing the tessellations was prepared. The test was applied to 40 eighth grade students. The findings show that students exhibited good performances in determining whether a pattern is tessellation and their performances were close to each other according to tessellation types. It was found that students' completion performances of the tessellation varied according to the tessellation types. In addition, the factors affecting these performances were also revealed as a result of the study.

**Key words:** geometry, transformation geometry, pattern, tessellation, students' understandings

## 1. Introduction

Math is the language of the universe and allows you to interpret real life. Perhaps the most concrete examples of the issues that mathematics has dealt with in real life are patterns. There is always a pattern in the order of our behavior during the day, inside a sunflower, in the snail shell, and the flow of water. Therefore, students develop a series of informal ideas about some concepts, such as patterns, gradually through their experiences in everyday life (National Council of Teachers of Mathematics [NCTM], 2000). In order to understand these relationships and transfer them to future thoughts (exp. Understanding the real life, functions, problem solving, algebraic thinking), patterns have been added to the content of mathematics teaching programs of countries (Australian Education Council [AEC] Curriculum Corporation, 1994; NCTM, 2000; Ministry of Education [MOE], 2009). Tessellations, which are the application of patterns in art, have been taken into consideration, and "transformation geometry", in which patterns and tessellations are covered, has been included in mathematics curricula implemented in Turkey as a sub-learning domain. In the mathematics curriculum from middle school, "Transformation Geometry", a sub-learning domain of the "Geometry and Measurement" learning domain, is taken into consideration from a very early age. In the $7^{th}$ grade, "determination of translation or reflection transformations in patterns, motifs, and other similar images" activities are recommended to achieve "being able to form the image of a planar shape that emerges as a result of successive translations and reflections" learning outcome. In the $8^{th}$ grade, "determination of transformations in various patterns and tessellations" activities are recommended to achieve "being able to form the images of shapes that emerge as a result of maximum two successive translations, reflections, or rotations" learning outcome. As is seen above, not only patterns but also tessellations formed with patterns are treated as part of mathematics education, and students are expected to look at certain tessellations from a mathematical perspective. Therefore, tessellation must be defined in the first place.

### 1. 1. Patterns and Tessellations

Patterns are defined as an orderly arranged repeated combination of a set of elements such as geometric figures, symbols, actions, or mathematical objects (e.g. numbers, shapes) (Fox, 2005;

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Water, 2004). For example, it is a pattern that we wake up in the morning, wear our slippers, wash our hands and face, and brush our teeth. Setting a square and a triangle in type square-triangle-square-triangle, respectively, is also a pattern. The order of numbers in the form of 2,4,6,8,10... is also a pattern. Figure 1 shows an expanding pattern.

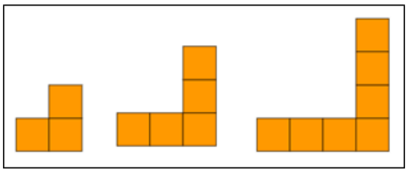


**Figure 1.** *Expanding pattern*

In mathematics, tessellations are repeated patterns of geometric shapes (tiles) that fill the plane with no gaps and overlapping tiles (Billstein, Libeskind & Lott, 2015). It should be noted here that there should be no gaps or shapes that should form a pattern so that the figures do not overlap as defined by the definition of tessellation. Here, each piece forming the pattern is called tile. Most importantly, the tilting must cover an infinite plane and this is easily achieved by finding a pattern that repeats in every direction (Eberle, 2014). Even if a shape can cover a random wide area, there is no general test to know if it can continue to cover the entire plane (Eberle, 2011). This is one of the obvious points of the tessellation theory (Schattschneider & Senechal, 2004). In the current study, the tessellation is considered as a structure consisting of a limited area that repeats in all directions indefinitely throughout the area.

The construction of tessellations can be done with one or more of the movements of translation, rotation, reflection, and translation reflectance. Any tile given in tessellation can be perpetually maintained in all directions (Değer & Değer, 2012). As can be understood from the definitions, the learning outcomes expected from the students about tessellations can be listed as follows:

1. Determine shapes/shapes to cover a surface with no space (tile creation)
2. Edit tiles so that they do not overlap while tiling

It can be understood that what is expected from the students about tessellation is only an area tiling. But what is expected is more profound. Tessellations should be understood both as a mathematical process and as a mathematical object. In the case of tessellations, we are dealing with a geometric construct that cannot be fully represented in finite space and therefore cannot be fully grasped by empirical abstraction alone (Eberle, 2011). Tessellations, which are the arrangement of shapes to form spatial patterns, involve patterning skills and knowledge of shape, space and angle (Waters, 2004). When the corners of the tiles are brought together, the total of angles should be $360^0$ to avoid overlapping tiles and no gaps. While the angle in tessellations is a concept that can be used with older age group students, the concepts of translation, rotation, reflection, pattern skills and knowledge of shape, space can be used with smaller age groups. Therefore NCTM (2006) recommends that the tessellations be examined in the fourth grade because children of this age can begin to study the transformations. At this age, children can coordinate shapes to global patterns, understand symmetry, and create patterns in a variety of complex ways. As can be seen, students from the fourth grade can be asked to think about mathematical meanings. This deep sense is emphasized by the The Republic of Turkey Ministry of National Education [MNE] (2009) by giving examples of teaching practices for younger age groups (about 9-13 years). MNE (2009) suggests that in teaching tessellations, “activities should be aimed to create images of shapes as a result of two consecutive translations, reflection or rotation” and “activities should be aimed to determine translation or reflection transformations in patterns, motifs, and similar visual images“. The translation, reflection or rotation movements mentioned here should be done in such a way that there will be no gaps on the surface and the shapes will not overlap. Therefore, students should know this dimension of the description of tessellation. Besides, students are expected to determine a pattern, motif and similar images of translation or reflection transformations. This has an important role in the continuation of the existing tessellation.

Considering all of this, two questions about the students' tessellation performance are taken into account:

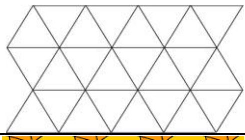
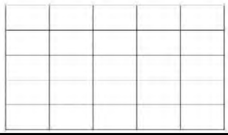
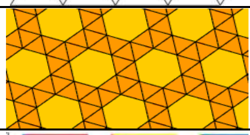
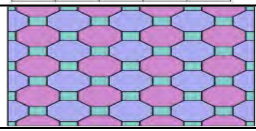
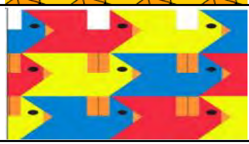
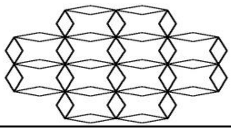
1. Do the students know the definition of tessellation encompassing the characteristics highlighted above? Or do they see every motif as a tessellation?

2. Can the unit shape (tile) which is formed to tessellation with various movements of translation, rotation and reflection be determined?

Considering these dimensions, it can be said that the performance expected from the students regarding the tessellation is "identifying the tessellation" and "completing the tessellation".

Since tessellations are composed of geometric figures, they can be categorized based on the properties of the geometric figures used in forming them. In this sense, in mathematical terms, tessellations fall into three categories: regular, semi-regular, and irregular (Van De Walle, 2004).

**Table 1.** *Tessellation Types*

| Tessellation Types | Property | Examples |
|---|---|---|
| **Regular Tessellations** | Only one of three regular polygons (square, triangle, and hexagon) is used. |   |
| **Semi-Regular Tessellations** | More than one different regular polygon is used. |   |
| **Irregular Tessellations** | Irregular polygons are used. |   |

As it is clear from the regular tessellation examples given in the table 1, these tessellations consist of polygons whose edge lengths and angles are equal. The edge lengths and angles of the figures used in semi-regular tessellations are equal, but, differently from regular tessellations, they are formed by using more than one regular polygon. While regular and semi-regular tessellation motifs are made up of regular polygons, irregular tessellations include irregular polygons.

Main purposes in adding tessellations to mathematics curricula was to reveal the cognitive schemes of students, to improve their psychomotor skills, and to teach mathematics based on the daily life activities from abstract to concrete (MNE, 2009). Besides, tessellation designs combine mathematics with art (Hatfield, Edwards, Bitter & Morrow, 2000; O'Daffer, Charles, Cooney, Dossey & Schielack, 1998) and reveal the incredible beauty of geometry in areas such as textiles, interior design, industrial design. Escher's patterns can be given as examples. The student's aesthetic preferences stand out in the tessellation activities (Eberle, 2011). Besides to aesthetic preferences, students need to decide which geometrical shapes are more appropriate for decorating a particular area and to take into account the characteristics of the shapes (Callingham, 2014). To fill the plane without overlapping or gaps, shapes must have dimensions of up to exactly $360^0$ when arranged around a point. This requires students to analyze shapes. The square grid, which is the simplest tessellation, is a basic concept in mathematics and is essential for basic and graphic studies. In this context, it can be said that understanding what tessellations mean is a prerequisite, to create tessellation and to continue the tessellation. In this context, students need to sort out on the works related to the improvement of their performance of the tessellation to determine which kind of tessellations they consider to be decorative and which kind of tessellations cannot be carried on. When the literature is examined, it is known that working with tessellations helps students to understand many concepts related to geometry (Van Hiele-Geldof etc. Fuys, 1984; Fuys, Geddes & Tischler, 1988; Serra, 1993; Wheatley & Reynolds, 1996). If tessellation is a concept that aims to improve students' understanding of geometrical shapes, children may be asked to define their understanding of tessellations, because misunderstandings of this subject may affect the development of other spatial ideas (Callingham, 2004). The studies show that most of the students in primary school can only describe the tessellations in a visual way (Callingham, 2004). It is known that students can make tessellations using geometric objects and shapes from the first grade (Eberle, 2014; Kılıç, Köse, Tanışlı & Öndaş, 2007; O'Daffer et al. 1998; Ward, 2003). On the other hand, studies showed that children do not fully understand finite-square sequences before the third or fourth grade (Battista, Clements, Arnoff, Battista & Borrow, 1998; Outhred & Mitchelmore, 2000). It is also known that it was difficult for students to understand the infinite area of tessellations (Eberle, 2011; Marchini, 2003; Vitale & Zinder, 1991). The students think that the tessellation is finished when the paper is finished and the spatial infinity cannot make sense and few students make creative attempts to add new paper (Marchini, 2003). When tiling a limited area, students work harder with triangles than with rectangles (Owens & Outhred, 1998). This indicates that students' success in the different tessellation types may change. It is known that students are more successful in identifying tiles and transformations in the tessellations they are familiar with, but sometimes being familiar with the shape causes them to ignore the transformations and focus on the shape, sometimes lead to high-level thinking of complex designs (Callinghan, 2004) and create irregular tessellation (Callinghan, 2004; Kılıç et al., 2007). These results have emerged from studies in different fields rather than studies focusing directly on student achievement in the tessellation types. For example, Callinghan (2004) focused on the classification of students' understanding of tessellation by Van Hiele's Geometric Thinking Levels. In this context, at Level 0 according to Van Hiele Geometric Thinking Levels, students were expected to define the shapes in the tessellation and the combination of the unit shapes that forming the pattern. At Level 1, students were expected to define both shapes and movements used to transform shapes in an unconnected manner. At level 2, they are expected to explain, integrate, and make some attempt to measure the extent of the transformation, using the technical language level, such as “flip” or “slide”. Although the use of Van Hiele Geometric Thinking Levels implicitly reveals the above-mentioned results, Callinghan (2004) suggests that the nature of the misconceptions that some students have was not sufficiently defined by this approach. Therefore, studies are needed for the emergence of misconceptions of the students.

The relevant literature was reviewed, it was seen that there is a need for studies on whether success has changed according to the tessellation types and the factors affecting the performances of creating, completing and determining the tessellation. In this context, it is important to examine whether the students see different types of tessellation as tessellation and whether their performance in different tessellation types has changed. In order to address this gap in the literature, this study aims to investigate "the tessellation performance of 8th grade students and factors affecting their performance."

The sub-problems discussed in this context are as follows:

1. How are the performances of the students in determining whether a pattern is a tessellation according to the tessellation types? What are the factors that affect this performance?

2. What are the performances of the students completing tessellations given according to the tessellation type? What are the factors that affect this performance?

As can be seen from the above discussion, patterns and tessellations are one of the topics that constitute the basis of Middle School Mathematics Education. In this context, it is thought that the findings obtained to answer research questions will serve to develop the instructional design.

## 2. Method

In this study, the performance of the students in regular, semi-regular and irregular tessellation is revealed. This study is a descriptive study since there is no intervention in existing situations in the process.

## 2. 1. Participants

In the 2013–2014 educational year during fall semester, 40 eighth-grade students (approximately 14 years old) from a public school participated in this study. School was determined by an examination called 'Nationwide Standardized Level Determination Exam' as medium achievement school levels. Considering the idea that different skills could be have by schools having different achievement levels, the attention was given while selecting schools as medium achievement level. In order not to associate results depending only one class, two volunteer teachers' classes were selected. The reason why 8$^{th}$ grade students are chosen is because of these students have received training for tessellations in 6, 7 and 8 grades. All students were trained in the same curriculum. Of these 40 students, 17 are girls and 23 are boys.

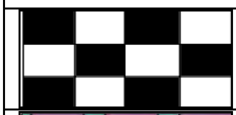
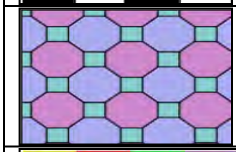
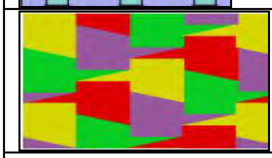
## 2. 2. Data Collection Tool

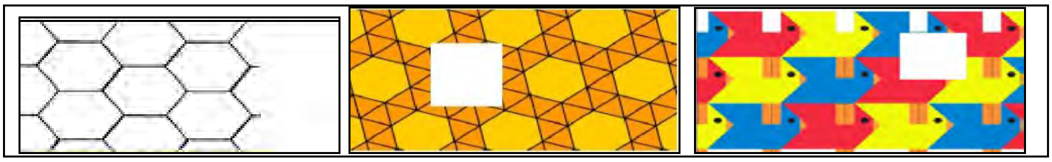
Various methods could use to determine the students' understanding of concepts. Diagnostic tests, concept cartoons, prediction-observation-explanation, concept maps, interviews are some of them. The development of diagnostic tests takes time, but it is easy to implement and interpret. Diagnostic tests provide convenience in interpreting students' understandings if the study carrying out with crowded groups. Diagnostic tests may include multiple-choice and open-ended questions. In multiple-choice diagnostic tests, it is difficult to understand the reason for the students' answer. In diagnostic tests involving open-ended questions, it is easy to understand the students' thinking because students were asked to explain the reasons for their answers. In this context, it is considered appropriate to use a diagnostic test with open-ended questions since this study will work with the crowded student group and the tessellation performance test has been developed. The steps that follow when developing a diagnostic test are as follows:

- Determination of the expected performance of the students related to tessellations
- Preparation of questions which have potential to determine performances
- Obtaining expert opinion to ensure validity
- Implementation of the pilot study
- Determination of reliability

In accordance with the purpose of the study and the explanations made in the introduction section, the performances expected from the students were gathered under two main themes: "determining whether or not a motif is tessellation" and "completing the tessellation". In this context 2 main questions were prepared. The questions prepared are given in Table 2.

**Table 2.** *Sample Questions*

| **1)** *Please mention your decision on whether or not the motifs given below are tessellations and explain your decisions with relevant reasons.* | | |
|---|---|---|
| **Motif** | **Tessellation**<br>**Why?** | **Not A Tessellation**<br>**Why?** |
|  | | |
|  | | |
|  | | |
| **2)** *Please complete the tessellations given below.* | | |



The researchers prepared equal number of sub-questions concerning regular (R), semi-regular (SR), and irregular (IR) tessellations for each main question. In this direction, in question 1, there are a total of 6 tessellation samples; two of R, two of SR and two of IR. In question 2, there are three tessellation samples, one of each tessellation type. The validity of the test has been assessed by an expert who has done his/her Ph.D. in mathematics education and studies on student understandings. Necessary revisions have been made. In the last version of the first question, the reason for giving two examples for each type of tessellation was to reveal the students' thoughts about tessellation definitions in detail. Determination of the students' thoughts in detail would give more information about the factors affecting the students' tessellation performances. For this reason, it was preferred to include two samples instead of one. As for completing the tessellations, it was thought that one sample would be sufficient for examine to students could determine the tiles whether or not. In this context, the test includes tasks of continuing the tile and completing the deleted part by determining the tile. It is stated that the use of manipulatives in tessellation building prevents the mental process related to tile building and drawing by hand allows analyzing the mental process of the child (Eberle, 2011; Owens & Outhred 1998). In this context, students were asked to draw while completing the given tessellations. There was no specific reason why completion tasks continue to decorate or leave the deleted part as completion. In both types of tasks require the students to determine the tile. As a result of the pilot study conducted with three students, it was observed that these expectations were met and the factors affecting the performance and performance of the students were demonstrated. In this respect, a "Diagnostic Test for Determining Tessellation Performance" consisting of a total of 9 tessellation samples has been established. The test was applied to fifty grade 8 students who would not be included in the main study to determine the reliability of the test. The calculated Cronbach Alpha reliability coefficient is 0.7. In this context, it can be said that the test is reliable.

### 2. 3. Data Gathering and Analysis Process

The diagnostic test was administered to the students by their own mathematics teachers in their own classroom environments for one course hour. The answers of 40 participating students to the sub-questions were examined under three categories: correct, incorrect, and unanswered. For each main question, the percentages of correct answers, incorrect answers, and unanswered questions to R, SR, and IR tessellations were determined. Accordingly, variations in the performance by tessellation type were analyzed. Moreover, the incorrect answers of the students were examined in an attempt to qualitatively determine the factors influential on performance based on the research questions.

## 3. Results

In this section, the performance of the students to determine whether or not the motif is tessellation according to tessellation types and to complete the tessellation. Table 3 below shows the quantile of the performance of the students in determining whether or not a given motif is a tessellation in terms of R, SR, and IR tessellations..

**Table 3.** *The Quantile of the Students' Performance in the First Question in Terms of R, SR, and IR Tessellations*

| Tessellation Type | Correct (f) | % | Incorrect (f) | % | Unanswered (f) | % | Total (f) |
|---|---|---|---|---|---|---|---|
| **R** | 72 | 90 | 7 | 8.75 | 1 | 1.25 | 80 |
| **SR** | 60 | 75 | 18 | 22.5 | 2 | 2.5 | 80 |
| **IR** | 64 | 80 | 12 | 15 | 4 | 5 | 80 |

The table 3 shows that 80 answers were taken for each tessellation type though 40 students participated in the study. This is because; there were two motifs for each tessellation type in the first question. Based on the correct answers of the students, the highest performance in determining whether or not a given motif was a tessellation was found to be in R tessellations. The performances were almost the same in SR and IR tessellations and they were lower than that in R tessellations. When the justifications of the students who answered wrongly were examined regardless of the type of tessellation, it was seen that there were three reasons affect the performance of the students: *"do not know the relationship with tessellation and pattern", "can not determine the tile forming the pattern", "misunderstanding about surface tiling"*. The justification examples of the students on the tessellations could see in Figure 1, Figure 2, Figure 3 and Figure 4.

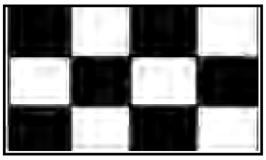


**Figure 2.** *R Tessellation*

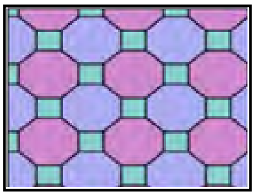


**Figure 3.** *SR Tessellation*

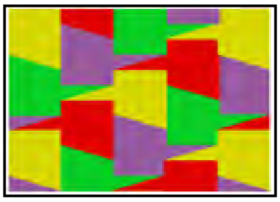


**Figure 4.** *IR Tessellation*

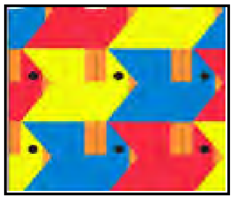


**Figure 5.** *IR Tessellation*

The justifications of the students who think that the motifs are not tessellation are given below.

**S4:** "Figure 2 and Figure 3 are not tessellations. They are a pattern. Figure 4 and Figure 5 are tessellations."

**S38:** "None of them are tessellations. These are patterns."

**S11:** "These motifs are not tessellations. They are a pattern. They all have a certain order."

Although the S4 did not realize that Figure 4 and Figure 5 are patterns, he can easily see the patterns of Figures 2 and Figure 3. He stated that the pattern cannot be tessellation. As it is seen, students think that *patterns cannot be tessellation*. These thoughts lead them to the wrong answer in determining whether the pattern is tessellation or not. The justification given by the S9 who gave the wrong answers due to a different reason is given below.

**S9:** "Figure 2, Figure 3 and Figure 5 are tessellations. Because patterns have a specific order and pattern. I can continue this motif because it is a tessellation. Figure 4 is not tessellation. Because there is no order."

As can be understood from the explanations of S9, the student pays attention to whether the motifs consist of a pattern, and indicates that motifs are tessellations if the tiles can determine easily. In other words, the student knows the relationship between pattern and tessellation. However, since the student cannot determine the rule of the tessellation given in Figure 4, this pattern is not a tessellation for him. It is seen that students *"can not determine the tiles that will cover a surface with no gaps*". Besides, when examine S9s' answer to the last pattern, the student states that the rule is certain and he can continue the pattern. Therefore, the pattern is a tessellation. The justifications given by S5, who gave incorrect answers for a different reason, are given below.

**S5:** "Figure 1 is a tessellation. It was in harmony and the page was filled (student refers to the box where the shape is given as a page). Figure 3 is not tessellation. Because the page is not full. Figure 4 is tessellation. Because the tessellation has a certain order and filled the page. Figure 5 is tessellation. Because the tessellation has a certain order and filled the page."

When the justifications of S5 are examined, it can be seen that, when deciding whether or not a motif is a tessellation, it is observed that is there a certain order and is the area where the pattern given filled. It is understood that the student perceives the *activity of covering a surface specified in the definition of tessellation as covering the whole area where the pattern is located*.

Table 4 below presents the quantile of the students' performance in completing tessellations in terms of R, SR, and IR tessellations.

**Table 4.** *The Quantile of the Students' Performance in the Second Question in Terms of R, SR, and IR Tessellations*

| Tessellation Type | Correct (f) | Incorrect (f) | Unanswered (f) | Total (f) |
|---|---|---|---|---|
| **R** | 18 | 19 | 3 | 40 |
| **SR** | 8 | 30 | 2 | 40 |
| **IR** | 21 | 18 | 1 | 40 |

When Table 4 is examined, it is seen that there are a total of 40 answers for each type of tessellation in parallel with the number of participants. This is because there is only one motif for each type of tessellation in the test for this question. When the number of correct answers given by the students is examined, it is seen that the best performance in completing a tessellation is in IR tessellation. It can be said that the performance in tessellation R is close to the IR tessellations. It is noteworthy that the number of students performing correctly is around 50%. On the other hand, the performances in SR tessellations are poor. When the justifications of the students who answered wrongly were examined regardless of the type of tessellation, it was seen that there were three reasons affect the performance of the students: "*misidentification of tile" "thinking that the gap should be completed with the tile", "lack of psychomotor skills"*.

It is observed that some students were not able to complete the patterns correctly when they were asked to complete the empty parts of the motifs. Examples of students who cannot complete the tessellations correctly are given below.

**Figure 6.** *Drawing of S35*

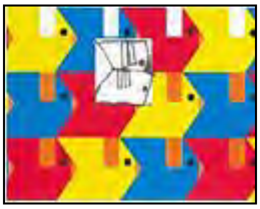


**Figure 7.** *Drawing of S12*

When Figure 6 is examined, it is seen that S35 continues the columns consisting of triangles by ignoring the hexagon formed in the middle of the triangular patterns. When Figure 7 is examined, it is seen that S12 tries to complete the pattern with two tiles instead of one. As can be understood from this, the students could not complete the tessellations correctly because they misinterpreted the pattern. The drawings made by S27 and S13 in the same tessellations are given in Figure 8 and Figure 9.

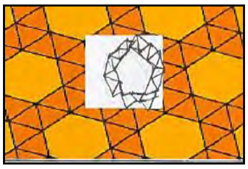


**Figure 8.** *Drawing of S27*

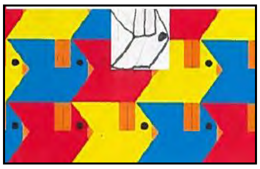
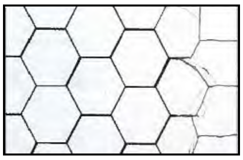


**Figure 9.** *Drawing of S13*

Figure 8 and Figure 9 shows that students determine the main shape/tile forming the tessellation correctly. Students think that the empty part of the tessellation should be filled with tile and make drawings in this direction. As can be understood from this, the students were unable to complete the tessellations correctly because they thought that “the gap should be completed with the tile.” The drawings of S8 and S38 are given in Figure 10 and Figure 11.



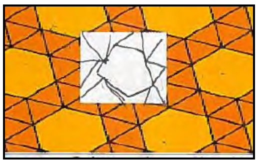


**Figure 10.** *Drawing of S8* **Figure 11.** *Drawing of S38*

Figure 10 shows that S8 can determine the main form of tessellation is a hexagon, but cannot draw the hexagon correctly. Similarly, the drawing of S38 given in Figure 11 shows that the student can determine the pattern of the triangles forming the tessellation but cannot draw the triangles properly. As can be understood from this, the students were unable to complete the tessellations correctly due to the "lack of psychomotor skills".

## 4. Conclusions and Discussion

Considering the findings, it was seen that the students exhibited good performance (over 70% correct answers) and their performances are close to each other according to tessellation types (90% for R, 75% for SR and 80% for IR). In the literature, studies emphasize that if students have familiarity with tessellation tiles that can have both positive and negative effects on their performances (Callinghan, 2004), as well as the fact that working with rectangles is easier than working with triangles (Owens & Outhred, 1998). In this study, R and SR tessellations consist of shapes that students are familiar with (less complex shapes such as squares, regular hexagons). IR tessellations are made up of irregular shapes which students have more difficulty in understanding. However, as a result of the study, the performance of the students is close to each other for all types of tessellation. In this study, different from literature, students were asked to interpret a tessellation rather than create. This result reveals the importance and necessity of evaluating students' tessellation performances in different activities. Student performances can be investigated by considering different activities in future studies.

This study shows that student performances do not differ according to the tessellation types, but also significant results related to students' understandings. When the justifications of the students about the tessellations were examined, it was seen that their performance in determining whether the pattern is tessellation is negatively affected due to reasons such as "*do not know the relationship with tessellation and pattern", "can not determine the tile forming the pattern", "misunderstanding about surface tiling"*. Although the student's understanding of tessellation has not investigated in the literature directly, the students' understandings of tessellation was tried to be explained in the studies. Studies show that students have difficulty in making sense of the infinite area of tessellations (Eberle, 2011; Marchini, 2003; Vitale & Zinder, 1991). A similar result was seen in this study. The students thought that the tessellation means a finite surface tiling and it is enough to completely cover the surface of tiles where the tessellation in. In other words, students interpret the meaning of surface tiling as a finite area tiling. On the other hand, studies showed that students had difficulty in creating tessellation because they could not determine the appropriate tile (Callingham, 2004; Kılıç et al., 2007). In this study, it was seen that the students thought that the pattern was not tessellation because they could not determine the tile. This result shows that students have difficulties in going from part (tile) to whole (tessellation) and from whole to part. In this context, it is important to carry out educational studies both from part to whole and from whole to part. Different from literature, this study also found that students had difficulties with the relation between pattern and tessellations. As students did not know the relationship between tessellation and pattern, they consider the patterns as a tessellation that they are familiar with and the tiles could be easily identified. And they consider that the motifs that they could not determine the pattern are tessellations. In this context, it can be said that students had misunderstandings about the mathematical tessellation and they did not know the deep mathematical meaning underlying tessellations.

It was seen that the students' tessellation completing performance changed according to the tessellation typesf (45% for R, 20% for SR, 52.5% for IR). It is remarkably that the best performance in tessellation completion is IR tessellation. However, the completion rate of R tessellation was also close to the completion rate of IR. It can be thought that the sample R and IR tessellations, which were discussed in this study, were caused easily determine the tile and easy draw by the students' familiarity with the tessellations. Besides, these tessellations were created by translations. In the tessellation given as an example to SR, it was a kind of tessellation that contains triangles that could be drawn easily. But it contains rotation and reflections. It was expected that the tessellation would affect students' performances if it contains more than one movement of translation, rotation, and reflection. Therefore, it could be said that the samples discussed in the study brought restriction in the context of a comparison of success according to tessellation types. However, an important result in this study was that the students' completing performance was low regardless of the tessellation types. In this context, the reasons that affect the performance of the students were also examined in detail. The reasons for not completing the tessellation, regardless of the tessellation types, were observed to have three factors: *"misidentification of tile" "thinking that the gap should be completed with the tile", "lack of psychomotor skills"*.

Studies showed that the students have difficulty in creating tessellations because they could not determine the tile (Callingham, 2004; Kılıç et al., 2007). In studies, students were asked to determine the tile to cover a surface and apply the movements of translation, reflection, and rotation suitable for this tile to cover the surface. Therefore, students first determine the tile in which the selected movement or movements would be applied and then tried to cover the surface. In this study, students encountered an unusual activity. Students were asked to determine the tile that creates a different tessellation. In this context, it can be said that the students have difficulties in both determining the tile and tiling because of the "misidentification of the tile". Thus, it can be said that the activities carried out to complete the tessellation from the tile cause the students to perceive the tile as a whole. Students could not determine the certain parts of the tile. As a result of this study, it was observed that students had difficulties because of their "thinking that the gap should be completed with the tile". This result obtained by students to make sense of the deep meaning underlying the decorations is able to add new approaches to the teaching of tessellation. Besides, it was observed that the performance of the students completing the tessellation was negatively affected by the "lack of psychomotor skills". One of the aims of integrating tessellations into teaching curriculums is to develop psychomotor skills (MNE, 2009). This result showed again the importance of drawing activities in the teaching of tessellations. It is stated that drawing activities were effective in making sense of the mathematical structure underlying the tessellations rather than manipulative use (Eberle, 2011; Owens & Outhred, 1998). In this study, the students were asked to draw the deleted parts of the existing tessellations rather than to make tessellations, and the difficulties of the students were determined. In this context, the potential of drawing to reveal the understandings of students has been revealed once again. This indicates the necessity of tessellation activities that give students more drawing opportunities.

In this study, it was not investigated whether the factors affecting the performances of the students in the tessellations have changed in the context of tessellation types. However, in light of the literature and the results obtained from this study, it can be said that the understanding of the students may vary according to contexts. This suggests that detailed studies on student conceptions in tessellations containing mathematical deep meanings should be continue. It can be said that the results obtained in this context are the basis for further studies.

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## Authors

**Funda Aydın-Güç,** Giresun University, Giresun (Turkey). E-mail: fundaydin05@gmail.com & funda.guc@giresun.edu.tr

**Mihriban Hacısalihoğlu-Karadeniz,** Giresun University, Giresun (Turkey). E-mail: mihrideniz@gmail.com

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