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Connections Between Real Analysis and Secondary Mathematics

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Abstract

In this study, we examine the connections between real analysis concepts and secondary mathematics content. This is accomplished by conducting a textbook analysis of a commonly used real analysis textbook. Concepts from the real analysis textbook were mapped to the Common Core State Standards-Mathematics (2010) to determine the relationship between real analysis and high school mathematics. The results of this textbook analysis suggests that several connections between real analysis and high school mathematics do exist with a particular emphasis on functions. Implications, limitations, and future research are also discussed.

Keywords: Real Analysis, Textbook Investigation, Advanced Mathematical Knowledge

Introduction

Real analysis is a course that nearly all mathematics majors and some mathematics education majors are required to take (Conference Board of the Mathematical Sciences, 2012). Standard topics covered in real analysis include the real number system, functions and limits, topology of the real numbers, continuity, differential and integral calculus for functions of one variable, infinite series, and uniform convergence (Bartle & Sherbert, 2011). This course is often viewed by pre-service high school mathematics teachers as daunting and disconnected from practice (Goulding, Hatch, & Rodd, 2003; Wasserman, Villanueva, Mejia-Ramos, & Weber, 2015). However, perceptions of this disconnect are incorrect since there are many explicit connections between what is learned in an introductory real analysis course and what is taught in high school mathematics courses (Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos, Weber, 2017).

Students in real analysis study the structure of the real number line and its subsets (Bartle & Sherbert, 2011). Analogously, high school mathematics teachers must develop student conceptions of the real number system as early as algebra. The convergence of sequences, which is studied rigorously in real analysis, also appears in the high school curriculum. Those who teach precalculus need to have a firm understanding of limits and continuous functions, both of which are studied intensely in an introductory real analysis course. Concepts that play a major role in calculus, such as differentiation, integration, and infinite series, make up the standard curriculum for the typical real analysis course (Bartle & Sherbert, 2011). Since these topics are foundational in the study of calculus, one may argue that calculus teachers ought to have a background in real analysis. However, let us consider the average high school mathematics teacher, who teaches courses such as algebra, geometry, and precalculus (not calculus). How can taking a course in real analysis benefit this teacher?

Problem Statement

If explicit connections exist between the content of real analysis and high school mathematics, it is essential that those who teach calculus, precalculus, and algebra have a firm understanding of this subject (Wasserman et al., 2017). However, there exists little research on the connection between learning real analysis and teaching high school mathematics, despite findings from various studies that assert student achievement is related to the content knowledge of their teachers (Ball, Thames, & Phelps, 2008; Begle, 1972; Hill, Rowan, & Ball, 2005; Monk, 1994). Although these studies have made significant contributions to our understanding of the need for strong content knowledge for teachers, we still know very little about the relationship between learning real analysis and teaching high school mathematics. Therefore, the goal of this study was to investigate this relationship so that we can ultimately better prepare teachers to teach high school mathematics. Specifically, the following research question will be addressed. What are the connections between real analysis and secondary mathematics?

Textbook Investigation

A textbook investigation was conducted to determine which concepts from real analysis are connected to high school mathematics. This textbook investigation was done using a qualitative analysis, which can be described as a systematic procedure for reviewing or evaluating documents (Creswell, 2013). Advantages of this method of qualitative analysis include efficiency, availability, cost-effectiveness, lack of obtrusiveness, stability, exactness, and coverage (Bowen, 2009). Although typically used for triangulation, the main purpose of this textbook investigation was to help establish the connections that exist between real analysis and high school mathematics content. This is similar to how Wasserman (2016) established connections between abstract algebra and high school mathematics. For each high school mathematics standard, which serves as a proxy for content in his and my work, Wasserman (2016) asked how teaching of the standard may be informed by teachers' knowledge of X (some given content in real analysis). The goal of this phase of the study is to find all possible links between high school mathematics standards (content) and teacher knowledge of X (in real analysis).

Textbooks tend to have a structure consisting of a variety of features which may have a significant impact on the audience (Valverde, Bianchi, & Wolfe, 2002). Textbook analysis provides a means by which these various features can be investigated. However, with all the possible features of a textbook to investigate, deciding which are worth analyzing may be difficult. To solve this problem, O'Keefe (2013) developed a framework consisting of four key features of textbooks (content, structure, expectation, language) built on the work previously done by the 2002 TIMSS study. In terms of the mathematics textbooks, content can be defined as the underlying mathematical concepts, whereas structure describes how these concepts are sequenced. Expectation refers to what is expected of the reader and language is comprised the mathematical notation and symbols used. Based on an extensive review of the literature, O'Keefe (2013) found that these four components contribute to the overall effectiveness of a textbook and I used.

Introduction to Real Analysis by Bartle and Sherbert (2011) is a standard text used in many undergraduate real analysis courses. This was text was chosen because it contains topics which align with the Conference Board of Mathematical Sciences' (2012) description of real analysis. Also, this text is typically used at the university where the participants of this study attended college and studied real analysis. Not all topics in this text may be directly connected to high school mathematics content. However, real analysis, along with abstract algebra, form the basis of advanced mathematics studied by mathematicians and are often considered foundational courses in the education of mathematics majors. Real analysis specifically diverges from the study of abstract algebra by investigating the structure behind continuous and infinitesimal quantities.

Introduction to Real Analysis (Bartle & Sherbert, 2011) contains the following six major topics: topology of the real numbers, sequences and series, limits, functions, differentiation, and integration. It should be noted that differentiation and integration have been omitted from this analysis, but would be useful for those who teach calculus. In fact, since real analysis can be thought of as advanced calculus (this course is often listed under this name at many universities), one may argue that all calculus teachers should take this course. However, the aim of this study

is to investigate the relevance of studying real analysis for all high school mathematics teachers, not just those who teach calculus. Therefore, only connections between real analysis and the CCSS-M (2010) will be analyzed. However, it should be noted that the study of real analysis is much more comprehensive than the study of calculus. The notion of rigor and proof is prevalent throughout the study of real analysis, with one of the primary goals of this course to prove various properties of the real numbers and real-valued functions, such as the field axioms, completeness axiom, and the Intermediate Value Theorem. This rigorous treatment of the real numbers can help teachers understand how to prove what is unstated in high school in order to avoid false simplifications and to be able to answer questions from students seeking further understanding. Moreover, these topics provide opportunities to make use of original historical sources, which can motivate the theory and make real analysis seem less disconnected from high school mathematics.

For each of these four major topics (topology of the real numbers, sequences and series, limits, and functions), relevant concepts were mapped to mathematics standards as described by the Common Core State Standards-Mathematics (CCSS-M, 2010). CCSS-M (2010) were chosen due to their broad adoption in the United States, as they provide an accurate representation of the mathematical concepts that should be discussed in a typical high school classroom and thus serve as a proxy for the content being taught. Applying the framework developed by O'Keefe (2013), the following components of each relevant topic were considered: underlying mathematical concepts, how these concepts are sequenced, what is expected of the reader, and notation and symbols used in communicating this concept. The topic is then discussed through the lens of how I conceived that the teacher might use it in classrooms, and why the topic is, or could be needed, when teaching the content (standards).

Results

From my analysis of *Introduction to Real Analysis* by Bartle and Sherbert (2011), I found evidence of several connections between standards (content) from real analysis and high school mathematics, which helps to validate the claim made by the Conference Board of the Mathematical Sciences (2012) which states that all teachers should study real analysis as part of their teacher preparation program. Although some of this content is specifically relevant for

those who teach precalculus, such as limits and exponential and trigonometric functions, there are still several concepts which are directly related to other high school mathematics topics. In fact, a total of 22 connections between real analysis concepts and CCSS-M (2010) were discovered.

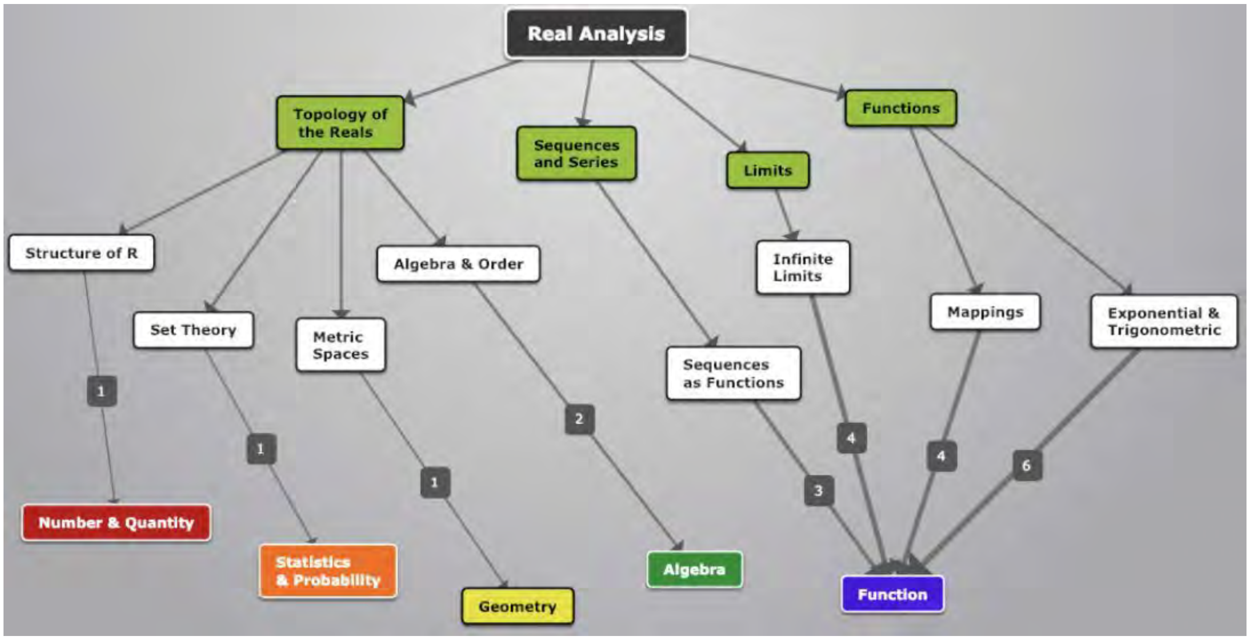

Figure 1
Mapping Real Analysis Topics to CCSS-M

Table 1 below shows the specific standards each real analysis topic was mapped to. Additionally, a summary of the mapping of real analysis concepts to domains of the Common Core State Standards-Mathematics can be seen above in Figure 1. The number displayed on each edge of the graph directed from the real analysis topics to the Common Core State

Standards-Mathematics domains indicates how many standards from the domain each real analysis topic was mapped to. The orange boxes represent the domains taken from the Common Core State Standards Mathematics. Note that at least one topic from the text mapped to a domain of the Common Core State Standards-Mathematics. Moreover, the notion of function appears to be the most relevant domain from the Common Core State Standards-Mathematics which is discussed in the book.

Prior to discussing limits and functions, the authors (Bartle & Sherbert, 2011) spend time building up the structure of the real numbers. Having a deep understanding of this structure is relevant for anyone who teaches high school mathematics, from algebra to precalculus and beyond. Additionally, this text emphasizes sequences and series, which is another topic relevant for all teachers of mathematics. In high school, students are often asked to find both the recursive and explicit formulas for sequences and then use these formulas to find a particular term.

Additionally, it is common for students to explore the relationship between sequences and functions (CCSS-M, 2010). For example, a teacher may plan a lesson where students discover that arithmetic and geometric sequences are essentially linear and exponential functions, respectively. Therefore, it is essential that teachers understand that sequences are just a specific class of functions, where the domain is restricted to the natural numbers.

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Table 1
List of Real Analysis Topics Mapped to CCSS-M

Real Analysis Topic	Common Core State Standard
Structure of the Real Numbers	CCSS.MATH.CONTENT.HSN.RN.B.3
Algebraic and Order Properties of the Real Numbers	CCSS.MATH.CONTENT.HSA.SSE.B.3 CCSS.MATH.CONTENT.HSA.REI.B.3
Set Theory	CCSS.MATH.CONTENT.HSS.CP.A.1
Metric Spaces	CCSS.MATH.CONTENT.HSG.GPE.B.4
Sequences as Functions from the Natural Numbers to the Real Numbers	CCSS.MATH.CONTENT.HSF.IF.A.3 CCSS.MATH.CONTENT.HSF.BF.A.2 CCSS.MATH.CONTENT.HSF.LE.A.2
Infinite Limits	CCSS.MATH.CONTENT.HSF.IF.B.4 CCSS.MATH.CONTENT.HSF.IF.C.7.C CCSS.MATH.CONTENT.HSF.IF.C.7.D CCSS.MATH.CONTENT.HSF.IF.C.7.E
Functions as Mappings	CCSS.MATH.CONTENT.HSF.IF.A.1 CCSS.MATH.CONTENT.HSF.IF.A.2 CCSS.MATH.CONTENT.HSF.IF.B.5 CCSS.MATH.CONTENT.HSF.BF.B.4
Exponential Functions	CCSS.MATH.CONTENT.HSF.IF.C.7.E CCSS.MATH.CONTENT.HSF.IF.C.8.B CCSS.MATH.CONTENT.HSF.BF.B.5 CCSS.MATH.CONTENT.HSF.LE.A.1
Trigonometric Functions	CCSS.MATH.CONTENT.HSF.TF.C.8 CCSS.MATH.CONTENT.HSF.TF.C.9

Discussion

Interestingly, this investigation suggests that a real analysis course intended for teachers should focus heavily on functions (17 connections). For both mathematicians and mathematics teacher educators, this means that a real analysis course designed for teachers should focus on developing a deep understanding of function and making explicit connections to the CCSS-M (2010) which address functions. However, it should be noted that real analysis should not be simply thought of as a list of topics that teachers should know. Rather, there should be an emphasis on rigor and proof with respect to these topics. While studying real analysis, teachers should be given the opportunity to prove various theorems about the real numbers and real-valued functions. For example, proving the Intermediate Value Theorem may help teachers to make sense of the graphical methods for solving equations that they teach to their students.This and many other theorems studied in real analysis form the basis of much of what is studied in high school mathematics. Like the work done with abstract algebra by Wasserman (2016), the results of this textbook investigation help to push our understanding of content knowledge for teachers forward, ultimately informing the mathematical preparation of teachers.

Since only one representative real analysis textbook was considered for the textbook investigation, a more complete study to investigate several commonly used real analysis textbooks could be beneficial. To determine which subset of introductory real analysis textbooks to analyze, brief surveys could be sent out to mathematicians at several universities who often teach courses in real analysis. A decision would need to be made about the number of textbooks to investigate, but the top few most commonly used textbooks could then be analyzed to investigate the connections between real analysis and the CCSS-M (2010). Additionally, this same mapping process could be done with textbooks focused on other advanced mathematics, such as abstract algebra, number theory, or axiomatic geometry.

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