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PARTITIVE FRACTION DIVISION: REVEALING AND PROMOTING PRIMARY STUDENTS' UNDERSTANDING

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Abstract

Students show deficient understanding on fraction division and supporting that understanding remains a challenge for mathematics educators. This article aims to describe primary students' understanding of partitive fraction division (PFD) and explore ways to support their understanding through the use of sequenced fractions and context-related graphical representations. In a design-research study, forty-four primary students were involved in three cycles of teaching experiments. Students' works, transcript of recorded classroom discussion, and field notes were retrospectively analyzed to examine the hypothetical learning trajectories. There are three main findings drawn from the teaching experiments. Firstly, context of the tasks, the context-related graphical representations, and the sequence of fractions used do support students' understanding of PFD. Secondly, the understanding of non-unit rate problems did not support the students' understanding of unit rate problems. Lastly, the students were incapable of determining symbolic representations from unit rate problems and linking the problems to fraction division problems. The last two results imply to rethink unit rate as part of a partitive division with fractions. Drawing upon the findings, four alternative ways are offered to support students' understanding of PFD, i.e., the lesson could be starting from partitive whole number division to develop the notion of fair-sharing, strengthening the concept of unit in fraction and partitioning, choosing specific contexts with more relation to the graphical representations, and sequencing the fractions used, from a simple to advanced form.

Keywords: Understanding, Partitive, Fraction division, Unit rate, Design research

Abstrak

Siswa menunjukkan pemahaman yang kurang pada materi pembagian pecahan dan mendukung pemahaman tersebut masih menjadi tantangan bagi pendidik matematika. Artikel ini bertujuan menjelaskan pemahaman pembagian pecahan partitif siswa sekolah dasar dan merumuskan cara untuk mendukung pemahaman tersebut melalui penggunaan pecahan secara terurut dan konteks yang berkaitan dengan representasi grafik. Dalam sebuah penelitian desain, empat puluh empat (44) siswa terlibat dalam tiga tahap eksperimen pengajaran. Hasil kerja siswa, transkrip rekaman diskusi kelas, dan catatan lapangan dianalisis secara retrospektif untuk menguji lintasan hipotesis pembelajaran. Terdapat tiga temuan utama berdasarkan eksperimen pengajaran. Pertama, konteks tugas, hubungan konteks dengan penggunaan representasi grafik, dan urutan pecahan yang digunakan sangat mendukung pemahaman siswa terkait pembagian pecahan partitif. Kedua, pemahaman siswa yang terbangun pada masalah yang bukan *unit rate* belum membantu siswa dalam menyelesaikan masalah *unit rate*. Terakhir, siswa belum mampu menentukan representasi simbolik dari permasalahan *unit rate* dan menghubungkannya dengan pembagian pecahan. Dua temuan terakhir menjadi bahan pertimbangan terkait *unit rate* sebagai bagian pembagian pecahan partitif. Merujuk pada temuan tersebut, empat cara ditawarkan untuk mendukung pemahaman pembagian pecahan partitif siswa, yaitu pembelajaran dimulai dengan pembagian partitif bilangan asli untuk mengembangkan konsep pembagian adil, memperkuat konsep satuan pecahan dan partisi, memilih konteks yang berhubungan dengan penggunaan representasi grafik, dan mengurutkan pecahan yang digunakan dari yang sederhana sampai yang sulit.

Kata kunci: Pemahaman, Partitif, Pembagian pecahan, Perubahan satuan, Penelitian desain

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Prior studies have revealed the students' errors in fraction division (FD) which reflect the lack of conceptual understanding on the topic (Tirosh, 2000; Cramer, Monson, Whitney, Leavitt, & Wyberg, 2010; Aksoy & Yazlik, 2017). Several aspects relate to the students' problems in learning fraction division, namely mathematics teachers' content knowledge (Ma, 2010), the instructions which promote the conceptual understanding of students (Hu & Hsiao, 2013), and the complex nature of fraction division (Prediger, 2006; Ma, 2010). These aspects imply that the success of the students' comprehension of fraction division mainly lies in the hands of teachers; Their way of arranging the content and learning-environment has a high impact on the understanding of the students. In summary, it can be pointed out that mathematics teachers need to have a relational understanding of the fraction division and have to be capable of designing learning environments with activities that promote the relational understanding of the students to promote an adequate understanding of the topic. The second point as the focus of the present study was in contrast to what we found in classroom observations as part of preliminary phase of design research wherein, for example, the learning activities were focused on procedural aspect instead of developing students' understanding and did not start with contextual problems which facilitate students to employ their prior knowledge to solve the problem and develop mathematical knowledge.

In the literature, FD has diverse conceptualizations (Sinicrope, Mick, & Kolb, 2002; Gregg & Gregg, 2007; Lamon, 2012; Adu-Gyamfi, Schwartz, Sinicrope, & Bossé, 2019). Sinicrope et al. (2002) categorize FD into five conceptualizations, namely measurement, partitive, unit rate, the inverse of an operator multiplication, and the inverse of a Cartesian product. Several authors categorize unit rate as part and the result of Partitive Fraction Division (PFD) (Gregg & Gregg, 2007; Lamon, 2012; Jansen & Hohensee, 2016). The present study adapts the latter conception of PFD which provides a fair-sharing situation and involve unit rate. In PFD, the number of groups and the total amount to be equally shared are given but the amount in each group is not known. It can be represented as *total ÷ number of groups = number in each group* (Petit, Laird, Marsden, & Ebby, 2016). An example of PFD in the context of fair-sharing is "*Anna has 3/4 of a chocolate bar that has to be equally shared with three friends. How many parts of the chocolate does each one of the friends get?*"

The measurement type of fraction division tends to have more attention than the partitive one in textbooks. Ott, Snook, and Gibson (1991) assert that most of literature and textbooks tend to ignore the partitive division with fractions. The chapter of fraction division in Indonesia primary school mathematics textbooks also shows less attention on partitive problems (Wahyu & Mahfudy, 2018). Zaleta (2006) argues that measurement division with fractions is more common than the partitive interpretation since the latter is problematic to imagine in natural context the number of groups is not the whole numbers. For a viable conceptual understanding, both types of fraction divisions are equally important (Flores, 2002). The apprehension of PFD should also be promoted since it plays a significant role in helping students to understand other mathematical concepts such as rate (Sinicrope et al., 2002; Lamon, 2012), slope, probability, and proportional reasoning (Hohensee & Jansen, 2017). Van de Walle (2004) suggests to mix up of the types of division problems in the instruction equally. Petit et al. (2016) also emphasize the promotion of various types of fraction division to ensure the students do not overgeneralize one way of thinking division problems.

Most of the studies on PFD focus on prospective mathematics teachers' content-related understanding (Simon, 1993; Nillas, 2003; Zembat, 2004; Gregg & Gregg, 2007; Lo & Luo, 2012; Jansen & Hohensee, 2016; Hohensee & Jansen, 2017). For instance, Jansen and Hohensee (2016) investigated the nature of prospective primary teachers' conceptions of partitive division with fractions. A small number of studies focuses the understanding of students (Okazaki & Koyama, 2005; Zaleta, 2006; Muchsin, Hartono, & Putri, 2014). Okazaki and Koyama (2005) designed three division problems and three didactical activities to support the 5th graders in constructing the meaning of division with decimals through overcoming their difficulties. The study aimed to analyze students' logical reasoning in understanding division with decimals (partitive form). Zaleta (2006) inquired into invented computational strategies developed by the 6th graders in solving four fraction division problems, two of which are partitive problems. Muchsin et al. (2014) employed three problems of partitive fraction division with a duration context to identify students' strategies.

The current study notably differs from prior studies regarding three aspects (Okazaki & Koyama, 2005; Zaleta, 2006; Muchsin et al., 2014). Firstly, the use of a less strenuous or sequential form of fractions as the starting point. Secondly, the use of specific, context-related graphical representations to support students' understanding as well as using contexts as a starting point in the sense of Realistic Mathematics Education (RME) (Freudenthal, 1981). Thirdly, a proposal for alternative ways of promoting the students' understanding of PFD. We argue that starting with a fraction dividing a whole number (5/10÷5) is much easier than both dividend and divisor being a fraction (1/4÷1/2 or 1½÷3/5). Furthermore, the use of different graphical representations and the linking of different representations (e.g. the contextual, verbal, and graphical representation) is highly recommended by researchers for fostering learning processes about fractions (Post & Reys, 1979; Van de Walle, Karp, & Bay-Williams, 2010; Prediger, 2013). We designed the learning activities that feature sequenced fractions and context-related graphical representations for fostering the students' understanding. Drawing from conducted teaching experiments with designed learning activities, we synthesize those outlined ways for students to understand PFD. Following this, two questions are addressed in this article: (1) how do primary students understand PFD through the designed learning activities? and (2) what are the characteristics of learning activities intended to support students' understanding of PFD?

The current study was conducted in the light of RME theory (Freudenthal, 1981; Gravemeijer, 1994) which emphasizes the use of familiar contexts embedded in sequential activities as the starting point of mathematics instruction (Gravemeijer, 2011; Widjaja, 2013) and models in solving mathematics problems (Gravemeijer, 1999). Many studies have shown how the principles of RME: guided reinvention through progressive mathematization, didactical phenomenology, and emergent models, transform the classroom setting and improve the students' learning processes from a mechanistic way of learning to a more interactional learning-environment promoting students' co-constructive and everyday-experience-based understanding process (Ronal, 2014; Wahyu, 2015;

Nasution, Putri, & Zulkardi, 2018; van Gallen & van Eerde, 2019).

METHOD

Design research (DR) was employed in the current study with three stages: preparation and design (preparing the experiment), three cycles of teaching experiments, and retrospective analyses between and after the teaching experiments (Bakker, 2004; Gravemeijer & Cobb, 2006). Gravemeijer and Prediger (2019) argue that RME requires the teachers to help students develop their mathematics knowledge, while at the same time the teachers focus on the learning goals. This leads to an interactive process in which the teachers adjust to the students' thinking. For this process, design research is a companion to RME in developing local instruction theories, which can be frameworks of references for teachers. Moreover, DR fitted the aim of the study, which is to produce design principles for teaching and learning of PFD and to reconstruct the students' understanding of the topic in the primary schools. To achieve the aim, a qualitative analysis was conducted, discussed and resulted in a Hypothetical Learning Trajectory (HLT) for PFD, which consists of the learning goal, learning activity, and hypotheses about the learning processes. The learning goals were to enable the students to understand PFD by (1) solving the problem-context with specific graphical representations; (2) determining number sentences (symbolic representations) of the problems; and (3) linking solutions including graphical representations with the symbolic representations.

Problem-Context

The design of the problem-context in this study went through two stages. At the first stage, we produced three problems for the first two teaching experiments (Table 1). The problems included fractions divided by a whole number which results in fractions using two contexts (Time and Chocolate sharing). Problem 1.1 is in the form of **a**/**b** ÷ **c**, **c** does not divide **a**, meanwhile problem 1.2 and 1.3 along with problems in the second experiment are **a**/**b** ÷ **c**, **c** divides **a**.

At the second stage, the fractions used in the third teaching experiment have been redesigned in terms of the content and sequence. One might notice a difference in fractions content used in the first two teaching experiments and third teaching experiment. Analyses of the first two teaching experiments as well as further theoretical findings showed a necessity to redesign the tasks: For a viable conceptual understanding of PFD, a broadening of the tasks seemed to be required. These redesigned and broadened teaching experiments consisted of optimized problem contexts (Chocolate sharing) and more PFD-relevant content: The students were given ten problems about partitive whole number division, fractions divided by the whole number, and fractions divided by fractions. An example for theoretical aspects influencing the task design was the finding that the use of partitive whole number division may help the students construct the meaning of partitive division or fair-sharing before working with fractions. Hence, we expected the students not to treat partitive division with fractions as the measurement type of fraction division – a finding we observed in the early teaching experiments. Another example for theoretical aspects influencing the third teaching experiments was that researches such as van de Walle et al. (2015) strongly suggest the teachers to introduce and link the whole number division to fractions division and PFD also includes fractions as the divisor or unit rate problems (Gregg & Gregg, 2007; Lamon, 2012).

Participants and Classroom Context

The first teaching experiment involved six 5th-grade students with differences in mathematical performance (measured by their school grades) who were purposively selected to examine whether and how the designed learning activities worked with regard to students' usual mathematical performance. We were aware that school grades do not represent mathematical competence objectively. The school grades were merely an indicator and we did not formulate any hypotheses about the conceptual understanding being dependent from school grades. In the second teaching experiment, twenty-eight 5th-grade students participated in a whole-class setting with a focused group (5 students). The third teaching experiment included ten 4th-grade students. The students were in the last term of year four and taking part in an enrichment program. The three fraction operations (addition, subtraction and multiplication) were taught in the program before the teaching experiment. We argue that the PFD problems are eligible to be given to the ten students in the third teaching experiment for two reasons: pre-requisite topics (addition, subtraction and multiplication of fractions) have been given and classroom contexts they had prior to teaching experiment are similar to the students involved in the foregoing teaching experiments.

In our design experiment, PFD problems were given in phases of individual work following an initial phase of classroom discussion. We concentrated our research on the individual responses to the designed problems in order to get in-depth insights in the students' understanding reported in Table 1.

Table 1. The problem-context of PFD

Teaching Experiment	The Contextual Problems	Context	Symbolic Representations
First	1.1. Dwi has 3/4 hours to solve 5 problems in a math assignment. If she uses equal time for each problem, how many hours can she give to each?	Time	3/4 ÷ 5
	1.2. Juz'an has 5/6 of a chocolate bar that has to be shared with his five friends. How many parts of the chocolate does a friend get?	Chocolate sharing	5/6 ÷ 5
	1.3. In a math competition, the students were given 2/3 hours to solve 2 problems. If each problem is given equal time, how many minutes are required for one problem?	Time	2/3 ÷ 2
Second	2.1. Dwi has 5/10 hours to solve 5 problems in a	Time	5/10 ÷ 5

Teaching Experiment	The Contextual Problems	Context	Symbolic Representations
	math assignment. If equal time is given to each problem, how many hours required to solve one problem? 2.2. and 2.3 are similar to 1.2 and 1.3	Chocolate sharing and time	5/6 ÷ 5; 2/3 ÷ 2
Third	Excerpt of Chocolate Puzzles problems:		
	3.1. Mother has 6 chocolates that have to be shared equally to her three children. How many chocolates does each of her children get?	Chocolate sharing	6 ÷ 3
	[…]		
	3.5. Beta has 2/3 parts of a chocolate bar that have to be shared equally to his 3 classmates. How many parts of the chocolate bar does each friend get?	Chocolate sharing	2/3 ÷ 3
	*3.6.b. Caca has 3/4 parts of a chocolate bar that have to be placed in three cake boxes. 1/4 of the first cake box is filled with the given chocolate bar. How many chocolate bars are needed to fill up the cake box fully?	Chocolate sharing	3/4 ÷ 1/4
	[…]		

*Problems 3.6.a-3.6.d were initially unit rate problems used in lesson 2 of the third teaching experiment, then they were revised (Table 2) for lesson 3.

The classroom context of the study in the three teaching experiments was: (1) the students were used to having mechanistic way of learning instead of an interactional way of learning, (2) prior lessons focused on procedural knowledge rather than developing conceptual understanding, and (3) prior to the teaching experiment, the use of contextual problems as the starting point of mathematical topics was not a common way of starting lessons, and the students were not actively encouraged to use graphical representations. Furthermore, contextual problems were only used for application at the end of the lesson, not for starting conceptual understanding processes based on everyday-experience.

Data Collection and Analysis

Data collected in the current study was students' works on the given tasks, recorded classroom discussion, and field notes which capture crucial moments of students' responses. Data analysis followed the paradigm of qualitative analysis by aiming at examining and contrasting the hypotheses of learning in HLT with the actual learning process, whether or not the enacted hypotheses lead to achieving the learning goals. The data was retrospectively analyzed in five cyclic steps: (1) students' works on the tasks in each teaching experiment were examined regarding the use of graphical representations and symbolic representation created by the students to solve the tasks; (2) recorded classroom discussion was transcribed and sequenced afterwards (together with the field notes); (3) critical moments were selected from the transcripts and field notes to examine the students' reasoning on linking graphical-representation-based solutions with the symbolic representation. Oral or verbal explanatory utterances of students could not be gathered in each teaching experiment since we could not carry out all of the one-to-one interviews. Nevertheless, we made sure that the students' written works were complete at least and verbal explanations could be used for further insights (see for example Figure 3 and Figure 6); (4) students' works and critical moments of observed and transcribed utterances were triangulated to draw inference on learning goals; and (5) the achievement of the learning goals, based on the outcomes in a class test about fraction division, was utilized to evaluate the matching of learning hypotheses with the students' actual learning trajectory.

RESULT AND DISCUSSION

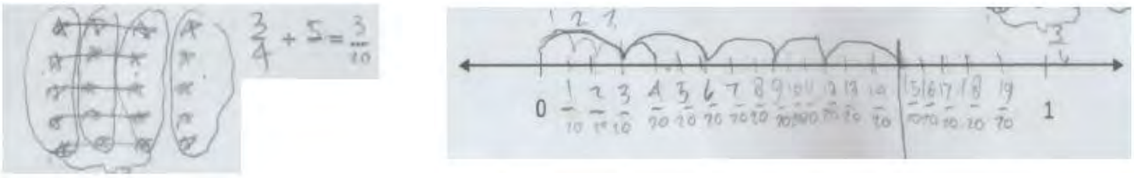
In this part, we firstly provide the results from the teaching experiments in the form of students' in-group and/or individual work and present interpretations of the learning process (supported by the students' works and parts of recorded classroom discussions or field notes). Following this, we formulate important findings in each teaching experiment based on the interpretations, afterwards we conclude the main findings with regard to our research questions. In the last step, we thoroughly discuss the main findings of all teaching experiments and the impact of the proposed design principles on the learning trajectories.

The First Teaching Experiment

Problem 1.1 was given to be discussed in the group. It was conjectured that the students could write a number sentence for the problem. This conjecture was in line with the real process. They could write correct number sentence of the problem, 3/4 ÷ 5. Another conjecture was that the students might not be able to draw multiple graphical representations, or they could only draw one representation to visualize the problem. However, the conjecture did miss the real process. The students even did not have any idea on how to represent 3/4 hour in the suggested graphical representations (Figure 1). Consequently, this outcome dismissed the next conjecture that the students might be able to solve the problem through the aid of a graphical representation.

Figure 1. The proposed graphical representations to help solving group problem

To support the students, we gave them a graphical representation in form of dots (Figure 1) and gave hints such as *one dot represents 1/12* and then asked how to represent the fraction 3/4. It took longer than we expected for most of the students to understand that 3/4 equals to 9/12. After that, we related the graphical representation to the problem by asking more generally how the five given problems in math assignment associated with the 9 dots in the model. Unfortunately, none of the students gave a proper answer to that question. This leads us to the conclusion that we need to be more precise as well as focused in our impulses to relate the contextual problem with the graphical representation and the symbolic representation. In the end, it provided helpful to use not one, but different kinds of graphical representations (with different levels of abstractions): It enabled the students to choose between the representations and link them to the mathematical problem (Figure 2).



(a) The works of group 1

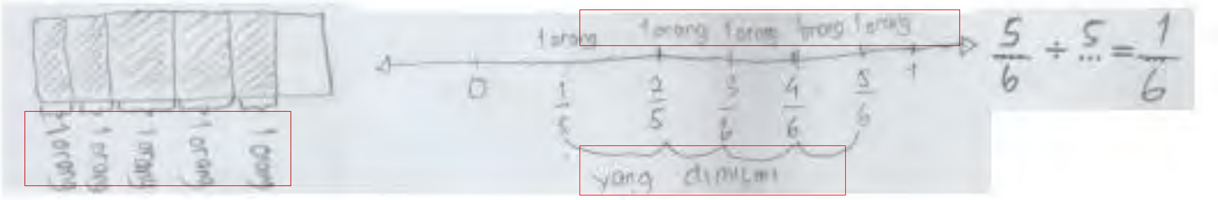
(b) The works of group 2

Figure 2. The students' works on the problem using multiple graphical representations

Not only did the students use multiple graphical representations to solve the problem but also linked it to the symbolic representation as shown in the following part of transcripts.

R : *Could you explain how 3/4 ÷ 5 results in 3/20 using a set of stars?*
S1.G1* : *Group the stars into four then determine 3/4 of it. Then 15 stars are shared equally to 5. Hmm… (pauses some seconds)*
R : *Could anyone help your friend?*
S2.G1 : *I, Sir (Raise his hand). One math problem gets 3 stars as in the fraction 3/20*
R : *That is great*
*Student 1 in Group 1

In order to examine further the students' understanding, each student was given an individual task (Problem 1.2 and 1.3). Surprisingly, all students had no problems in solving problem 1.2 using self-developed graphical representations (Figure 3). For problem 1.3, all students could determine the symbolic representation and construct their preferred graphical representation, mostly being area models (Figure 4). However, five of them focused on finding the quotient using graphical representations but missed answering the question of the problem (which is to determine how many minutes were need for solving one math problem).



Translation: 1 orang (1 friend) Given chocolate

Figure 3. The sample of one student's work on problem 1.2

There are important findings in the first teaching experiment: (1) in the first group, the students could not come up with a single graphical representation to visualize 3/4 hours. It leads us to the presumption that the relation of 'hour' to either the number line, sets of objects, or area model is disconnected because of contextual obstacles. It implies that contexts affect the way students are using (or not using) with the graphical representation; (2) the students could not directly determine 3/4 from the sets of objects or that 3/4 equals 9/12, which hindered them in using a graphical representation to solve the problem; (3) the students' works on the individual problem 1.2 indicate that the context of chocolate sharing is much more familiar and easier to connect with graphical representations (especially in form of areas); and (4) despite the fact that the students found it difficult to relate the hour context with a graphical representation in problem 1.1, it was not an issue any more in problem 1.3.

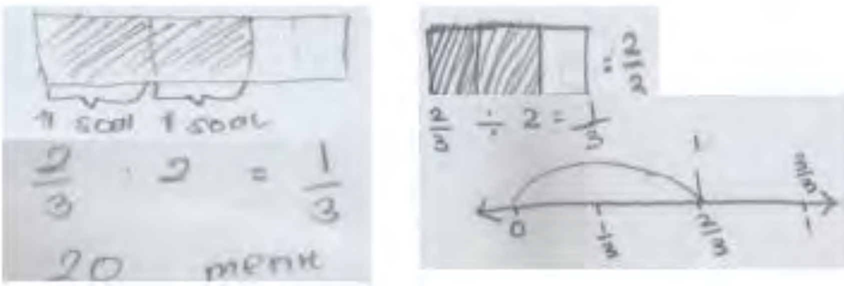


Figure 4. The sample of students' work on problem 1.3

We assume that there might be an impact of students' prior works on the same context. We conclude that a *careful use of contexts being related to more suitable graphical representations (context-related graphical representations) and the choice of fractions significantly contributes to the students' understanding of the partitive fraction divisions*. The context of sharing 5/6 cakes to 5 persons is easily related to graphical representations than fairly sharing 3/4 hours to 5 mathematics problems: The context of sharing cakes seems to be more appropriate to foster students' understanding of partitive fraction division. Furthermore, we observed that the use of specific fractions may help the students in creating graphical representations: Constructing a graphical representation about dividing 2/3 hour to 2 mathematics problems is not as tricky as constructing a graphical representation of 3/4 hour divided to 5 mathematics problems.

The Second Teaching Experiment

The students were grouped in groups of five students, one group being the focus-group. The activities in the focus-group were recorded as video and audio data, whereas the other four groups were only recorded by one static camera. The different treatment was necessary due to the use of the whole classroom, which involves 28 students, and it exceeded the resources to have all groups recorded separately. The students in the focus-group were carefully selected to be representative enough in the study and thus were matched in terms of mathematics ability.

One problem

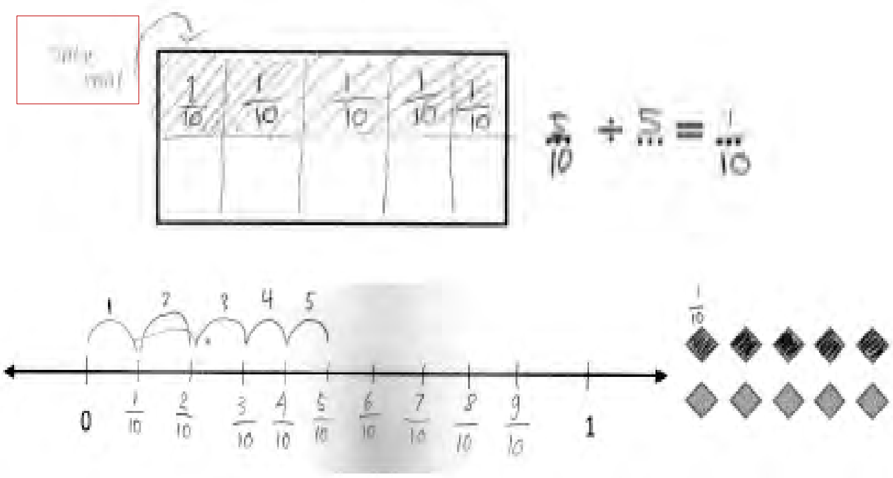


Figure 5. The works of students in one of the non-focused groups for problem 2.1

Problem 2.1 was given to all groups. Figure 5 and Figure 6 show the works of two groups on this problem. *It reveals that graphical representations could be easily constructed to solve a problem involving fractions 5/10 ÷ 5.* The focus-group could determine immediately that 1/10 hour was needed for one problem. The students in the focus-group *could relate the 5 problems and the given time (5/10 hour) by using the graphical representation as displayed in Figure 5 and Figure 6*. Each divided part of the rectangle (Figure 6) stands for one problem.

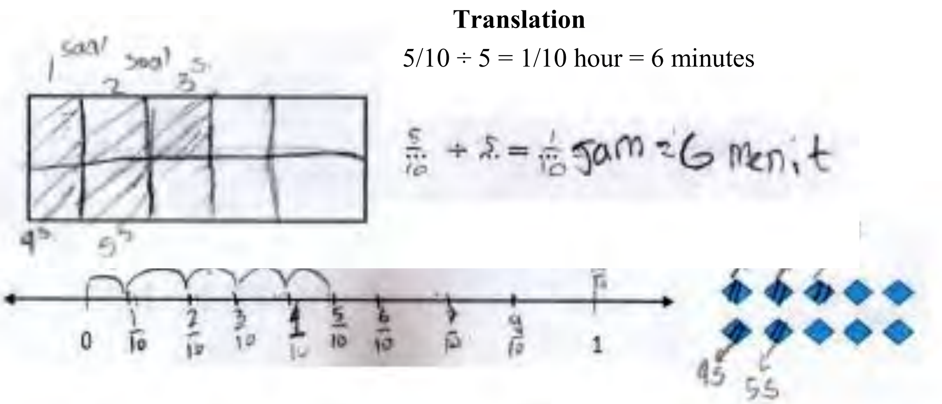


Figure 6. The works of students in the focused group for problem 2.1

The mathematical problem was different from problem 1.1 (in terms of chosen fractions), where the students *could not directly associate 3/4 with 5 problems by using the graphical representation*. A further analysis was required to find out that 3/4 can be written as 15/20. The students also had no problems in solving problem 2.2, being given in the first teaching experiment also (Figure 7). Problem 2.1 and 2.2 were given and discussed in one lesson.

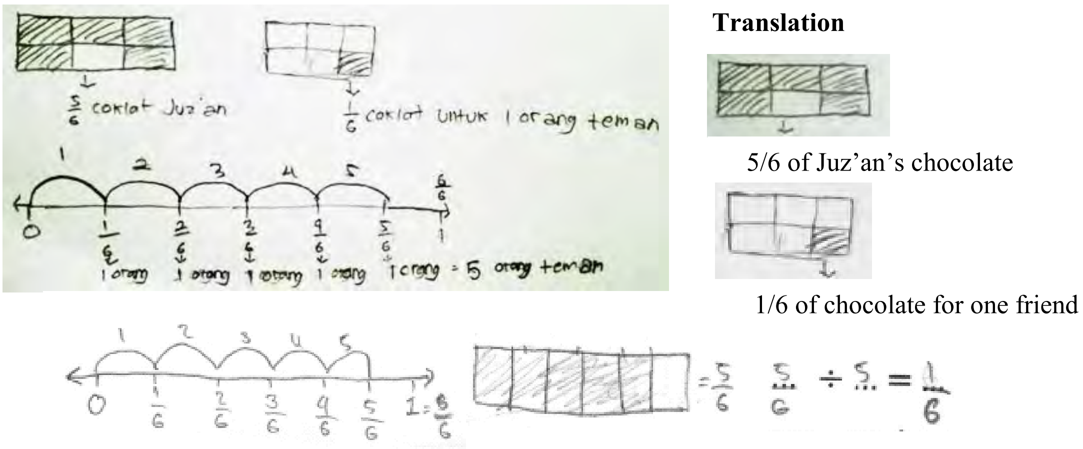


Figure 7. The sample of students' works on problem 2.2

Problem 2.3 was given in the next lesson. Each students' work on the problem shows a contradictory result. There were only eight students who properly used a graphical representation to solve the problem. Figure 8 and Figure 9 shows excerpts of two students' works. In the focus-group, three of four students were amongst eight students who correctly solved the problem using graphical representations.

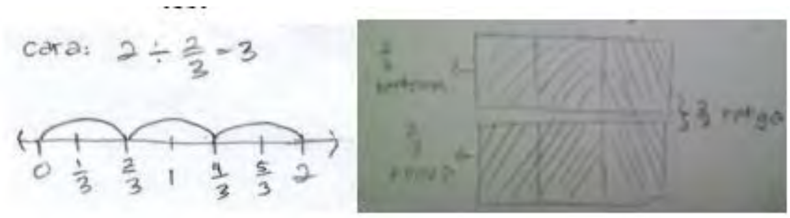
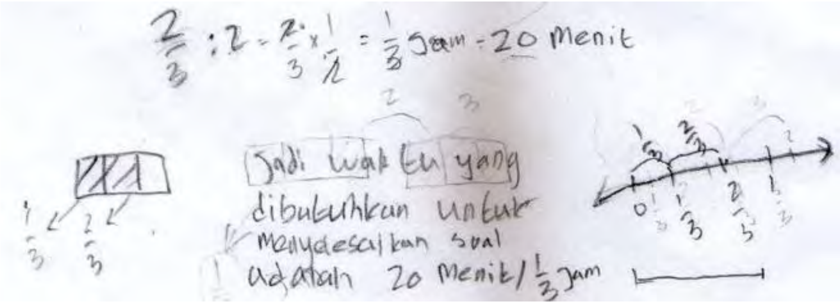


Figure 8. The sample of a student's work in a non-focus-group for problem 2.3

The second teaching experiment uncovers several findings: (1) using similar contexts and different contents of fractions (5/10 ÷ 5), the students were able to solve the problem using graphical representations immediately. It shows us that, although with similar contexts as used in the first teaching experiment, the use of **a**/**b** ÷ **c** (**a** is divisible by **c**) is easier for students than 3/4 ÷ 5 (**a**/b ÷ **c**, **a** is not divisible by **c**). Thus, using a specific sequence of fractions to teach PFD seems important in supporting the students' understanding of PFD. This assumption is also supported by the students' works on the problems 1.2 and 2.2 which consisted of similar fractions; (2) the cause for the students' work on problem 2.3 showing different results compared to problem 1.3 might be the attention that the teachers put on the unfocused groups.

Drawing from the findings of the two prior teaching experiments, in particular using context-related graphical representations and a better sequence of factions for the PFD problems and further analysis of the PFD concept (Sinicrope et al., 2002; Gregg & Gregg, 2007; Lamon, 2012), we extended the problem-contexts and changed the context into the chocolate context for the third teaching experiment. The primary consideration behind the use of the chocolate context was the fact that students could be more used to this context as well as our observation that working with non-cake contextual problems lead to students' difficulties in self-developed graphic representations. Besides the area model we picked up from the students' use, we included three more graphical representations of chocolates: The chocolate bar (area model), chocolate stick (number line), and chocolate chip (sets of objects) (Petit et al., 2016) so that the students were supported in using more than the area model.



Translation

1/3 hour = 20 minutes

So, the time required to solve one problem is 20 minutes/ 1/3 hour.

Figure 9. A student's work in the focused group for problem 2.3

The Third Teaching Experiment

In this teaching experiment, the students were initially given ten problems. The problem 3.1 is a whole number of partitive division and problems 3.2-3.10 were PFD-problems, four of them being unit rate problems. The problems were solved and discussed in two lessons. Additionally, one lesson was designed to discuss the revision of unit rate problems since the students had shown difficulties understanding the unit rate problems in the previous lessons.

For the first lesson, we focused on problem 3.1-3.5, which consists of sequenced fractions as follows: 6 ÷ 3, 5/6 ÷ 5, 1/2 ÷ 3, 4/7 ÷ 2, and 2/3 ÷ 3. The sample word problem is that *Budi has 4/7 of a chocolate chip that has to be shared equally with his two friends. How many part(s) of the chocolate chip does each friend get?* Solving problems with these new contexts involving whole number divisions was not complicated for the students. Eight students drew area models (Figure 10) to represent the chocolate; meanwhile, two students used sets of objects (chocolate chip) to determine the amount of chocolate each child gets. The idea of providing the students with partitive whole number division problem is to develop the notion of partitive or fair-sharing. Accordingly, the notion could be extended to the partitive division with fractions.

Translation: anak (child)

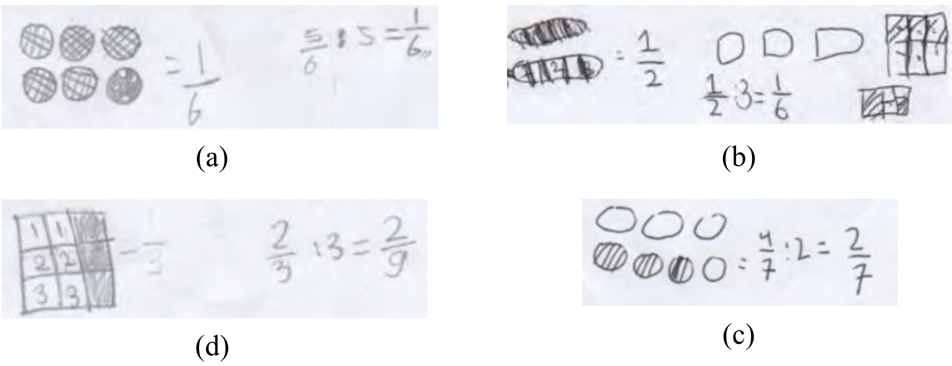
Figure 10. A student's answer for whole number partitive division

As in the two preceding teaching experiments, the students also found it easy to write 5/6 ÷ 5 as a number sentence and draw an area model as well as a set of objects (Figure 11.a) to solve problem 3.2. When a student was asked how she determined 1/6, she explained that "Ani has 5 Choco left (pointing to five similar shaded parts). One friend gets this (points to one); it is 1/6." For problem 3.3, a student we approached already drew two chocolate sticks and determined 1/2 of it (Figure 11.b). He then partitioned one chocolate stick into five. The following transcript shows our discussion.

R : *Could you explain your drawings?*
S : *I divided a chocolate into five to share to three friends*
R : *Why did you make five parts of it when there are only for three friends?*
S : *... (He took about a minute to rethink his drawing then he suddenly changed the chocolate stick into 3 parts and wrote the associated numbers). It is three parts for three friends, not five parts.*
R : *Could you draw a rectangle to solve the problem?*
S : *... (Drawing a rectangle and partition it into six parts, Figure 11.b). Non-shaded parts are for three friends, each gets 1/6*

Of 10 students, only one student used a graphical representation and yet determined the quotient incorrectly. All students could construct graphical representations to solve the two remaining problems (Figure 11.c and Figure 11.d).

In the second lesson, the fractions used for unit rate problems (3.6.a - 3.6.c) were 3/4 ÷ 1/4, 3/4 ÷ 2/3, and 3/4 ÷ 1½. Problem 3.6.d asks the students to decide which cake box is the smallest one based on their solutions to three prior problems. The excerpt of the word problem is shown in Table 1 (3.6.a).



(a) (b) (d) (c)

Figure 11. Students' selected answers for the PFD problems

All students had no ideas on how to solve problem 3.6.a, from drawing a suitable graphical representation and determine a related symbolic representation. In this case, we led the students to focus on drawing graphical representations of available chocolate with a cake box. The transcript below reveals a student's difficulty with the problem.

S : *... (Drawing the representation of chocolate and cake box, Figure 12)*
R : *How the box was 1/4-filled with 3/4 of a Choco bar?*
S : *... (Drawing right-side area model where one of four parts shaded to represent 1/4-filled box)*
R : *How it relates to 3/4 of a Choco bar?*
S : *... (Thinking some seconds). It is difficult. I have no idea to solve it.*
R : *Could you draw 3/4 of a choco filled in the box?*
S : *... (he partitioned the model vertically to have 3 parts)*
R : *What do you think about the remaining unshaded parts? What amount of chocolate does it represent?*
S : *... (writing 3/4 in each 1/4 of the box)*
R : *So, how many chocolates are needed to fill up the whole cake box?*

S : *... (Thinking some seconds). I do not know Sir.*
R : *Well, look at the upper rectangle. How many parts of a chocolate?*
S : *... (marking every four parts in the right-side area model which represents one chocolate) Three chocolates Sir.*
R : *Could you write number sentence for the problem as you did in previous problems?*
S : *... (Reading problem 3.6.a) Sir, I do not know. It is different problem.*

We proceeded to the second problem and found that the students were again not able to understand the problem even with our support. Problem 3.6.b and 3.6.c are more complex than problem 3.6.a since it involves non-unit fraction as the divisors, which means that the students had to deal with remainders. Considering students' difficulties, we tried finding alternatives to revise the problems.

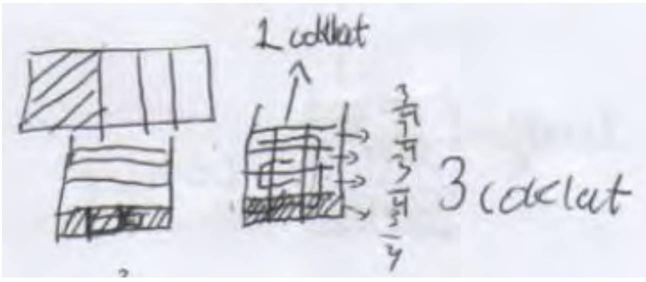


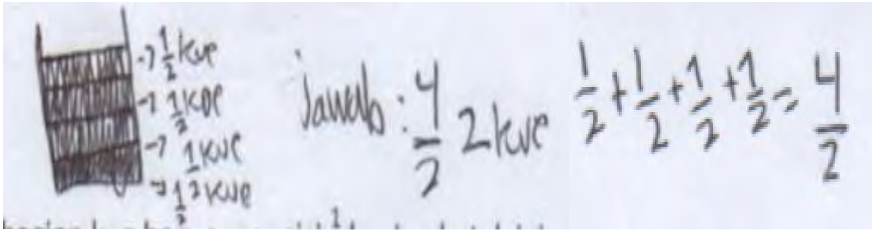
Figure 12. Student's answer for the initial unit rate problem

The third lesson was purposefully arranged for discussing the revised unit rate problems. The findings in the second lesson showed students having troubles in understanding unit rate problems. In this case, we tried to make the unit rate problems easier by modifying the number of cakes to be filled in the Cakebox (Table 2). Our new word problem then was "*3/4 of a cake only fill up 1/4 of a Cakebox, how many cakes are needed to fill up a whole Cake box?"*.

Table 2. Revised unit rate problems

Part	Available cakes	Filled Cakebox	Symbolic representations
A	1		1 ÷ 1/4
	1/4	1/4	1/4 ÷ 1/4
	1/2		1/2 ÷ 1/4
	3/4		3/4 ÷ 1/4
B	1		1 ÷ 1/2
	1/4	1/2	1/4 ÷ 1/2
	1/2		1/2 ÷ 1/2
	3/4		3/4 ÷ 1/2
C	1		1 ÷ 3/4
	1/4	3/4	1/4 ÷ 3/4
	1/2		1/2 ÷ 3/4
	3/4		3/4 ÷ 3/4

For part A and B (as depicted in Table 2), the students managed to understand and solve the problems quickly since the filled box represents a unit fraction which lead the students to use the repeated addition strategy (Figure 13). The problems in part C include non-unit fractions as the divisor. As a result, the former strategy is not applicable and the students are *required to understand the unit and remainder*. Although the problems are more complicated than the previous problems, the students' experiences of working with part A and B seemed quite helpful in determining how to fill the box with one given cake. *However, the students remained unable to determine the symbolic representations and how it relates to the problem-context.*



Translation: kue (cake) Answer: 4/2 2 cakes …

Figure 13. Students' works for revised unit rate problems (1/2 ÷ 1/4)

The third teaching experiment reveals some new findings: (1) the whole number partitive division seems to be a good conceptual base to understand fair-sharing-division tasks involving fractions as divisors. Understanding how to divide 6 chocolates equally to 3 children will help students thinking about how to share 5/6 of a chocolate equally to 5 friends; (2) the use of context-related graphical representations supports the students' in using self-developed representations; (3) the sequence of fractions plays an essential role in helping the students understanding tasks with PFD as shown in non-unit rate problems and revised unit problems. This means that at the early stages of instruction working with problem-contexts with fractions like 5/6 ÷ 5 is easier than 4/7 ÷ 2 or 1/2 ÷ 3 and 2/3 ÷ 3. Similarly, directly dealing with 3/4 ÷ 1/4 in a unit rate context is more difficult than a sequence as shown in Table 2; (4) unit rate problems pose a higher level of difficulty for primary students in terms of connecting it to fraction division problems (symbolic representations) and working with graphical representations compared to non-unit rate problems. We confirm the finding that unit rate should be considered as the different conceptualization from PFD (Sinicrope et al., 2002); (5) the students were able to understand the non-unit rate problems by using graphical representations to solve the problems, determining the number sentences appropriate to the problem context, and linking number sentences with developed graphical representations. However, with revised unit rate problems in the third cycle of our teaching experiments, the students could only solve the problems using graphical representations.

The results of the three cycles of teaching experiments in partitive division with fractions consisted of three main findings: (1) the context of the task, the relation of the context to the used graphical representations, and the sequence of fractions used support the students' understanding of PFD; (2) the students' developed understanding in non-unit rate problems seems not to be supportive for solving unit rate problems; and (3) the students are incapable of determining symbolic representations from unit rate problems and linking the problems to fraction division problems. Drawing from our findings, the following design principle (van den Akker, 1999; Bakker, 2018) can be formulated to support students' understanding of PFD:

> *Given that the students have no experience in contextual problems and graphical representations in their prior learning trajectories of fraction and if they instead focused the operational and procedural aspect of learning mathematics, then following aspects should be considered in planning a teaching-and-learning-environment to support the students' conceptual understanding: (1) Starting from whole number partitive division to develop the sense of fair-sharing or partitive division, (2) strengthening the concept of unit in fraction and partitioning to help the students in determining the unit as well as the remainder, (3) choosing appropriate contexts as well as related graphical representations, and (4) sequencing the fractions used, from a simple to advanced form being adjusted to the context.*

Although further research seems necessary, these initial findings should be discussed with mathematics educators and teachers, who intend to design mathematics activity involving PFD.

It is argued that the use of students' familiar contexts and graphical representations is a viable starting point for the learning process. A familiar context is a context embedded in the mathematics tasks (contextual problems) which is accessible mentally or psychically by the students to help them solve the tasks, for example drawing graphical representations. Gravemeijer and Doorman (1999) argue that contextual problems are an important starting point in RME since they function as anchoring points for a guided reinvention of mathematics. Several other studies reveal that the didactical use of graphical representations helps the students construct mathematical meaning (van den Heuvel-Panhuizen, 2003; Cramer, Wyberg, & Leavitt, 2008; Cramer et al., 2010; Wahyu, Amin, & Lukito, 2017). One of the criteria to select the contexts is its potential to be modelled with graphical representations (Fosnot & Dolk, 2002). We argue that the extent to which the contexts are accessible or familiar to the students depends on the students' daily experiences. For example, we decided to employ the time context in the early design of the problems since it is a typical daily context being used in prior study (Muchsin et al., 2014). However, the students were struggling with the representation of the 'hour' in the first teaching experiments although being familiar with the time-context, which we assume may be connected to the fact that '3/4 hour' cannot be represented well by the area model nor a set of objects. In contrast, when given a cake context, the students 'by default' drew a rectangle to represent 5/6 of a chocolate. Our conclusion is that in order to successfully *model* the problem-contexts, the contexts should be familiar to the students as well as be related to specific graphical representations.

Beside of the context-related graphical representations, the sequence of fractions used does seem to matter in supporting students' understanding (Gregg & Gregg, 2007; Schwartz, 2008; van de Walle, Karp, & Bay-Williams, 2015). The fractions introduced in the primary schools are sequenced, for example at first unit fractions are taught in the lower grade (grade 3) and then followed by non-unit fractions (NCTM, 2000; Common Core State Standards Initiative). In the first teaching experiment, problem-contexts involving 5/6 ÷ 5 were easier to solve than 3/4 ÷ 5. Two of the fractions used in the teaching experiment, 5/10 ÷ 5 and 3/4 ÷ 5, seem to require distinct approaches from students even though they have the same task contexts. In the last teaching experiment, we re-arranged the sequence. A legitimate quest might be, why 5/6 ÷ 5 is introduced at the beginning and then followed by the task 1/2 ÷ 3, which include unit fraction. We observed that when using graphical representations to solve 5/6 ÷ 5, the students could directly determine the size of each group since the numerator is divisible by the divisor (Figure 11.a). Thus, it seems appropriate to use these fractions as the starting point in the learning process. In contrast, solving 1/2 ÷ 3 by using graphical representations seemed to require the partition of 1/2 in order to relate with the divisor (Figure 11.b). Table 2 presents the sequence of fractions for the unit rate, which significantly differs from the fractions used in the second lesson if compared to the last teaching experiment. Drawing from the findings of the current study, prior relevant studies (Gregg & Gregg, 2007; Wahyu et al., 2017), and theoretical considerations (Schwartz, 2008; van de Walle et al., 2015), sequencing fractions seems very important to support students' understanding of fraction operations in general and PFD in specific. Other studies did not examine the sequence of fractions, but they provided fruitful insights concerning students' works through problem-context on PFD (Muchsin et al., 2014; Roni, Zulkardi, & Putri, 2017; Rianasari & Julie, 2018). Thus, a further study is required to examine whether the use of sequenced fractions correlates to the effectivity of teaching and learning fractions operations.

It is characteristic of partitive division that the number of groups is known, but the size of each group must be determined (Gregg & Gregg, 2007). A typical partitive fraction division is **a**/**b** ÷ **c**, either **c** divides **a** or not (Sinicrope et al., 2002; Adu-Gyamfi et al., 2019). It occurs to be problematic for the learners if the divisor is not a whole number in a fair-sharing context. However, Ott et al. (1991) argue that determining the size of one group or set is equivalent to calculating the unit rate. Likewise, the partitive division can be perceived as determining the unit value (quantity per one unit), the divisors being a whole number or not (Okazaki & Koyama, 2005 cited Vergnaud, 1983). Regardless of the normative difference of PFD conceptualization derived from a theoretical viewpoint, comparing the way students understand unit rate and non-unit rate problems from a descriptive point of view as it is done in this study can help to understand the individual notions of understanding partitive fraction division. On the one side, for example in problem 3.5 (Table 1), episodic situations (Staub & Reusser, 1995) could help to comprehend the way students construct a mathematics problem model or number sentences regarding the operation 2/3÷3. On the other side, for example in problem 3.6.b (Table 1), the students could not even find a *textbase* (Kintsch & Greeno, 1985) to reach text comprehension as a basis to the episodic situation, and thus could not construct a problem model (Staub & Reusser, 1995). We argue that the students are capable of solving the problem 3.6.b by using the strategy of the addition or multiplication of fractions since more chocolate is needed to fill up the whole cake box. Future studies are undoubtedly required to identify whether or not this gap is purely the issue of students' ability in working on both problems rather than different fractions which seem to hinder students' connection of fair-sharing to the unit rate.

There are several limitations to the current study. Firstly, although we attempted to sequence the fractions in the problem-context, not all types of fractions are covered, such as a mixed number (2 3/4), in fair-sharing. For this reason, this study does not provide insights into how students solve the partition-problem of mixed numbers to determine the size of one group. Secondly, the current study was focused on encouraging the use of graphical representations to solve the problem-context, so it did not provide opportunities for students to solve the problems employing other strategies such as formal algorithm as we found in the problem 2.3 of the second experiment. These two cases could be an essential entry for future studies.

CONCLUSION

This article explicates primary students' understanding of partitive division with fractions and proposes design principles to promote their understanding. Students' understanding seemed to be linked to three criteria; using graphical representations to solve mathematical problems, determining symbolic representations, and linking graphical representation-based solutions of the problem-context with the symbolic representations. The three aspects could be reconstructed in the students' solutions for the non-unit rate problems, but the latter two criteria could not be reconstructed in the learning trajectories for the unit rate problems. Furthermore, the teaching experiment revealed that students' understanding of fair-sharing is not helpful in understanding the unit rate (further research is needed to confirm this observation). Students' understanding of fair-sharing and fraction division as well as unit rate problems seems to be supported by the use of context-related graphical representation and sequenced fractions.

The current study resulted in developing four important design principles for the instruction of partitive fraction division, namely begin with partitive division with whole numbers to develop a basic conceptual understanding of the fair-sharing, re-introduce and reinforce the concept of unit and partition of fractions, utilize context-related graphical representations, and carefully sequence the fractions. A further teaching experiment is required to try out the design principles with the developed problem-contexts for PFD.

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