# Angling for the Right Result: Students' Conceptualizations of Angles

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*The concept of angles is important for future geometric knowledge (Arslan et al., 2016; Moore, 2013; Yigit, 2014). However, although Piaget (1948) suggests angles lead to the discovery of lines, angles are typically taught later in schools, after points, lines, and planes (Charles, 2011). Therefore, the way in which angles are taught can affect how students conceptualize angles. This study investigates such phenomena by using APOS theory and a preliminary genetic decomposition. Data was collected from a fifth grade student, a seventh grade student, and a twelfth grade student. Results show: the fifth grader developed a 2- and 1-line schema for angles; the seventh grader developed a 2-line schema; and the twelfth grader developed a 2-, 1-, and 0-line schema. Overall, the students contradicted the genetic decomposition model due to the fluidity of their thinking. This study provides teaching implications as well as areas for future research.*

Keywords: *geometry, angles, genetic decomposition, APOS theory*

In geometry, angles are an important concept for students to understand in order to progress in their geometric knowledge (Yigit, 2014). According to Piaget (1948), children begin drawing shapes as scribbles. They then draw curves in order to make different shapes. From curves, children later abstract them into straight lines. Figuratively, this indicates that angles lead to the discovery of straight lines and not the reverse. However, it appears that in some curriculum angles are formally introduced after rays, lines, and line segments (Charles, 2011). It also appears teachers develop many different views of angles including erroneous ones (Arslan et al., 2016; Kontorovich & Zazkis, 2016), resulting in student confusion and misconceptions (Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990). Due to these erroneous views, students' abilities to pursue further geometric topics such as angle measurement, right triangles, and trigonometry are hindered (Arslan et al., 2016; Moore, 2013; Yigit, 2014). However, few studies have identified specific ways in which students conceptualize angles by including the specific mental actions. More insight could be gained by looking at the specific mental actions and operations students construct during their experiences with angles.

The purpose of this study is to investigate students' conceptualizations of angles. The goal is to generally describe how students think about and define angles. Therefore, the paper is guided by the question: How do students conceptualize angles? This study examines three students' conceptualizations of angles during a clinical interview. By using APOS theory as a theoretical framework, a model of students' conceptualizations of angles is proposed, along with a hypothesis for each student. For example, I hypothesize that students who view angles as static figures, involving no mental actions, will be limited in their understanding of angles; students who view angles as dynamic figures, involving mental actions, will have a more sophisticated understanding of angles Analysis of data and results will be discussed. Based on the results, the model will be revised to accommodate new insights into students' views of angles.

## Review of Literature

As previously mentioned, angles are an important concept for students to understand in order to progress through many other mathematical concepts. However, there are many factors that influence students' views such as how teachers define, identify, and classify angles (Arslan et al., 2016; Kontorovich & Zazkis, 2016; Yigit, 2014). When investigating students' conceptualizations of angles, it is important to explore each of these areas to identify where misconceptions might arise and where appropriate conceptions are constructed. Therefore, more insight could be gained by investigating the various definitions, types, and classifications of angles.

### Definitions of an Angle

In their textbook, Henderson and Taimina (2005) define angles as "the union of two rays (or segments) with a common endpoint" (p. 38). Charles (2011) also defines an angle as two rays with the same endpoint. This indicates that students must understand what rays or line segments and endpoints are before they can understand angles. In another textbook, Greenberg (2008) uses the following definition for angles: "An angle with vertex A is a point A together with two distinct non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the sides of the angle) emanating from A" (p. 18). From this definition, again students must understand what a ray, point, and vertex are to understand angles. However, this definition goes even further than the previous one in that the rays cannot be opposite. This means that a straight line is not an angle. Similarly, Long (2009) defines an angle as "two rays with the same endpoint that do not lie on the same line" (p. 23), further emphasizing that straight lines are not angles. In contrast, the Merriam-Webster dictionary provides several definitions for an angle including the following: (a) a corner, (b) a figure formed by extending two lines from the same point, and (c) a turn (angle, 2017). Depending on the accepted definition, students can develop different conceptualizations of angles.

**Types of Angles**

Piaget, Inhelder, and Szeminska (1960/1981) also discuss the different stages students progress through as they learn about angles and angle measurement. In Stage I, students ages four to five draw and measure angles from visual estimates. As they progress to Stage II at around age six to seven, they begin using tools, but still only view the angle as a perceptual figure (Piaget et al., 1960/1981). In Stage III, students age seven to eight develop more complex skills and try to use slope to measure angles. However, they still fail at recognizing angle measure as the openness or linear separation between the two lines. It is only in Stage IV that students ages ten and above can begin to coordinate many actions within different dimensions to understand angles as the linear separation between two lines. These more developed students no longer see angles and their measurements as figural, but as operations and actions (Piaget et al., 1960/1981). Therefore, it would be beneficial to investigate how textbooks and teaching materials define an angle to see how angles are taught, thus providing insight into how students conceptualize angles.

**Dynamic and Static Angles.** Related to the description given by Piaget et al. (1960/1981) of how students view angles differently, many other studies have shown that students typically develop two views of angles: static and dynamic (Bütüner & Filiz, 2016; Clements & Burns, 2000; Kontorovich & Zazkis, 2016; Smith et al., 2014). Static angles are viewed as a geometric shape or figure (Kontorovich & Zazkis, 2016; Smith et al., 2014) in which the student focuses on what the angle looks like in relation to the location or position of the sides (Bütüner & Filiz, 2016). In this sense, static angles are simply pictorial or figurative representations. As a result, students can have a difficult time identifying angles in different positions, such as 0o, 180o and 360o angles (Smith et al., 2014). In their study, Bütüner and Filiz (2016) discovered that high achieving sixth grade students developed many misconceptions related to angles, leading to a static view of angles. Although these students were high achieving students, they did not perform well on tasks and the total average of percent correct for the six tasks was thirty-four percent. This supports the idea that static angles do not lead to a deeper understanding of angles.

In contrast, dynamic angles involve movement or actions, in which students sweep, drag, or rotate a line or ray to make an angle (Clements & Burns, 2000). Additionally, students use body movements or gestures to represent dynamic angles (Clements & Burns, 2000; Smith et al., 2014). As a result, students are able to understand more variety of angles and gain a deeper understanding of angles. For example, Clements and Burns (2000) discovered when students use these dynamic actions for angles, they are able to develop schemas that can then be utilized in later situations. As a result, students gain a deeper understanding that promotes more accurate angle measures and connections between concepts (Clements & Burns, 2000).

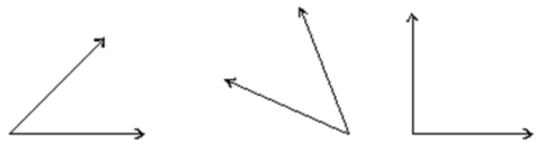
**Angle as a Union, Wedge, or Turn**. Referring back to the various textbook definitions, angles can be broken down into three main categories: union, wedge, or turn. An angle as a union is seen as the intersection of two rays or line segments at a common vertex (Mitchelmore & White, 1998; Keiser, 2004). Based on this definition, this would be classified as a static angle (Clements & Burns, 2000; Browning et al., 2008). Here, students view the position of the angle and not the movement that was involved in the creation of the angle (Mitchelmore & White, 2000). This is what is typically known as the standard angle in which both sides of the angle are visible (Mitchelmore & White, 1998, 2000).

An angle as *wedge* is seen as the space or region between two rays or line segments (Browning et al., 2008; Yigit, 2014). This could be considered the openness or linear separation as described Piaget et al. (1960/1981), and is also based on the position of the rays or lines segments (Yigit, 20104). However, there is no movement related to the position; this type of angle would still be considered a static angle. Despite the lack of motion, the conceptualization of this angle can lead to a deeper understanding of angles, resulting in more accurate angle measure (Browning et al., 2008; Wilson, 1990).

Finally, an angle as a turn represents the amount of rotation between two lines, or the rotating of one line (Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Yigit, 2014). A turn would represent a dynamic angle as it could be described by using a sweeping motion for the amount of rotation (Browning et al., 2008; Clements & Burns, 2000; Kontorovich & Zazkis, 2016; Smith et al., 2014). By developing an angle as a turn and emphasizing the dynamic motion, students are able to develop a more abstract conceptualization of angles (Mitchelmore & White, 2000), leading to higher performance and greater learning gains (Smith et al., 2014).

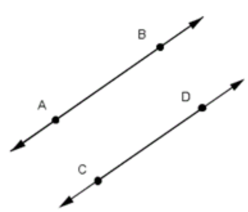
**2-line, 1-line, and 0-line Angles**. Yigit (2014) classifies three distinct types of angles: 2-line, 1-line, or 0-line angles. 2-line angles are angles such that both sides of the angle are visible and easily identifiable. With 2-line angles, students can clearly see the two rays or lines that result in the creation of the angle. For example, this would be the traditional and most widely accepted view of an angle such as the definitions of an angle provided by Greenberg (2008) and Long (2009). In this case, 2-line angles could be considered unions or wedges, depending on how a student views the angle. Additionally, these angles would be considered static angles since there is no action involved in the creation of the angle. It is simply a pictorial or figurative representation. Figure 1 represents those types of angles.

*Figure 1*: 2-line Angles



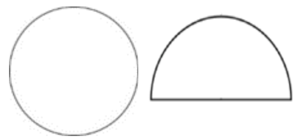
*1-line angles* are formed by two opposite rays, creating a straight line. These angles measure $180^{o}$ (Yigit, 2014). In contrast to the traditional view of angles, these angles are straight lines, such as those represented in Figure 2. As a result, some people would classify these not as angles, but rather as lines. However, according to the definition of an angle given by Henderson and Taimina (2005) along with the classification of 1-line angles, these figures are indeed angles. Similarliy, these types of angles could also be considered unions or wedges depending on how students look at them. However, these angles still contain no actions and would also be considered pictorial or figurative representations. In this case, 1-line angles would be considered static angles as well.

*Figure 2:* 1-line Angles




*0-line angles* are formed by rotating a line about a point. These angles are those that measure $360^{o}$ or are simply viewed as turns in which there are no visible lines or sides to the angle (Yigit, 2014). Similar to 1-line angles, these are also in contrast to the traditional view of angles such as those represented in Figure 3. Whereas some people would classify these figures as a circle and a semi-circle, Merriam-Webster defines an angle as a turn (angle, 2017), thus indicating these figures are 0-line angles and could be classified as turns. It is evident these angles are action-based due to a rotation. In this case, 0-line angles would be classified as dynamic angles.

*Figure 3:* 0-line Angles



As previously mentioned, depending upon the accepted definition of an angle, students can view angles as either dynamic or static. Furthermore, they can conceptualize angles as either a union, wedge, or a turn, in addition to the more complex conceptualization of angles as 2-line, 1-line, or 0-line angles. From these more distinct classifications, a connection can be established where static angles include unions and wedges, as well as 2-line and 1-line angles. It then follows that dynamic angles include turns and 0-line angles (Yigit, 2014). Contingent upon the student's view of angles, this will lead to different schemas in which students then use to solve more difficult geometric problems.

**Misconceptions**

Due to these conflicting angle concepts, student might develop misconceptions during their learning process. The major misconception students might develop is that angles are only static figures that are represented by the intersection of two lines or rays (Bütüner & Filiz, 2016). When trying to identify angles, students might only select those with two visible rays or lines, and not straight lines or circles (Clements & Battista, 1989, 1990). Furthermore, students might have a difficult time recognizing angles in different orientations, rather than the typical horizontal angle orientation (Bütüner & Filiz, 2016). Another misconception students might develop, in relation to angle measurement, is that angle measurement can be determined by measuring the side lengths of the angles (Bütüner & Filiz, 2016), rather than measuring the openness or linear separation between the two rays or lines (Piaget et al., 1960/1981). Students might also think about angle measure being related to the size of the arc representing the angle or the amount of space between the sides of the angle (Bütüner & Filiz, 2016).

Depending on the constructed schemas, students can conceptualize angles in many different ways while also allowing room for various misconceptions. As a result, it is important to investigate and determine the mental structures and mechanisms students construct and utilize in order to conceptualize angles.

## Theoretical Framework

### APOS Theory

APOS theory is a theoretical framework used for researching student thinking and learning. It was created as an extension of reflective abstraction as a way to understand students' mathematical thinking (Yigit, 2014). The goal is to "create a model to investigate, analyze, and describe the level of students' mental constructions of a mathematical concept" (Yigit, 2014, p. 712). Within this model, researchers identify actions, processes, and objects that lead to the development of a schema for a mathematical concept. More specifically, researchers describe the "Actions that a student needs to perform on existing mental Objects and continues to include explanations of how these Actions are interiorized into Processes" (Arnon et al, 2014, p. 28). Then from this model, researchers can test it and determine the different levels of student thinking.

At the action level, students take different steps to perform certain procedures when triggered by an external stimuli (Yigit, 2014, p. 712). They transform objects based on certain rules and instructions (Dubinsky & McDonald, 2001). Then, once the action is "repeated, reflected upon, and/or combined with other actions" (Yigit, 2014, p. 713), students develop an internal construct called a process. At this level, students no longer need an external stimuli to perform procedures, but rather can think about performing the actions without physically doing so (Dubinsky & McDonald, 2001). Next, students encapsulate the process into an object (Yigit, 2014) when they view it as an entity and "realize that transformations can act on it" (Dubinsky & McDonald, 2001, p. 3). Students can then organize their actions, processes, and objects into a schema in order to accommodate "the new knowledge discovered from the mathematical problem" (Yigit, 2014, p. 713). By looking at these different levels, researchers can track student learning and determine whether or not students have developed appropriate schemas for mathematical concepts, allowing for predictions of success or failure (Dubinsky & McDonald, 2001), thus gaining insight into student learning.

**Genetic Decomposition**

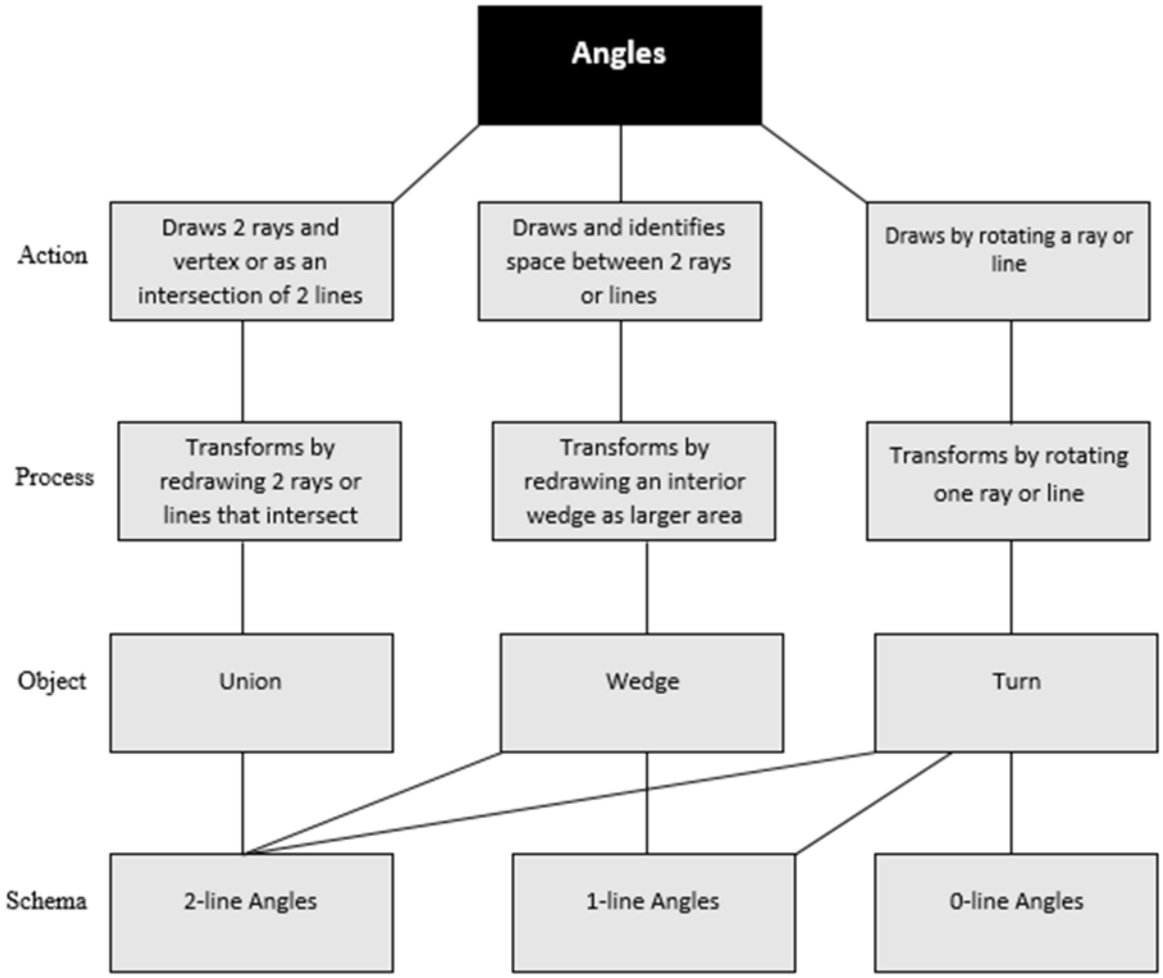
When trying to study and understand student learning through APOS theory, researchers might consider a genetic decomposition. A genetic decomposition is a model of students' mental structures concerning how they construct a mathematical object (Arnon et al., 2014). It provides insight into "the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept" (Arnon et al., 2014, p. 27). For example, a genetic decomposition includes actions, processes, objects, and schemas students use in the construction of mathematical concepts (Arnon et al., 2014), thus highlighting its important role in APOS theory.

By using a genetic decomposition, researchers can develop "hypotheses about how learning mathematical concepts can take place" (Dubinsky & McDonald, 2001, p. 4), which can then be used to predict future phenomena and mental constructions (Arnon et al., 2014), thus providing greater insight into students' understandings of mathematics. Furthermore, the model can help identify gaps in students' mental processes and identify areas of misconceptions. This can help guide future research and inform teachers' instructional designs that support the development of students' mathematics, ultimately creating a bridge between theory and practice (Arnon et al., 2014). Also by looking at a genetic decomposition of a mathematical concept, researchers can begin to understand what skills and understandings students must possess and in what order they should be developed, thus informing teachers' curriculum and pacing. Therefore, a genetic decomposition can play an important role in understanding students' learning of a mathematical concept by serving as a model that can be tested for viability (Arnon et al., 2014), and then used to predict future thinking and learning (Dubinsky & McDonald, 2001).

**Model of Students' Learning of Angles**

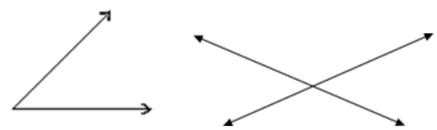
Based on the aforementioned definitions and types of angles, there are many different structures students can develop during their learning of angles. However, one possible genetic decomposition model for students' learning of angles is represented in Figure 4. By following APOS theory and considering the different types of angles, this genetic decomposition highlights the actions, processes, objects, and schemas students construct during their learning of angles.

*Figure 4.* Proposed Genetic Decomposition of Angles




**Action.** During the action stage, students construct and identify angles by performing procedures (Yigit, 2014), either by using different tools or sensory-motor skills. When asked to draw an angle, students might construct an angle by freehand drawing two lines or rays that intersect (Figure 5). This represents the action related to the concept for union or wedge angles, specifically the step by step instructions (Dubinsky & McDonald, 2001) used to draw two rays that intersect at a vertex. Students could also use tools such as a ruler to draw straight lines to represent this action. Here, even though these angles are static, the student must perform mental actions in order to construct an angle as related to the concept of a union or wedge angle.

*Figure 5.* Action for Drawing an Angle as Intersection of Two Lines or Rays

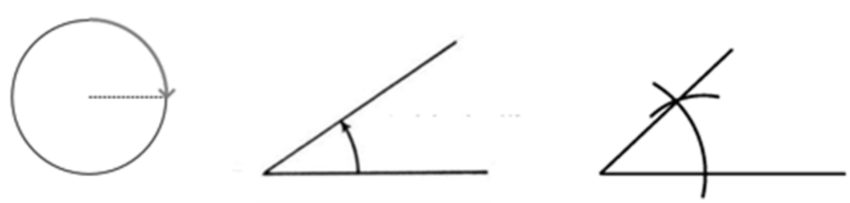


Furthermore, when asked to identify the angle, students might point to the complete picture they drew, indicating that the action of drawing the intersection determines an angle. Students might also identify the angle as the space between the two lines, or the amount of space that fills the area (Browning et al., 2008) by using arcs to represent the action related to the wedge angle (Figure 6).

*Figure 6.* Action for Identifying Angle as Space between Two Lines or Rays

However, students could also construct an angle by rotating a line or by sweeping the line (Clements & Burns, 2000). This could be done visually, by using sensory-motor actions of the finger, or by using a compass to draw an arc representing the sweep. Clearly, this represents a dynamic angle involving the action of rotation. Such actions are represented in Figure 7, where lines with arrows represent rotations or sweeps.

*Figure 7*. Dynamic Action for Rotation of Line or Ray



**Process**. Once students have performed these different construction actions and then repeated the same steps, reflected upon their actions, and/or combined with additional actions (Yigit, 2014), students can interiorize those actions and develop a cognitive process for the construction of angles. During this process, students can act upon, transform, and deconstruct angles (Yigit, 2014) without having an external stimuli (Dubinsky & McDonald, 2001). Students no longer have to work through all the steps to construct an angle from scratch, but can perform more complex actions by building upon on existing actions (Arnon et al., 2014). This is similar to the act of counting on, where students can begin at nine and add five more instead of starting over at one. Furthermore, after having developed a process, students can think about reversing the construction or composing it with other processes (Dubinsky & McDonald, 2001).

With this development of a process, students can transform the angles, construct larger or smaller angles, and give explanations as to why the new angle is larger or smaller (Yigit, 2014). For example, if asked to draw a larger angle, students would not need to redraw an original angle, but could use their existing one to complete a bigger one (Yigit, 2014). Students who drew angles such as those in Figure 5 could redraw the lines or rays going in a different direction to indicate a larger angle. Students would think about a larger angle as one who has lines or rays going in a different direction as the original angle (Figure 8), and can visually tell the angle is larger (Browning et al., 2008).

*Figure 8.* Process to Redraw Lines or Rays that Intersect

Similarly, students who drew angles such as those in Figure 6 could redraw their angles to represent a larger area or space between the lines or rays. These students would look for an interior region (area identified with an arc) that represented a larger space than the interior region of the original angle (Figure 9). Students would also look at the openness of the angle to determine the new angle had a wider opening (Clements & Battista, 1989).

*Figure 9.* Process to Redraw a Larger Interior Region


Finally, students who drew angles such as those in Figure 7 could redraw their angles by rotating or turning the lines or rays further than the original angle (Clements and Battista, 1989). Students would think about the amount of rotation needed in order to create a larger angle. Students could also think about how far the line is swept in order to create a larger angle (Figure 10).

*Figure 10.* Process to Redraw a Larger Rotation


**Object.** After students have developed processes for constructing and transforming angles, they can begin viewing the angle as a single entity (Arnon et al., 2014). In doing so, the processes are encapsulated into a mental object (Arnon et al., 2014) such that students view the angle as a cognitive object. At this point, students can coordinate the actions of drawing with the processes of transformation to create a mental object of the angle, instead of having to go through each action process in sequence. When the angle is viewed as an object, students can visualize the actions and processes used to create an angle. This is different than just seeing a picture of an angle. After a mental object is made, students can further act upon and transform the object (Arnon et al., 2014).

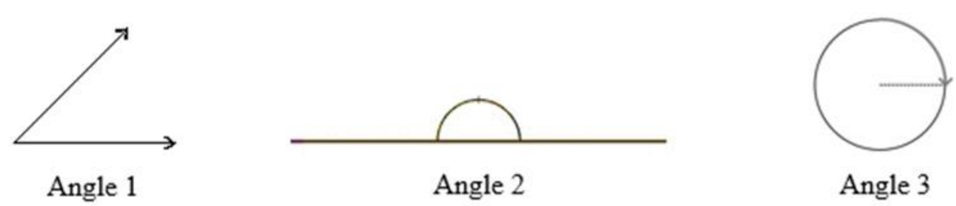
Based on the different definitions and types of angles, students develop three different angle objects. The first type of object students can construct is a union. This is where students construct, identify, and transform angles as the union of two lines or rays (Mitchelmore & White, 1998; Keiser, 2004). Once students coordinate the action of drawing two lines or rays that intersect with the process of redrawing a line or ray going in a different direction, students can further act upon the object by yet again transforming it and/or comparing it to other angles (Yigit, 2014). Again, students do not need to redraw angles and complete all the steps involved in the action and process stage, but can use the mental object of the union angle to further compare and transform angles.

Another type of angle object students might construct is a wedge. This is where students recognize angles as the space or region between two lines or rays (Browning et al., 2008; Yigit, 2014), or also as the openness or linear separation between the two sides (Clements & Battista, 1989; Piaget et al., 1960/1981). However, they simply do not see the wedge as a region with no underlying actions or processes. Instead, students can visualize all the actions and processes involved to create the wedge. Once they have encapsulated those actions and processes, they can then take the wedge as an object and further act upon it.

The final type of object students can construct for angles is a turn. Students who view angles as a turn can see the actions and processes of rotating a line or ray (Yigit, 2014; Clements & Battista, 1989, 1990). At this point, students can visualize the sweeping action and process of a line or ray in construction of an angle (Clements & Burns, 2000). Students can then perform more transformations and actions upon the object to create more angles and solve more complex problems. Although not the same objects in the Piagetian sense, the turn would technically be the only true mathematical object since it contains actions of rotating which can be composed with other actions and even be reversed.

**Schema.** After establishing angles as a mental object in terms of either a union, wedge, or turn, students can then organize and coordinate all the actions, processes, and objects to accommodate new knowledge (Yigit, 2014). At this stage, students have internalized their actions, processes, and objects to create a working system called a schema (Dubinsky & McDonald, 2001). This schema serves as a framework for future problem solving (Dubinsky & McDonald, 2001). Depending on the actions, processes, and objects, students can develop different schemas for different angles. For example, once students have gone through the actions and processes to establish angles as a mental object as a union, they might develop a schema for 2-line angles. Due to the nature of drawing angles as two lines or rays, students rely on seeing two distinct lines or rays when solving angle problems (Yigit, 2014). If those lines are not visible and do not match their mental object of an angle as a union, they can only solve problems when there are two visible lines or rays. Students would only be able to perform tasks including Angle 1, but not Angles 2 or 3 in Figure 11.

*Figure 11.* Schemas for 2-line, 1-line, and 0-line Angles



At this stage, one could classify this angle schema as only involving situational angles (Mitchelmore & White, 1998). Here students rely on a "physical angle situation" (Mitchelmore & White, 1998, p. 21) and look for similarities between their mental object and their current experience or situation. This is a very basic schema for solving angle problems but one that is most likely to develop in students due to the traditional definition of angles being formed by two non-opposite rays as defined by Greenberg (2008) and Long (2009).

On the other hand, if students have completed actions and processes to construct angles as a mental object as a wedge, they might develop a schema for 1-line angles. With this schema, students would be able to solve the same problems as with a 2-line schema, but would include a wider variety of angles. Instead of being dependent on two visible lines, students can now solve problems when only one line is visible. For example, when presented the angles in Figure 11, students who have developed a 1-line angle schema would be able to complete problems involving Angles 1 and 2, but not Angle 3. Since these students would most likely define angles as formed by two rays or lines, similar to the definition provided by Henderson and Taimina (2005), students would recognize that Angle 2 is formed by two opposite rays and therefore is an angle. However, students would not recognize two rays in Angle 3 and therefore not identify it as an angle.

With a 1-line angle schema, one could classify these students as being able to solve problems involving contextual angles (Mitchelmore & White, 1998). Although students look for similarities between the problem and their mental object, different situations may call for different actions. Therefore, "the actions performed and their purpose vary widely from situation to situation" (Mitchelmore & White, 1998, p. 21). Whereas a situational angle only applies to that one situation, a contextual angle "must represent that which is common to all situations in the context and not any specific situation” (Mitchelmore & White, 1998, p. 22). As a result, students who develop mental objects of angles as wedges can develop both a 2-line and 1-line angle schema, depending on the situation they are presented.

Lastly, if students develop mental objects of angles as a turn, they might develop a schema for 0-line angles. In contrast to schemas for 2-line and 1-line angles, students no longer rely on any visible lines. They now focus on the dynamic motion of sweeping or rotating as a means perform, act upon, and transform angles during different situations and problems (Keiser, 2004). However, this is not to say that students cannot also solve problems involving 1-line or 2-line angles. In fact, once students have developed a mental object of a turn, students could later reconstruct an angle as the intersection of two lines (Piaget, 1948). In addition, these students would most likely agree with Merriam-Webster’s definition of an angle as being either a corner, extension of two rays or lines, or a turn. Therefore, these students would be able to complete problems involving Angles 1, 2, and 3 in Figure 11. Due to this 0-line angle schema, students are now able to solve problems involving abstract angles (Mitchelmore & White, 1998). This means that students can solve problems involving angles from all different contexts. As a result, students can develop deeper knowledge of angles and solve more complex problems. A 0-line angle schema does not limit students to visualizations but allows for more abstract thinking. Overall, students who develop mental objects of angles as turns can develop a 2-line, 1-line, or even a 0-line angle schema.

By using this genetic decomposition, one can gain insight into how students conceptualize angles. Ultimately, students who create mental objects of angles as a union are limited with their figurative schema development, resulting in only a 2-line angle schema. This restricts students to look for visible lines or rays and focus on situational angles. Students who create mental objects of angles as a wedge can either develop a 2-line or 1-line angle schema, and depending on the situation, students activate different actions and processes, leading to a different schema. Finally, students who create mental objects of angles as a turn can progress further in their learning by developing a 0-line angle operational schema, but may as well develop a 2-line or 1-line angle schema, depending on the context. These students focus on dynamic motion and are able to solve more abstract problems. To clarify, as students progress to more complex schemas, moving from left to right in Figure 4, they begin to develop more operative schemas. For example, a 2-line angle schema is a more schema, whereas a 0-line angle schema is more operational since it involves the action of rotation. Based on this genetic decomposition, I hypothesize: (a) students who construct angles as an union object will be limited to developing a 2-line angle schema; (b) students who construct angles as a wedge object will develop a 1-line angle schema, but also a 2-line angle schema; and (c) students who construct angles as a turn object will be capable of developing a 0-line angle schema, but also a 1- and 2-line angle schema.

## Methods

One purpose of a genetic decomposition model is to gain insight into student's conceptualizations of a particular mathematical concept (Arnon et al., 2014). After developing the aforementioned genetic decomposition, I conducted interviews with three students to test the model to determine if it was indeed a viable model for students' conceptualizations of angles. I sought to gain insight into how different students with different geometry backgrounds understood angles, and to determine if there was a difference in their conceptualizations.

**Participants**

Three students were recruited from the same school district by making initial contact with parents and guardians I knew who had children in fifth, seventh, or any high school grade. Participation was voluntary for the students, per IRB approval. The first student Olivia (pseudonym) was a ten-year-old fifth grader who had no formal Geometry instruction. The next student Daniel (pseudonym) was a thirteen-year-old seventh grader, currently enrolled in an Algebra I course, who had some informal Geometry instruction. The third student Emma (pseudonym) was an eighteen-year-old high school senior, currently enrolled in a Geometry course, who had formal Geometry instruction. These students were purposefully chosen based on their geometry instruction background and also to provide data concerning students' conceptualization of angles. By selecting a fifth-grade student with no formal Geometry instruction, a seventh grader with some informal instruction, and a twelfth grader with formal Geometry instruction, data can be collected to demonstrate that as students are exposed to more formal Geometry instruction, they begin to develop more complex conceptualizations of angles.

**Data Collection and Analysis Methods**

To test the model, I conducted one clinical interview with each student in hopes of gaining insight into each student's thinking, reasoning and development of mental structures (Clement, 2000). Students sat for one interview session, lasting thirty minutes on average. Students were presented a total of 14 tasks related to angles and angle measures (see Appendix A for sample tasks). Tools (e.g, ruler, compass, and protractor) were given to students that could be used during the tasks. The tasks were constructed based on years of experience as a mathematics educator and knowledge of previous work done on angle conceptualizations (Bütüner & Filiz, 2016; Clements & Burns, 2000; Kontorovich & Zazkis, 2016; Mitchelmore & White, 1998;

Moore, 2013; Piaget et al., 1960/1981; Smith et al., 2014; Wilson, 1990; Yigit, 2014). They were designed to assess the student's conceptualization of angles following APOS theory. The students were also asked questions related to their responses to gain insight into their thinking. The location of the interview setting was determined by the parents and guardians to ensure students would be comfortable during the interview.

During the interviews, students were audio recorded and videotaped. I transcribed each interview and digitized each students' written work. I then coded each transcript and identified episodes that demonstrated students' thinking. Next, I examined each students' work in relation to their verbal descriptions to make a hypothesis about their learning. I also tested each hypothesis with the proposed genetic decomposition model to determine if the students' thinking aligned with or contradicted the model. This was a means to determine if the model was viable or if modifications needed to be made in order to accommodate the differences in student thinking. Finally, I compared and contrasted each student's conceptualization of angles.

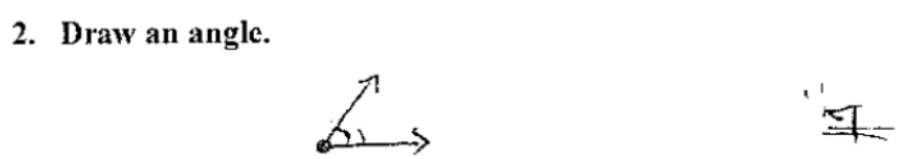
## Results

From the data, results indicate that each student's thinking was very different and at sometimes fluid. Students moved between different actions, processes, and objects that resulted in the development of different schemas. It was often difficult to identify stable moments during each student's thinking process. However, this section describes each student's thinking during the interviews. A discussion will be provided in how each student's thinking compared to the proposed genetic decomposition. Finally, each student's conceptualization of angles will be compared to identify similarities and differences between students.

**Olivia**

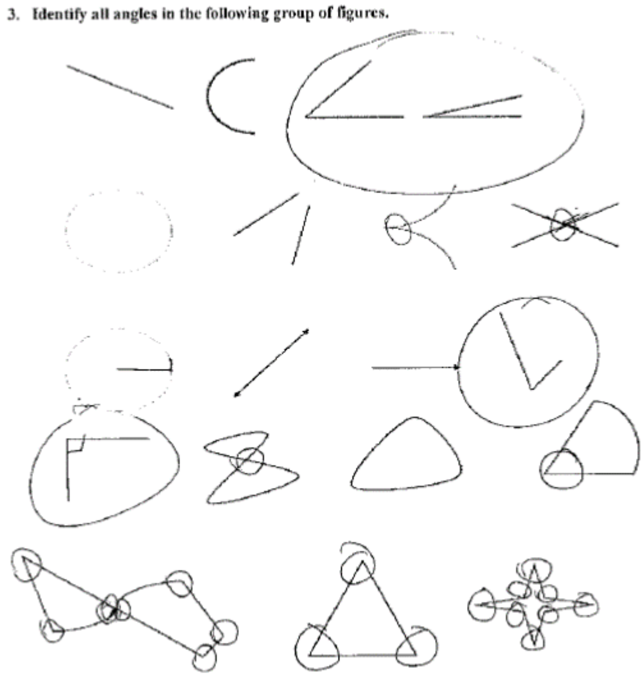
I first asked students to define an angle to gain insight into how they describe angles. Olivia's initial response indicated she was thinking about an angle as an intersection when she said "an angle is two line segments that meet at a like point", but she was also thinking about an angle as a rotation when she said "and they can range anywhere from I guess 0 to 180." She used hand motions while providing her definition by making her hands into two sides of an angle and then demonstrated the lines segments ranging from 0o to 180o. This indicated she was thinking about an angle in a dynamic manner. However, when asked what she meant by range and her hand motion, Oliva responded "It can be measured anywhere." This indicated her thinking did not support a rotation and supported the notion of a static angle. When Oliva drew an angle to support her definition, she drew two rays that intersected at a common vertex and indicated the angle with an arc (Figure 12).

*Figure 12*. Olivia's Angle Representation




This indicated she was thinking about an angle as an intersection, although her use of an arc to represent the angle as a curve made me question whether she was thinking about an angle as a rotation. Here, she appeared to be moving fluidly between actions involving a rotation and an intersection, indicating she was thinking about an angle both statically and dynamically. I then asked Olivia if she could identify all angles in a given group of figures. She identified and circled all angles represented by two lines or rays that intersect (Figure 13).

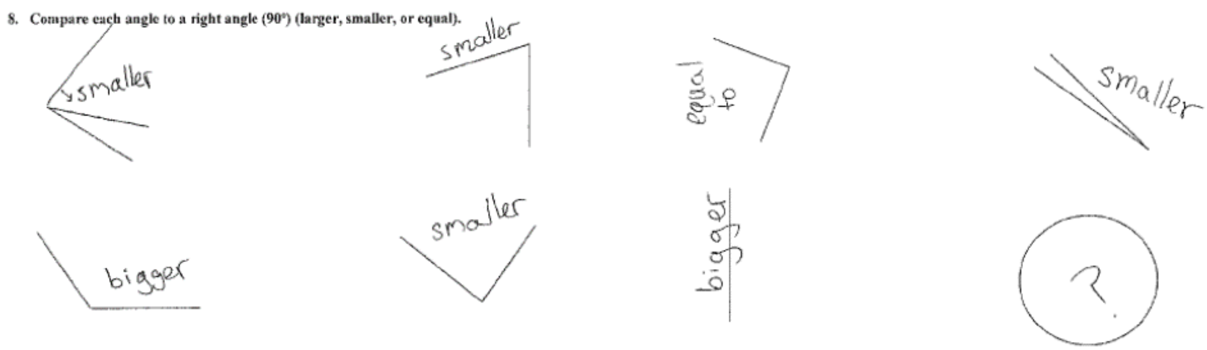
*Figure 13.* Olivia's Angle Identification




Her explanation of her actions supported the intersection notion when she explained the two separated lines was not an angle "because they haven't met at a like point yet." Therefore, I concluded Olivia was in the Action stage relating to intersections.

Next, I asked Olivia to copy given angles. She proceeded by using tools such as a protractor and a ruler to create precise copies. She then realized she drew one wrong, stating she drew an acute angle instead of an obtuse angle. She visually recognized her angle was not the same size as the original and needed to be revised. This indicated she was viewing an angle as a larger area between the two sides. However, when asked to draw a bigger angle, Olivia responded by using hand motions representative of a rotation and also made reference to longer side lengths when she said, "Like bigger as in size, length, or like degrees?" This response indicated she was thinking fluidly about angles as rotations, areas, and intersections as static and dynamic angles. However, later on, Olivia was unable to compare a circle and a semi-circle. This indicated she was not thinking about angles as a rotation, but was relying on the intersection and space between the two sides. To delve deeper into this matter, I asked Olivia to compare a set of angles to determine if it was bigger, smaller, or equal to a right angle (Figure 14).

*Figure 14.* Olivia's Comparison of Angles to a Right Angle




On this set, she did not use a protractor and visually determined the comparison. Once again, when presented with a circle, she became confused and wrote a question mark, claiming "I don't know how to measure a circle in degrees" (Excerpt 1).

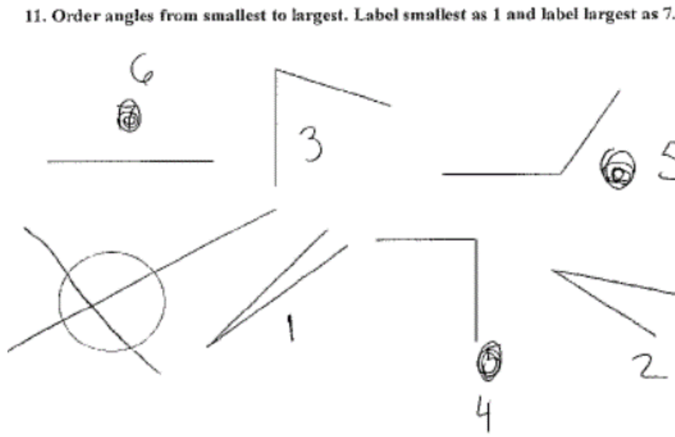
*Excerpt 1*. Olivia's Explanation for Comparisons

| | |
|---|---|
| Olivia: | Ok smaller. Smaller. Bigger. Smaller. Smaller. Equal to. Bigger. |
| Interviewer: | And why did you put a question mark there? |
| Olivia: | Because I don't know how to measure a circle in degrees. |
| Interviewer: | Ok and you said earlier a circle wasn't an angle right? |
| Olivia: | Yeah. |
| Interviewer: | Um and how did you determine whether these were smaller or bigger? |
| Olivia: | Um I could just tell like a 90 degree angle it would be, the line would be like here and this one is lower than that so. |

During these sets of tasks, it was unclear as to what formal processes Olivia had formed as a means to transform angles as she was fluid in her thinking moving between intersection and space. However, based on her responses and work, I concluded she was in the Process stage relating to a larger area.

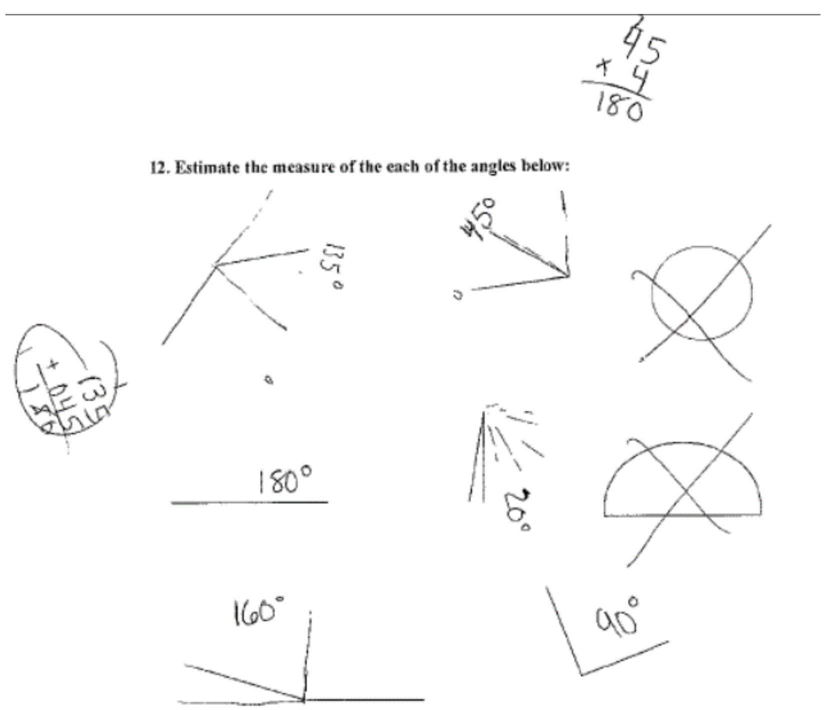
In the next set of tasks, I wanted to test Olivia's mental objects she had constructed for angles. I asked Olivia to order the angles from smallest the largest (Figure 15).

*Figure 15.* Olivia's ordering of Angles




She had difficulty since she did not think a circle was an angle and left it out of the sequence. After renumbering, she ordered the angles by visually comparing them. This indicated she was thinking about angles as a wedge since she "just looked at them and determined." I then asked Olivia to estimate measures of angles to further assess her wedge concept (Figure 16).

*Figure 16.* Olivia's Angle Estimations



In these tasks she used benchmarks such as $90^{o}$ and $180^{o}$ to assist in her estimation (Excerpt 2).

*Excerpt 2*. Olivia's Explanation for Estimating

| | |
|---|---|
| Interviewer: | So thinking about 90 degrees and 180 could you guess at what those measures were? |
| Olivia: | Yeah. So this is 90 and this is 180. This is in the middle so that would be what's? That's 90, that's 180 what's in the middle? 270/2 equals 135. Ok. But then does? 135 plus 45. (mumbles) Ok. So yep this is 180 degrees. This is 10, 20, 30, 40, 50, nope. So that looks like 20. That's like in the middle so that is. Here is (mumbles). Ok alright so this is here, 45. |

This was indicative of a dynamic angle in thinking about it being a wedge. However, again, she could not act upon the circle and semi-circle. This led me to believe she was treating angles as an intersection, but she could also work with a straight line which contradicted the intersection notion. This reflected her fluid thinking with treating angles as static and dynamic objects as both unions and wedges. However, since she was simply using visuals to compare sizes of angles, I finally concluded Olivia was in the Object stage relating to a union.

To conclude the interview, I asked questions to determine what schemas Olivia had developed in relation to angles and angle measure. She was able to complete all tasks where two sides of the angle were visible. She was also able to operate with straight lines, even though she did not originally identify those as angles. When presented with a circle or a figure where two lines were not clearly visible, Olivia could not complete the tasks. This evidence supports a fluid transition between a 2- and 1-line schemas. However, I argue that most evidence supports the notion that Olivia developed a 2-line schema for angles since she did not identify straight lines as angles and could not act upon semi-circles.

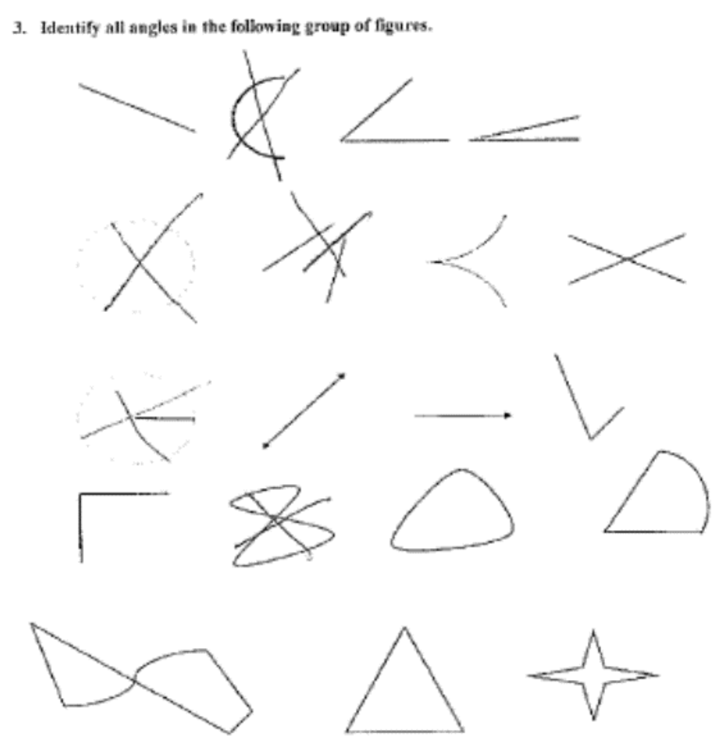
**Daniel**

Daniel's definition of an angle clearly indicated he was thinking about an angle as an intersection when he stated, "where two lines intersect." He was firm in his answer and had no other alternate explanation. When asked to draw an angle to support his definition of an angle, he quickly drew two rays that intersected (Figure 17). This indicated he was thinking about an angle as a static intersection.

*Figure 17.* Daniel's Angle Representation


I then asked Daniel to identify all angles in a given group of figures. He identified all angles represented by two lines or rays that intersect and crossed out ones that were not angles (Figure 18).

*Figure 18.* Daniel's Angle Identification



As long as he could clearly see the two lines intersecting, he claimed it was an angle, but if "the lines don't intersect" it was not an angle (Excerpt 3).

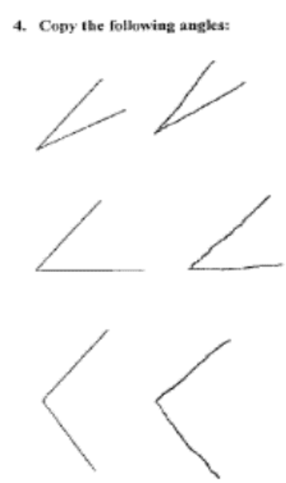
*Excerpt 3*. Daniel's Explanation for Action

| | |
|---|---|
| Interviewer: | Ok. So why did you say this one is not an angle? (points to circle) |
| Daniel: | Cause the lines don't intersect with each other. |
| Interviewer: | Ok. Uh what about that one? (points to figure 8) |
| Daniel: | Maybe because they come at a curved. |
| Interviewer: | Ok. And what about this one? (points to curve in top row) |
| Daniel: | The lines don't intersect. |

However, this contradicted some of his other answers. For example, he claimed a few figures with curved lines were indeed angles but said others were not even though they were similar such as the figure 8 and rounded triangle in the fourth row of Figure 18. It appeared he was relying on a visual representation for how an angle should look. Therefore, I concluded Daniel was in the Action stage relating to intersections.

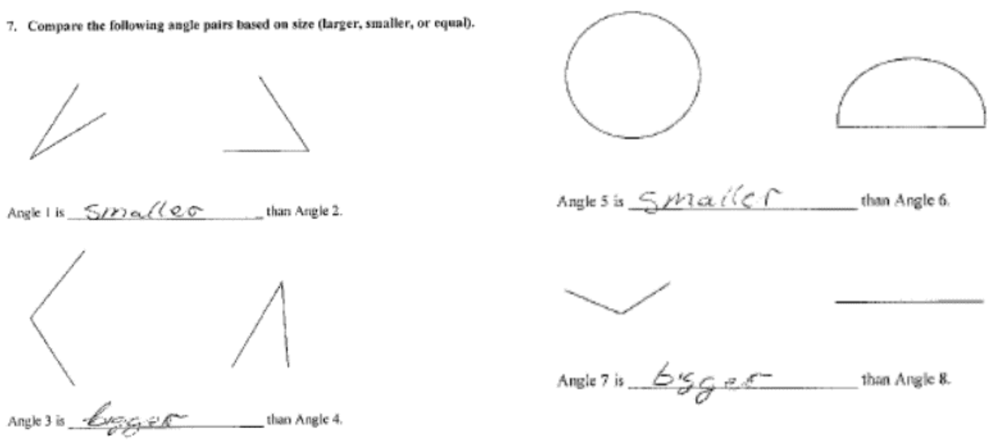
I next assessed Daniel's processes by asking him to copy given angles. He quickly drew his representation of a copy without the use of any tools (Figure 19).

*Figure 19.* Daniel's Copies of Angles




He justified his copies by saying "They look the same" and that he "just guessed" to determine the sizes. When asked to draw a bigger and smaller angle, Daniel quickly drew his estimated angles by visually comparing them. This indicated he was clearly thinking about the angles as static figures, but it was unclear as to whether he was relying more on the intersection or the space between the sides. To further assess his processes, I asked him to compare pairs of angles to determine if one was bigger, smaller, or equal to the other (Figure 20).

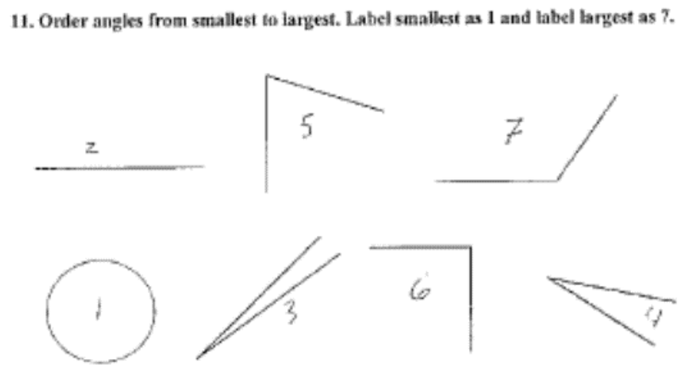
*Figure 20.* Daniel's Comparisons of Angles




In his explanation, Daniel referenced linear separation when he said, "the lines are closer together" and "this one is farther apart than that one", indicating he was thinking about angles as having a larger space. However, he then provided evidence he was thinking about angles as an intersection with responses such as "the lines do intersect." In later tasks, Daniel used a protractor and gave indication that he was thinking about angles as a rotation when he mentioned line movements. This supported the fluidity of his thinking as he moved between the notion of an intersection, space, and rotation. Nevertheless, he was still thinking about angles as static figures. Therefore, I concluded Daniel was in the Process stage relating to intersection since he relied heavily on two visible intersections.

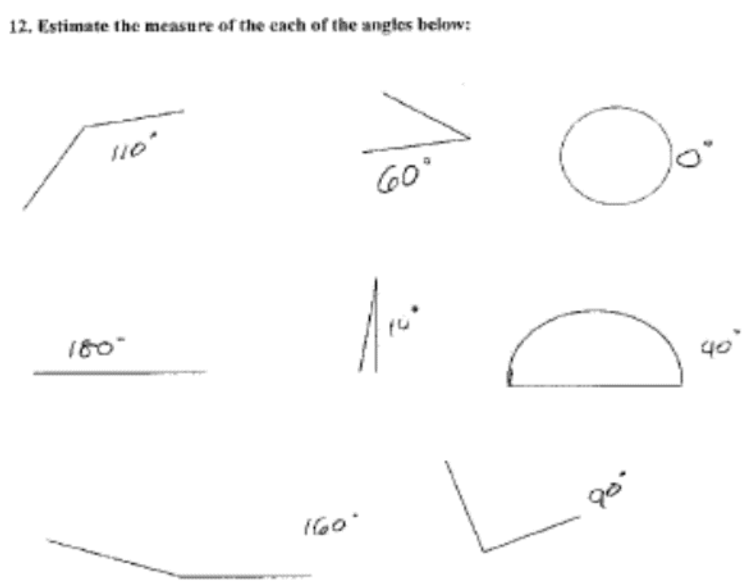
I then assessed Daniel's construction of mental objects by asking him to order a set of angles from smallest the largest (Figure 21).

*Figure 21*. Daniel's Ordering of Angles




He quickly ordered the angles with the circle being the smallest. In his explanation, he relied heavily on the notion of an intersection and the openness of angles by looking at whether or not the angle "was more open than the rest of them were." Again, he was moving fluidly between the static figure of a union and a wedge. In the next set, I asked Daniel to estimate measures of angles (Figure 22).

*Figure 22.* Daniel's Angle Estimations




He visually guessed as to what the angles would measure. In his explanation, he thought of "a right angle and tried to guess um how many degrees it would be off by 10s". However, once again, he could not act upon the circle due to the lack of intersecting lines. This indicated he was thinking about an angle as a static wedge. Although, he did change his reasoning for a straight line having $0^o$, and identified it as having $180^o$. He then changed his mind on the previous questions and reordered the angles so that the straight line was the largest, but the circle was still the smallest (Excerpt 4), supporting his fluid transition back to a union.

*Excerpt 4*. Daniel's Explanation for Estimating and Reordering

| | |
|---|---|
| Interviewer: | Ok and you said the circle has 0 because you said it had no angles. What about the line? |
| Daniel: | Well I changed my mind about it because on the protractor one side is 0 and the other side is 80, 180. So I tried doing it that way to see if I could remember it better. |
| Interviewer: | Ok. So thinking back to these questions |
| Daniel: | I would probably change my mind. |
| Interviewer: | So if you were to rearrange those. |
| Daniel: | I would probably switch, make this one 1 (circle), then 2, 3, um 4, 5, 6, and then 7. |
| Interviewer: | So this one would be your biggest one now? (points to line) |
| Daniel: | Yeah from 180. |

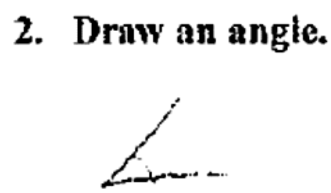
Based on the evidence provided, I concluded he had fully developed angles as a union Object, but also had partially developed angles as a wedge Object.

Finally, I tried to assess what schemas Daniel developed in relation to angles and angle measure. During the tasks, he was able to complete tasks and act upon angles that had two visible lines and treated them as static figures. This indicated he developed a schema for 2-line angles. When presented with a circle, he could not properly complete the task. Also, he originally thought straight lines were not angles or had $0^o$ angle. However, since he renegotiated his thinking and realized a straight line measured $180^o$, I think he partially developed a schema for 1-line angles since it was not developed until the protractor perturbed his thinking. Overall, I would say he had a fully developed 2-line schema.

**Emma**

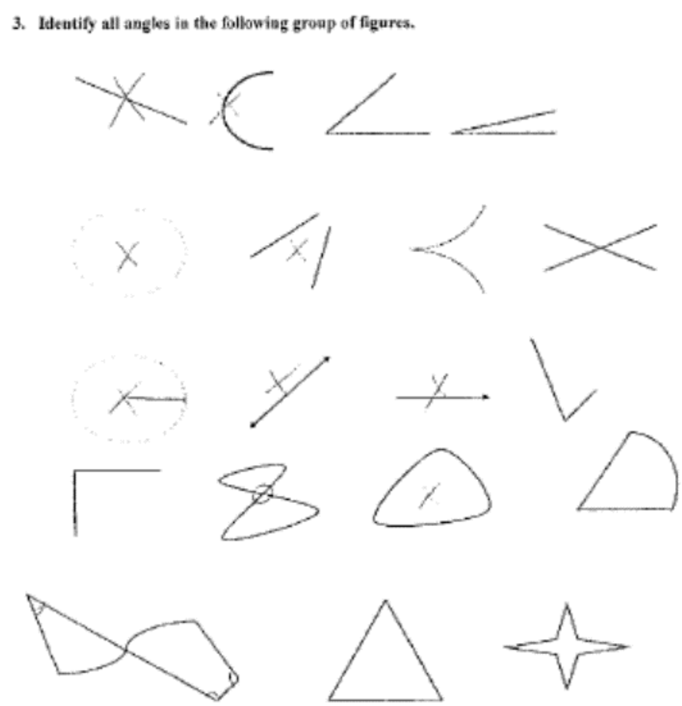
Emma defined an angle as" two lines open… at a certain degree." To support her definition, she drew and identified an angle by two intersecting rays representing it with an arc (Figure 23).

*Figure 23*. Emma's Angle Representation




This indicated she was thinking about a dynamic angle as the space between the two sides. When identifying angles, she marked out the figures that did not represent angles, leaving only those with two distinct intersecting lines (Figure 24).

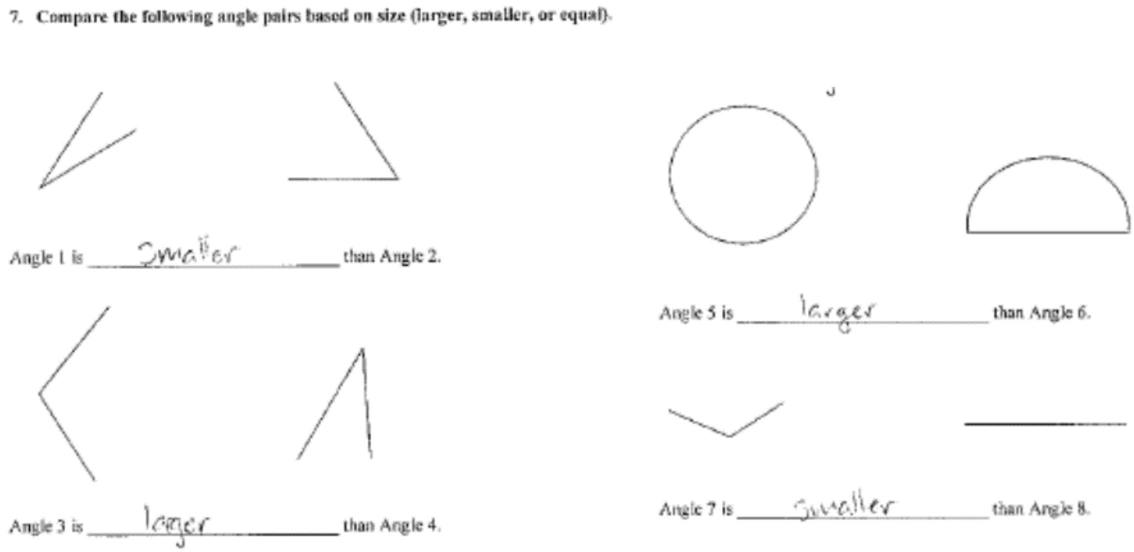
*Figure 24.* Emma's Angle Identification



Emma explained that angles need to have "two lines that meet at one point" and if "they don't' connect" then they are not angles. However, some of her explanations contradicted what she had identified as angles. For example, she claimed the circle with a radius was not an angle because there were not two straight lines. On the other hand, she identified several figures that had curved lines as angles and located the angle with an arc. This indicated Emma's thinking was fluidly moving between the notion of a dynamic angle as a space and a static angle as an intersection. Based on the evidence, I concluded Emma was more in the Action stage relating to space.

In the next set of tasks, Emma copied angles. Like Olivia, Emma proceeded by using tools such as a protractor and a ruler to construct copies. She was very precise in her efforts, taking over two minutes to copy three angles. When constructing larger and smaller angles, Emma stated an angle bigger than $180^{o}$ could not be drawn. She also said she could not draw an angle smaller than a circle since it was not a circle. However, after having a discussion about line movements, she concluded that $180^{o}$ would be an angle smaller than a circle. She stated, "Anything like 180 is smaller than 360." This episode represented the fluidity in Emma's thinking about angles as she was trying to coordinate a static intersection with a dynamic rotation, but she could not piece the two concepts together. Emma then compared pairs of angles to determine if one was bigger, smaller, or equal to the other (Figure 25).

*Figure 25.* Emma's Comparisons of Angles




Here, she visually estimated the angles and compared them, determine which ones were "not as wide as" others (Excerpt 5).

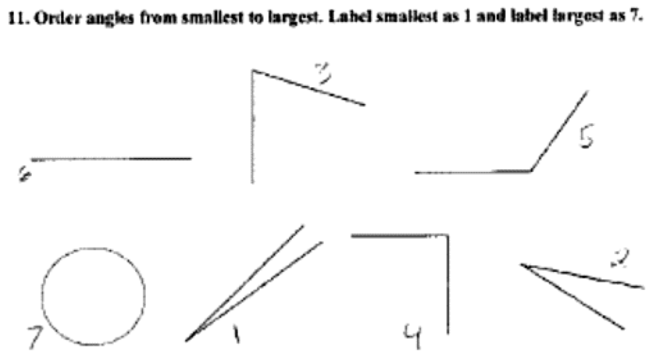
*Excerpt 5*. Emma's Connection with a Circle and Semi-Circle

| | |
|---|---|
| Interviewer: | How did you determine whether they were bigger or smaller? |
| Emma: | The, this just looks like it's not as wide as this one. |
| Interviewer: | Ok. |
| Emma: | And same for that one. Well this one looks larger than that one. |
| Interviewer: | Ok and what about the circle down there? Because you said oh now I get it. |
| Emma: | (laughs) uh It doesn't go all the way around so this is like 360 and this is only 180. |
| Interviewer: | Ok. So would you go back to the beginning and say a circle was an angle? |
| Emma: | Yes. |
| Interviewer: | Ok. And the last one down there? |
| Emma: | Uh this is 180 and that's smaller than 180. |

She discussed openness and rotations, providing evidence that she was transforming angles by comparing rotations and spaces, no longer relying on intersections. As such, she recognized circles as indeed angles, and changed her responses to previous questions. In the next sets of tasks of comparing angles to right angles, she again referenced openness and visual estimates. Based on this evidence, I concluded Emma had developed a Process relating to a larger interior area.

In order to assess Emma's construct of mental objects, I asked Emma to order the angles from smallest the largest (Figure 26).

*Figure 26.* Emma's ordering of Angles




She correctly ordered the angles with ease. In her explanation, she discussed how she looked at the linear separation to compare. She stated angle 1 was the smallest "Because it's the skinniest" and angle 7 was the largest because "Because it's 360 degrees and you can't get bigger than that." This indicated Emma had fluid thinking about angles as the static amount of space between the sides and no longer dynamic. Emma then went on to estimate measures of angles by using $90^{o}$ as a benchmark for her estimates, reinforcing the wedge concept. She compared static figures to the benchmarks by looking to see if "it's less than 90 degrees." Based on the evidence, I concluded Emma had arrived at the Object stage relating to angles as wedges.

To determine what schemas Emma had developed in relation to angles and angle measure, I asked Emma to complete tasks involving circles and lines. She was able to complete all tasks. Despite her contradictions in the beginning set of tasks, Emma could act upon circles and lines with ease. In the beginning, Emma demonstrated a schema for 2-line angles. However, as she progressed, she provided more evidence that allowed me to conclude she had developed a

0-line schema. However, I would argue there was some fluidity between a 1- and 2-line schemas throughout the interview.

**Comparison to Proposed Genetic Decomposition**

Based on the evidence, each student's thinking was different than the proposed genetic decomposition model. The proposed genetic decomposition represents a rigid model for student thinking. However, all students had fluidity in their thinking, contradicting the model. For example, Olivia began by being in the intersection Action stage. She then progressed into the space or area Process stage. However, according to the proposed model, this is not possible. Additionally, after having developed a process relating to area, Olivia then constructed angles as a union Object. This also is not possible according to the model. Finally, Olivia presented evidence that she had some development of a 1-line schema, which again is contradictory to the model. Similarly, Daniel also began in the intersection Action stage. He then progressed into the area Process stage, which as mentioned earlier is contradictory to the proposed model. From here, evidence showed Daniel developed angles as a wedge in the Object stage. This trajectory aligned with the model, and then followed into the development of a schema for 2-line angles.

However, Emma's case aligns more appropriately with the model. Looking at Emma's conceptualization of angles, she began in the Action stage related to space. She then displayed signs that allowed me to place her in the Process stage related to area. Further evidence showed she developed angles as a wedge Object. So far, Emma followed on track with the model. However, her trajectory fell off the model as she was able to develop a schema for 0-line angles. When comparing these three students to the model, it is evident that the students' fluid thinking does not align with the rigid proposed genetic decomposition. Therefore, there is a need for revisions to the proposed model to allow room for fluidity and may be taken up in the next iteration of this study.

**Comparison of Students**

While the students' thinking did not align with the proposed genetic decomposition, it would be beneficial to compare students to identify similarities and differences that could assist with revisions to the model. Olivia and Daniel had very similar learning trajectories as they progressed through the tasks. They started out in the same Action stage and moved into the same Process stage. Daniel solely thought of angles as static figures, while Olivia was fluid in her thinking, bouncing back and forth between static and dynamic. Despite Daniel's static view, he was able to construct angles as a wedge Object while Olivia constructed them as a union Object. Nevertheless, Olivia provided more evidence that she had developed a 2-line schema and a partial 1-line schema, while Daniel only developed a 2-line schema with several misconceptions in a 1-line schema. Also, Olivia was not able to conceptualize angles as circles and failed to complete tasks involving circles, whereas Daniel developed a misconception for circles as $0^{o}$ angles that allowed him to incorrectly complete tasks involving circles.

On the other hand, Emma had a different trajectory and started in the space Action stage. In doing so, she developed the most complex schema and aligned best with the model. Even though she developed the same Process as Olivia and Daniel, and the same Object as Daniel, she was able to develop a more complex schema than both students. She also thought of angles as both static and dynamic. As a result, she never developed angles as solely static intersections, which could have affected her schema development. She arrived at a schema that allowed her to correctly complete tasks involving circles. However, it is also important to note that she was very fluid in their thinking, allowing her to move back and forth between static and dynamic angles, ultimately developing a schema for 0-, 1- and 2-line angles.

## Discussion

This study was conducted to examine students' conceptualizations of angles. Based on the literature, I hypothesized: (a) students who construct angles as an union object will be limited to developing a 2-line angle schema; (b) students who construct angles as a wedge object will develop a 1-line angle schema, but also a 2-line angle schema; and (c) students who construct angles as a turn object will be capable of developing a 0-line angle schema, but also a 1- and 2-line angle schema. However, results from analysis showed that students' thinking was very fluid and allowed them to progress to new levels the model prohibited. As a result, both students, Olivia and Daniel, who constructed angles as a union developed 2- and 1-line schemas. Although both provided evidence for a partial 1-line schema, they were still able to progress past the model's restriction. Additionally, Emma constructed angles as a wedge but was able to develop a 0-line schema, also contradicting the model. Based on this evidence, the proposed genetic decomposition needs to be revised to represent students' fluidity in their thinking. However, more research should be conducted to test other ways students might think about angles. More genetic decompositions need to be developed in relation to angles to gain better insight into viable models.

When thinking about static and dynamic angles, Emma who never thought of angles solely as static figures was able to develop the most complex schema out of all three students, supporting previous research (Bütüner & Filiz, 2016). Olivia viewed angles as static and dynamic but leaned more so on the static side. Daniel however viewed angles as solely static and was able to only develop a full 2-line schema, whereas Olivia provided evidence that she had developed a partial 1-line schema. This concept of static and dynamic also needs to be included in the model to address the fluidity in students' thinking and to show that students can view angles as a union in both a static and dynamic manner as Olivia demonstrated. Future research should also focus on what affect this static and dynamic view of angles has on the development of schemas. Also, more research needs to be conducted to determine what actions student coordinate with static angles. Although they are classified as static angles, students still must coordinate actions to transform those static figures. Another concern is the use of APOS theory when trying to investigate student thinking concerning a static figure. If the figure truly is static, a different framework is needed to assess student thinking.

Finally, I chose three students from three different grade levels to see if Geometry instruction played a part in students' thinking. Olivia who had no formal Geometry instruction was able to develop a more complex schema than Daniel who had some informal Geometry instruction. This is contradictory to my hypothesis. However, Emma who had formal Geometry instruction developed the most complex schema out of all three students. This indicates more research needs to be conducted to investigate how angles are being formally and informally taught in schools.

## Conclusions

Understanding angles as a static figure or a dynamic action can be beneficial for educators. When students conceptualize angles as dynamic figures, they are able to gain a deeper understanding of angles that promotes greater future success (Clements & Burns, 2000). In this study, Emma and Olivia viewed angles as dynamic angles and were able to progress farther in their learning. Therefore, this suggests angles should be taught in more dynamic ways, such as including body movements to help support the dynamic motion (Clements & Burns, 2000; Smith et al., 2014), and not just a visual that leads to misconceptions such as in Olivia's case.

Additionally, the notion of an angle as a union, wedge, or turn is beneficial to understand, as it provides insight into misconceptions students might develop. For example, Daniel thought of an angle in a static manner, relying on the intersection of two rays or lines (Mitchelmore & White, 1998; Keiser, 2004), and progressed to the Object stage of angles as a union. Here, he was only able to develop a 2-line angle schema and could only identify angles with two visible rays, as previous studies have suggested (Bütüner & Filiz, 2016). It is also important to understand that although Emma developed angles as a wedge, recognizing the region between two rays (Browning et al., 2008; Yigit, 2014) by relying on linear separation as described by Piaget et al. (1960/1981), she was able to develop a 0-line schema for angles. This is contradictory to previous research which shows that students are able to progress further, develop a richer understanding of angle, and outperform others when they view angles as turns (Browning et al., 2008; Clements & Battista, 1989, 1990; Mitchelmore & White, 1998). Emma demonstrated that not only can students progress farther if they develop angles as a turn, but it is also possible for students who develop angles as wedges.

Finally, this study supports previous research that shows when students develop 0-line angle schemas, they are more equipped to solve more complex problems involving angles measures, right triangles, arcs, and even circles (Yigit, 2014). Emma was the only student who could solve problems involving circles, due to her 0-line schema. The other students were not able to solve such tasks since they had only developed a 1- or 2-line angle schema. Ultimately, this study supports previous research that shows the development of dynamic angles leads to deeper understanding and a more complex schema, but also provides insight into future research on how students conceptualize angles as either dynamics or static angles, and what implications come with such conceptualizations.

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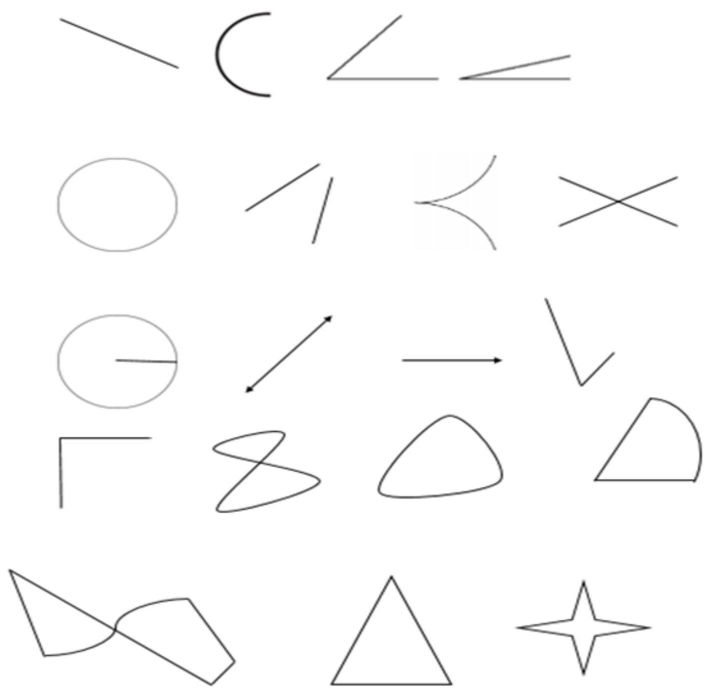
**Appendix A**
**Sample Tasks for Assessing Students' Conceptualizations of Angles**

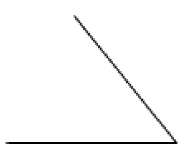
1. Define an angle.

2. Draw an angle.

3. Identify all angles in the following group of figures.

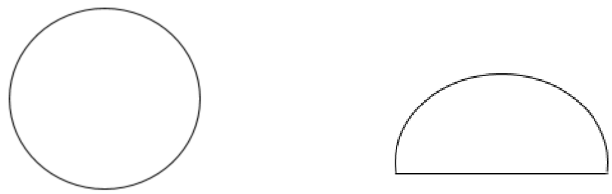


4. Draw an angle bigger than the one given:



5. Compare the following angle pairs based on size (larger, smaller, or equal).

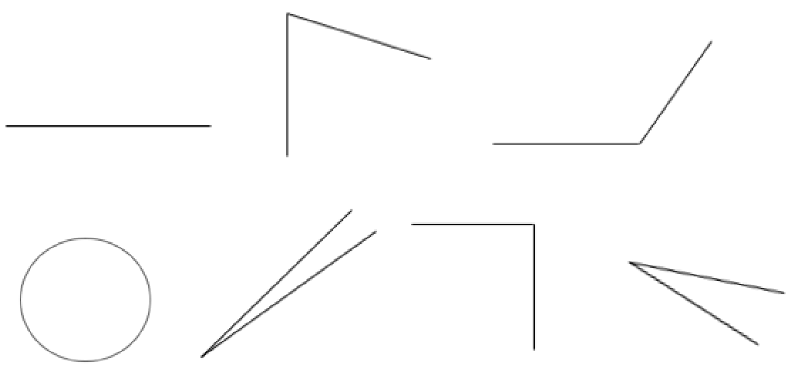
Angle 1 is ______________________________ than Angle 2.



Angle 3 is _____________________________ than Angle 4.

Angle 5 is _____________________________ than Angle 6.

6. Order angles from smallest to largest. Label smallest as 1 and label largest as 7.



7. Estimate the measure of the each of the angles below: