Research Report

Error Variance in Common Population Linking Bridge Studies

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RESEARCH REPORT

Error Variance in Common Population Linking Bridge Studies

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When an assessment undergoes changes to the administration or instrument, bridge studies are typically used to try to ensure comparability of scores before and after the change. Among the most common and powerful is the common population linking design, with the use of a linear transformation to link scores to the metric of the original assessment. In the common population linking design, randomly equivalent samples receive the new and previous administration or instrument. However, conventional procedures to estimate error variances are not appropriate for scores linked in a bridge study, because the procedures neglect variance due to linking. A convenient approach is to estimate a variance component associated with the linking to add to the conventionally estimated error variance. Equations for the variance components in this approach are derived, and the approximations inherently made in this approach are shown and discussed. Exact error variances of linked scores, accounting for both conventional sources of variance (e.g., sampling) and linking variance together, are derived and discussed. The consequences of how linking changes how certain errors are related is considered mathematically. Specifically, the impacts of linking on the error variance for the comparison of two linked estimates (e.g., comparing the mean score of boys to the mean score of girls, after linking), for the comparison of scores across the two samples (e.g., comparing the mean score of boys in the new administration or instrument to the mean score of boys in the old administration or instrument), and for aggregating scores across the two samples (e.g., the mean score of boys across both administrations or instruments) are derived and discussed. Finally, general methods to account for error variance in bridge studies by simultaneously accounting for both conventional and linking sources of error are recommended.

Keywords Bridge study; linking; error variance; standard error; variance; common population

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Bridge studies are used to link scores from a new administration or measurement instrument to scores from a previous administration or measurement instrument, with the aim of ensuring the comparability of scores between the new and previous administrations or instruments (Dorans, Pommerich, & Holland, 2007). Common uses of bridge studies include addressing changes to test content (J. Liu & Walker, 2007) and addressing changes to the mode of administration, such as paper based to digitally based administration (Eignor, 2007). If standard error variance estimation procedures are applied to scores in a bridge study, variance due to linking will be neglected, and the estimated error variances will not be accurate. The issue of error variance estimation in bridge studies is considered in this research report.

The common population or randomly equivalent group design involves a *target* sample of test takers receiving the old administration or instrument and a *source* sample of test testers receiving the new administration or instrument (Kolen & Brennan, 2014). Both samples are assumed to be drawn from a common population.

A transformation is applied to the scores from the source sample to provide scores from the new administration or instrument in the same metric as the previous administration or instrument. The original metric of the source sample scores is defined here as the *original metric*, and the metric of the target sample scores is defined as the *target metric*. Provided that criteria such as construct similarity are satisfied (Dorans et al., 2007), the transformation links the two administrations or instruments and allows scores between the two to be compared in a common metric (the target metric).

The transformation may take different forms, but among the most common is the linear transformation (Kolen & Brennan, 2014). When a linear transformation is used, the score for any given test taker i in the original metric, θ_i, is assumed to be linearly related to the score for the same test taker in the target metric, Y_i. Specifically,

$$\sigma_Y^{-1}\left(Y_i-\mu_Y\right)=\sigma_\theta^{-1}\left(\theta_i-\mu_\theta\right), \tag{1}$$

where μ_Y is the population mean of the source sample scores in the original metric, σ_Y is the population standard deviation of the source sample scores in the original metric, μ_θ is the population mean of the source sample

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scores in the target metric, and σ_θ is the population standard deviation of the source sample scores in the target metric.

In a common population design, both target and source samples are assumed to be drawn from the same population and therefore to have the same population mean and standard deviation. Specifically,

$$\mu_Y = \mu_X \tag{2}$$

$$\sigma_Y = \sigma_X, \tag{3}$$

where μ_X is the population mean of the target sample scores and σ_X is the population standard deviation of the target sample scores. X denotes scores from the target sample, which are already in the target metric. This equivalence is important, because it means that we have estimates of two population values for the source sample scores in the target metric.

Rearranging Equation 1 and substituting Equations 2 and 3 gives

$$Y_i = a\theta_i + b, \tag{4}$$

where the linking coefficients, a and b, are

$$a = \sigma_X \sigma_\theta^{-1} \tag{5}$$

$$b = \mu_X - a\mu_\theta. \tag{6}$$

In practice, the following estimators are used:

$$\hat{Y}_i = \hat{a}\,\hat{\theta}_i + \hat{b}, \tag{7}$$

$$\hat{a} = \hat{\sigma}_X \hat{\sigma}_\theta^{-1}, \tag{8}$$

$$\hat{b} = \hat{\mu}_X - \hat{a}\,\hat{\mu}_\theta, \tag{9}$$

where $\hat{\theta}_i$, $\hat{\mu}_\theta$, $\hat{\sigma}_\theta$, $\hat{\mu}_X$, and $\hat{\sigma}_X$ are estimators of θ_i, μ_θ, σ_θ, μ_X, and σ_X, respectively.

For example, linear transformations with the common population design have been used routinely in the National Assessment of Educational Progress (NAEP; Yamamoto & Mazzeo, 1992). Such bridge study designs were used when the item sampling procedure was revised in the 1988 NAEP assessments (Johnson & Zwick, 1990), when the designs of the 2004 long-term trend assessments were updated (Perie, Moran, & Lutkus, 2005), when the assessment framework was updated for the 2009 reading assessment (National Center for Education Statistics, 2009), and when the reading and mathematics assessments were transitioned from paper-based to digitally based (or computer-based) assessment in 2017 (Jewsbury, Finnegan, Xi, Jia, & Rust, 2019).

For the 2017 NAEP reading and mathematics assessments, the new assessment was digitally based, whereas the previous assessment was paper based. The source sample received the digitally based assessment, and the official reported scores were the digitally based results after the scores were transformed to the target metric: the previous, paper-based metric of prior NAEP assessments.

In the 2017 NAEP assessments, the variances of estimators were obtained by decomposing the variance into two variance components (Mazzeo, Donoghue, Liu, & Xu, 2018). First, conventional sources of variance, such as sampling, were estimated with standard procedures applied to the scores posttransformation ($\hat{Y}$). In this report, this variance component is denoted as the *prelinking variance component*, because it is the variance of the scores due to all sources of variance in the scores pretransformation. Second, the additional source of variance due to linking, or more specifically, the variance of the scores due to variance of $\hat{a}$ and $\hat{b}$ were estimated with a Monte Carlo-based estimator. Assuming that the covariance between the two components is zero, they were summed to obtain the total variance of the estimator.

In this report, I first describe the assumptions of the variance component decomposition approach, derive exact and approximate equations for the variance components, and discuss the importance of accounting for the covariance between the two components. Second, I derive the exact variance of estimators without decomposing the variance into components. Third, I explore how the transformation results have subtle but important consequences for comparing estimates posttransformation to other estimates from the same assessment or other assessments. Finally, I describe a method to estimate uncertainty of estimates that accounts for both sources of variance simultaneously, avoiding approximations inherent in the variance component decomposition approach, and which can be used to properly compare posttransformed estimates to other estimates with valid error variance estimates.

General Notes

The target and source samples may or may not be sampled independently. For greatest generalizability, independence of the target and source samples is not assumed throughout the present document. For independent target and source samples, the equations may be simplified by setting covariances between target sample estimators and pretransformation source sample estimators to zero. Relatedly, the distribution of the scores may or may not be normal. With normally distributed scores, estimators of the mean and the standard deviation are independent (e.g., Hogg, McKean, & Craig, 2013). Again, for greatest generalizability, normally distributed scores are not assumed throughout the present document. For normally distributed scores, the equations may be simplified by setting covariances of mean and standard deviation estimators from the same sample to zero.

Except for the Combining the Target and Source Samples section, all proofs in the present report begin with an equation for the posttransformation estimator as a function of the pretransformation estimator. All posttransformation estimators that have the same form as a function of their pretransformation estimator may have their posttransformation variance characterized with the same equations in these proofs, because no further properties are assumed.

Many estimators of interest have one of two forms as a function of the pretransformed estimator, which defines type AB and type A estimators as follows.

Definition 1. A type AB estimator is a posttransformation estimator, denoted as $\widehat{Y}_{\mathrm{AB}}$, that depends on the pretransformation estimator, denoted as $\widehat{\theta}_{\mathrm{AB}}$, by

$$\widehat{Y}_{\mathrm{AB}} = \widehat{a}\widehat{\theta}_{\mathrm{AB}} + \widehat{b}. \tag{10}$$

Type AB estimators include individual scores, sample means, and sample percentiles, as proven in the appendix. As the name implies, the variances of both $\widehat{a}$ and $\widehat{b}$ contribute to the variance of type AB estimators.

Definition 2. A type A estimator is a posttransformation estimator, denoted as $\widehat{Y}_{\mathrm{A}}$, that depends on the pretransformation estimator, denoted as $\widehat{\theta}_{\mathrm{A}}$, by

$$\widehat{Y}_{\mathrm{A}} = \widehat{a}\widehat{\theta}_{\mathrm{A}}. \tag{11}$$

Type A estimators include sample standard deviations, differences between type A estimates, and differences between type AB estimates, as proven in the appendix. As the name implies, the variance of $\widehat{a}$ but not the variance of $\widehat{b}$ contributes to the variance of type A estimators.

Variance Component Decomposition Approximation

Previous authors (e.g., Mazzeo et al., 2018) considered an approximation of $\widehat{Y}$ as a sum of (co)variance components. This approximation is convenient and intuitive and, although effective for the case considered by the authors, may not be appropriate for all estimators and all bridge study designs. The assumptions of this approach are made explicit and explored herein. Consider any estimator that is a function of the data (i.e., the individual score estimators),

$$\widehat{Y}\left(\widehat{Y}_1, \dots, \widehat{Y}_I\right), \tag{12}$$

where I is the number of students. Following Equation 7, we may also write $\widehat{Y}$ as a function of the pretransformed individual score estimators and the linking coefficient estimators:

$$\widehat{Y}\left(\widehat{a}, \widehat{b}, \widehat{\theta}_1, \dots, \widehat{\theta}_I\right). \tag{13}$$

The variance decomposition approach assumes first that

$$\widehat{Y}\left(\widehat{a}, \widehat{b}, \widehat{\theta}_1, \dots, \widehat{\theta}_I\right) = \widehat{Y}\left(\widehat{a}, \widehat{b}, \widehat{\theta}\right) \tag{14}$$

and second that

$$\widehat{Y}\left(\widehat{a}, \widehat{b}, \widehat{\theta}\right) = \widehat{Y}_{\mathrm{link}}\left(\widehat{a}, \widehat{b}\right) + \widehat{Y}_{\mathrm{prelink}}\left(\widehat{\theta}\right), \tag{15}$$

where the linking component, $\widehat{Y}_{\mathrm{link}}$, denotes terms of $\widehat{Y}$ involving the estimators of the linking coefficients, $\widehat{a}$ and $\widehat{b}$, and the prelinking component, $\widehat{Y}_{\mathrm{prelink}}$, denotes terms of $\widehat{Y}$ involving the pretransformation estimator, $\widehat{\theta}$.

These assumptions lead to the decomposition of the variance of $\widehat{Y}$:

$$\mathrm{var}\left(\widehat{Y}\right) = \mathrm{var}\left(\widehat{Y}_{\mathrm{link}}\right) + \mathrm{var}\left(\widehat{Y}_{\mathrm{prelink}}\right) + 2\mathrm{covar}\left(\widehat{Y}_{\mathrm{link}}, \widehat{Y}_{\mathrm{prelink}}\right), \tag{16}$$

where the linking variance component, $\mathrm{var}\left(\widehat{Y}_{\mathrm{link}}\right)$, is the variance due to variance of $\widehat{a}$ and $\widehat{b}$ and the prelinking variance component, $\mathrm{var}\left(\widehat{Y}_{\mathrm{prelink}}\right)$, is the variance due to variance of $\widehat{\theta}$.

The first assumption is true for both type AB and type A estimators (see Equations 10 and 11). However, the first assumption may not be true for other estimators, such as achievement level estimators.

The second assumption may be evaluated by using Taylor series expansion to linearize $\widehat{Y}$. The Taylor series of $\widehat{Y}$ at $\left[\widehat{a}, \widehat{b}, \widehat{\theta}\right] = [a, b, \theta]$ is

$$\widehat{Y}\left(\widehat{a}, \widehat{b}, \widehat{\theta}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left[\widehat{a} - a, \widehat{b} - b, \widehat{\theta} - \theta\right] \cdot \begin{bmatrix} \frac{\delta}{\delta \widehat{a}} \\ \frac{\delta}{\delta \widehat{b}} \\ \frac{\delta}{\delta \widehat{\theta}} \end{bmatrix} \right)^{n} \widehat{Y}(a, b, \theta), \tag{17}$$

which in general does not reduce to the form of Equation 15 unless

$$\frac{\delta^2}{\delta \widehat{a} \delta \widehat{\theta}} \widehat{Y}(a, b, \theta) = \frac{\delta^2}{\delta \widehat{b} \delta \widehat{\theta}} \widehat{Y}(a, b, \theta) = 0. \tag{18}$$

Equation 18 is not true for types AB and A estimators, for which the second-order partial derivatives of $\widehat{Y}$ with respect to $\widehat{a}$ and $\widehat{\theta}$ are not zero (see Equations 10 and 11). Nevertheless, Mazzeo et al. (2018) found that the variance decomposition approach was an extremely good approximation for estimators of means, which are a type AB estimator, at least for large and independent target and source samples.

When the assumptions of the variance decomposition approach are satisfied, Equation 17 simplifies to

$$\widehat{Y}\left(\widehat{a}, \widehat{b}, \widehat{\theta}\right) = \widehat{Y}(a, b, \theta) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\left[\widehat{a} - a, \widehat{b} - b\right] \cdot \begin{bmatrix} \frac{\delta}{\delta \widehat{a}} \\ \frac{\delta}{\delta \widehat{b}} \end{bmatrix} \right)^{n} \widehat{Y}(a, b, \theta) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\left[\widehat{\theta} - \theta\right] \frac{\delta}{\delta \widehat{\theta}} \right)^{n} \widehat{Y}(a, b, \theta), \tag{19}$$

which is of the form of Equation 15, where

$$\widehat{Y}_{\mathrm{link}}\left(\widehat{a}, \widehat{b}\right) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\left[\widehat{a} - a, \widehat{b} - b\right] \cdot \begin{bmatrix} \frac{\delta}{\delta \widehat{a}} \\ \frac{\delta}{\delta \widehat{b}} \end{bmatrix} \right)^{n} \widehat{Y}(a, b, \theta) \tag{20}$$

$$\widehat{Y}_{\mathrm{prelink}}\left(\widehat{\theta}\right) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\left[\widehat{\theta} - \theta\right] \frac{\delta}{\delta \widehat{\theta}} \right)^{n} \widehat{Y}(a, b, \theta) + \widehat{Y}(a, b, \theta). \tag{21}$$

Equation 21 helps explain the appeal of the linking and prelinking decomposition. If the variance of $\widehat{Y}$ is estimated via typical procedures used to estimate variance of scores, the variances of $\widehat{a}$ and $\widehat{b}$ will be neglected, as the transformation has already been applied, and an estimate of the prelinking variance component is obtained. Defining a second variance component associated with the neglected variance of $\widehat{a}$ and $\widehat{b}$ follows naturally and sensibly may be defined as the linking variance component. For example, the linking variance component may be estimated with a Monte Carlo estimator (Carlin & Louis, 1996; Mazzeo et al., 2018), and the prelinking variance component may be estimated with standard procedures on the scores posttransformation.

The linking and prelinking components for types AB and A estimators can be obtained by inserting their defining equations (Equations 10 and 11, respectively) into Equations 20 and 21:

$$\widehat{Y}_{\mathrm{AB,link}} = \widehat{a}\theta_{\mathrm{AB}} + \widehat{b} - a\theta_{\mathrm{AB}} - b, \tag{22}$$

$$\widehat{Y}_{\mathrm{AB,prelink}} = a\widehat{\theta}_{\mathrm{AB}} + b, \tag{23}$$

$$\widehat{Y}_{\mathrm{A,link}} = \widehat{a}\theta_{\mathrm{A}} - a\theta_{\mathrm{A}}, \tag{24}$$

$$\widehat{Y}_{\mathrm{A,prelink}} = a\widehat{\theta}_{\mathrm{A}}. \tag{25}$$

Inserting Equations 22 to 25 into Equation 16 produces intuitive expressions for the variance component decomposition approximation:

$$\mathrm{var}\left(\hat{a}\hat{\theta}_{\mathrm{AB}}+\hat{b}\right)\approx\mathrm{var}\left(\hat{a}\theta_{\mathrm{AB}}+\hat{b}\right)+\mathrm{var}\left(a\hat{\theta}_{\mathrm{AB}}+b\right)+2\mathrm{covar}\left(\hat{a}\theta_{\mathrm{AB}}+\hat{b},a\hat{\theta}_{\mathrm{AB}}+b\right) \quad (26)$$

$$\mathrm{var}\left(\hat{a}\hat{\theta}_{\mathrm{A}}\right)\approx\mathrm{var}\left(\hat{a}\theta_{\mathrm{A}}\right)+\mathrm{var}\left(a\hat{\theta}_{\mathrm{A}}\right)+2\mathrm{covar}\left(\hat{a}\theta_{\mathrm{A}},a\hat{\theta}_{\mathrm{A}}\right). \quad (27)$$

Obtaining expressions for the linking and prelinking variances and covariances for type AB and type A estimators follows from Equations 22 to 25 and the definition of variance:

$$\mathrm{var}\left(\hat{Y}_{\mathrm{AB,link}}\right)=\theta^2\mathrm{var}\left(\hat{a}\right)+\mathrm{var}\left(\hat{b}\right)+2\theta\mathrm{covar}\left(\hat{a},\hat{b}\right), \quad (28)$$

$$\mathrm{var}\left(\hat{Y}_{\mathrm{AB,prelink}}\right)=a^2\mathrm{var}\left(\hat{\theta}\right), \quad (29)$$

$$\mathrm{covar}\left(\hat{Y}_{\mathrm{AB,link}},\hat{Y}_{\mathrm{AB,prelink}}\right)=a\theta\mathrm{covar}\left(\hat{a},\hat{\theta}\right)+a\mathrm{covar}\left(\hat{b},\hat{\theta}\right), \quad (30)$$

$$\mathrm{var}\left(\hat{Y}_{\mathrm{A,link}}\right)=\theta^2\mathrm{var}\left(\hat{a}\right), \quad (31)$$

$$\mathrm{var}\left(\hat{Y}_{\mathrm{A,prelink}}\right)=a^2\mathrm{var}\left(\hat{\theta}\right), \quad (32)$$

$$\mathrm{covar}\left(\hat{Y}_{\mathrm{A,link}},\hat{Y}_{\mathrm{A,prelink}}\right)=a\theta\mathrm{covar}\left(\hat{a},\hat{\theta}\right). \quad (33)$$

The preceding expressions for the variance and covariance components are exact. However, the interpretation of these expressions may not be straightforward. Instead of the variance of the estimators of the linking coefficients ($\hat{a}$ and $\hat{b}$), expressing the variance of the linking component in terms of the variances of the population mean and standard deviation estimators ($\hat{\mu}_\theta$, $\hat{\sigma}_\theta$, $\hat{\mu}_X$, and $\hat{\sigma}_X$) is more familiar. Furthermore, as both the pretransformation estimator ($\hat{\theta}$) and the linking coefficient estimators share a common dependency on the source sample estimators of the population mean and standard deviation ($\hat{\mu}_\theta$ and $\hat{\sigma}_\theta$), expressing the variance of the prelinking component also in terms of the variance of the population mean and standard deviation estimators helps to interpret the covariance between the two components.

To express the prelinking variance component in terms of the variance of the population mean and standard deviation estimators, we can define the standardized score for estimands corresponding to a type AB estimator as

$$Z=\sigma_\theta^{-1}\left(\theta_{\mathrm{AB}}-\mu_\theta\right) \quad (34)$$

and for estimands corresponding to a type A estimator as

$$R=\sigma_\theta^{-1}\theta_{\mathrm{A}}, \quad (35)$$

with corresponding estimators

$$\hat{Z}=\hat{\sigma}_\theta^{-1}\left(\hat{\theta}_{\mathrm{AB}}-\hat{\mu}_\theta\right) \quad (36)$$

$$\hat{R}=\hat{\sigma}_\theta^{-1}\hat{\theta}_{\mathrm{A}}. \quad (37)$$

Inserting Equations 34 to 37 into Equations 22 to 25, and suppressing zero-variance terms to simplify, gives

$$\hat{Y}_{\mathrm{AB,link}}=\hat{\sigma}_X\hat{\sigma}_\theta^{-1}\left(\sigma_\theta Z+\mu_\theta-\hat{\mu}_\theta\right)+\hat{\mu}_X, \quad (38)$$

$$\hat{Y}_{\mathrm{AB,prelink}}=\sigma_X\sigma_\theta^{-1}\left(\hat{\sigma}_\theta\hat{Z}+\hat{\mu}_\theta\right), \quad (39)$$

$$\hat{Y}_{\mathrm{A,link}}=\hat{\sigma}_X\hat{\sigma}_\theta^{-1}\sigma_\theta R, \quad (40)$$

$$\hat{Y}_{\mathrm{A,prelink}}=\sigma_X\sigma_\theta^{-1}\hat{\sigma}_\theta\hat{R}. \quad (41)$$

The following approximations based on first-order Taylor series are derived for the variance and covariance components in terms of the variances of $\hat{\mu}_\theta$, $\hat{\sigma}_\theta$, $\hat{\mu}_X$, and $\hat{\sigma}_X$. A derivation of the general Taylor series-based approximation method that is applied in the following proofs is provided in the appendix.

Linking Component

Theorem 1. *For a type AB estimator, the linking variance component can be approximated as*

$$\begin{aligned}\mathrm{var}^{TS}\left(\widehat{Y}_{\mathrm{AB,link}}\right) &= Z^2\left[\mathrm{var}\left(\widehat{\sigma}_X\right) + (-1)^2 a^2\mathrm{var}\left(\widehat{\sigma}_\theta\right) - 2a\mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\sigma}_\theta\right)\right] \\ &+ Z\left[2a^2\mathrm{covar}\left(\widehat{\mu}_\theta, \widehat{\sigma}_\theta\right) + 2\mathrm{covar}\left(\widehat{\mu}_X, \widehat{\sigma}_X\right) - 2a\mathrm{covar}\left(\widehat{\mu}_X, \widehat{\sigma}_\theta\right) - 2a\mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\mu}_\theta\right)\right] \\ &+ \mathrm{var}\left(\widehat{\mu}_X\right) + (-1)^2 a^2\mathrm{var}\left(\widehat{\mu}_\theta\right) - 2a\mathrm{covar}\left(\widehat{\mu}_X, \widehat{\mu}_\theta\right). \end{aligned} \tag{42}$$

Proof. The variance of $\widehat{Y}_{\mathrm{AB,link}}$ is approximated by the variance of the first-order Taylor series of Equation 38 at $\left[\widehat{\mu}_\theta, \widehat{\sigma}_\theta, \widehat{\mu}_X, \widehat{\sigma}_Y\right] = \left[\mu_\theta, \sigma_\theta, \mu_X, \sigma_Y\right]$:

$$\mathrm{var}\left(\widehat{Y}_{\mathrm{AB,link}}\right) \approx \mathrm{var}^{TS}\left(\widehat{Y}_{\mathrm{AB,link}}\right) \tag{43}$$

$$= \nabla^T\widehat{Y}_{\mathrm{AB,link}}\left(\mu_\theta, \sigma_\theta, \mu_X, \sigma_X\right) \cdot \mathbf{\Sigma} \cdot \nabla\widehat{Y}_{\mathrm{AB,link}}\left(\mu_\theta, \sigma_\theta, \mu_X, \sigma_X\right), \tag{44}$$

where

$$\nabla\widehat{Y}_{\mathrm{AB,link}}\left(\mu_\theta, \sigma_\theta, \mu_X, \sigma_X\right) = \begin{bmatrix} \frac{\delta}{\delta\widehat{\mu}_\theta} \\ \frac{\delta}{\delta\widehat{\sigma}_\theta} \\ \frac{\delta}{\delta\widehat{\mu}_X} \\ \frac{\delta}{\delta\widehat{\sigma}_X} \end{bmatrix} \widehat{Y}_{\mathrm{AB,link}}\left(\mu_\theta, \sigma_\theta, \mu_X, \sigma_X\right) \tag{45}$$

$$= [-a \; -Za \;\; 1 \;\; Z]^T \tag{46}$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} \mathrm{var}\left(\widehat{\mu}_\theta\right) & \mathrm{covar}\left(\widehat{\mu}_\theta, \widehat{\sigma}_\theta\right) & \mathrm{covar}\left(\widehat{\mu}_\theta, \widehat{\mu}_X\right) & \mathrm{covar}\left(\widehat{\mu}_\theta, \widehat{\sigma}_X\right) \\ \mathrm{covar}\left(\widehat{\sigma}_\theta, \widehat{\mu}_\theta\right) & \mathrm{var}\left(\widehat{\sigma}_\theta\right) & \mathrm{covar}\left(\widehat{\sigma}_\theta, \widehat{\mu}_X\right) & \mathrm{covar}\left(\widehat{\sigma}_\theta, \widehat{\sigma}_X\right) \\ \mathrm{covar}\left(\widehat{\mu}_X, \widehat{\mu}_\theta\right) & \mathrm{covar}\left(\widehat{\mu}_X, \widehat{\sigma}_\theta\right) & \mathrm{var}\left(\widehat{\mu}_X\right) & \mathrm{covar}\left(\widehat{\mu}_X, \widehat{\sigma}_X\right) \\ \mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\mu}_\theta\right) & \mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\sigma}_\theta\right) & \mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\mu}_X\right) & \mathrm{var}\left(\widehat{\sigma}_X\right) \end{bmatrix}. \tag{47}$$

■

Theorem 2. *For a type A estimator, the linking variance component can be approximated as*

$$\mathrm{var}^{\mathrm{TS}}_{\mathrm{link}}\left(\widehat{Y}_{\mathrm{A}}\right) = R^2\left[\mathrm{var}\left(\widehat{\sigma}_X\right) + (-1)^2 a^2\mathrm{var}\left(\widehat{\sigma}_\theta\right) - 2a\mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\sigma}_\theta\right)\right]. \tag{48}$$

Proof. The variance of $\widehat{Y}_{\mathrm{A,link}}$ is approximated by the variance of the first-order Taylor series of Equation 40 at $\left[\widehat{\sigma}_\theta, \widehat{\sigma}_Y\right] = \left[\sigma_\theta, \sigma_Y\right]$:

$$\mathrm{var}\left(\widehat{Y}_{\mathrm{A,link}}\right) \approx \mathrm{var}^{TS}\left(\widehat{Y}_{\mathrm{A,link}}\right) \tag{49}$$

$$= \nabla^T\widehat{Y}_{\mathrm{A,link}}\left(\sigma_\theta, \sigma_X\right) \cdot \mathbf{\Sigma} \cdot \nabla\widehat{Y}_{\mathrm{A,link}}\left(\sigma_\theta, \sigma_X\right), \tag{50}$$

where

$$\nabla\widehat{Y}_{\mathrm{AB,link}}\left(\sigma_\theta, \sigma_X\right) = \begin{bmatrix} \frac{\delta}{\delta\widehat{\sigma}_\theta} \\ \frac{\delta}{\delta\widehat{\sigma}_X} \end{bmatrix} \widehat{Y}_{\mathrm{A,link}}\left(\sigma_\theta, \sigma_X\right) \tag{51}$$

$$= \left[-Za \quad Z\right]^T \tag{52}$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} \mathrm{var}\left(\widehat{\sigma}_\theta\right) & \mathrm{covar}\left(\widehat{\sigma}_\theta, \widehat{\sigma}_X\right) \\ \mathrm{covar}\left(\widehat{\sigma}_X, \widehat{\sigma}_\theta\right) & \mathrm{var}\left(\widehat{\sigma}_X\right) \end{bmatrix}. \tag{53}$$

■

The coefficients $(-1)^2$ are not simplified to represent that the source mean and standard deviation estimators negatively relate to the linking component, which is important later.

For means, $\widehat{Z}$ can be interpreted as an estimator of the standardized score with respect to the population mean and standard deviation estimators, or a Z-score. For standard deviations, $\widehat{R}$ can be interpreted analogously to $\widehat{Z}$ as the ratio of the standard deviation estimator over the population standard deviation estimator. For other parameter types, such as percentiles, the equations still apply, although the standardized score estimators, $\widehat{Z}$ and $\widehat{R}$, may not conform to the usual interpretation associated with standardized scores. Both $\widehat{Z}$ and $\widehat{R}$ are unitless and invariant to the transformation and may be calculated from either the pretransformed or posttransformed scores equivalently.

The a coefficients in the linking variance components (Equations 42 and 48) may be understood to ensure that all terms are in the target metric. For example, a variance of an estimator in the original metric, such as $\text{var}\left(\widehat{\mu}_\theta\right)$, has the a^2 coefficient, and a covariance involving one estimator in the original metric and one estimator in the target metric, such as $\text{covar}\left(\widehat{\sigma}_X, \widehat{\sigma}_\theta\right)$, has the a coefficient.

When the a coefficients are understood to ensure that all terms are in the target metric, the Taylor series-based approximation of the linking variance components is symmetrical, where the population mean and standard deviation estimators from both the target and source samples contribute equivalently. Note that the exact symmetry is due to the Taylor series-based approximation of the variance of $\widehat{\sigma}_\theta^{-1}$. However, this approximation is expected to be very precise in large samples (see Remark 1), suggesting that the exact linking variance component would be highly symmetrical.

The variances of the two mean estimators from the two samples contribute to the linking variance component of type AB estimators equivalently and independently of the standardized score. The linking variance component of type A estimators does not depend on the mean estimators and their variances. For both types AB and A estimators, the contribution of the variance of the standard deviation estimators is proportional to Z^2 (for type AB estimators) or R^2 (for type A estimators). The remaining terms are covariances between pairings of the mean and standard deviation estimators.

Overall, the linking variance components are quadratic functions of the standardized scores. For type A estimators, the linking variance component is minimum, at zero, when R is zero. For type AB estimators, if the covariance terms are zero, the linking variance component is minimum when Z is zero; however, if the covariance terms are nonzero, the linking variance component for type AB estimators may be minimum for a nonzero standardized score.

Remark 1. First-order Taylor series expansion is not exact when higher order partial derivatives are nonzero. Notably, the higher order derivatives of $\widehat{\sigma}_\theta^{-1}$ with respect to $\widehat{\sigma}_\theta$ are nonzero. The approximate variance of $\widehat{\sigma}_\theta^{-1}$ based on the first-order Taylor series at $\widehat{\sigma}_\theta = \sigma_\theta$ is

$$\text{var}^{TS}\left(\widehat{\sigma}_\theta^{-1}\right) = \sigma_\theta^{-4}\text{var}\left(\widehat{\sigma}_\theta\right). \tag{54}$$

Assuming that the third and fourth moments of $\widehat{\sigma}_\theta$ are zero, the approximate variance of $\widehat{\sigma}_\theta^{-1}$ based on the second-order Taylor series at $\widehat{\sigma}_\theta = \sigma_\theta$ is

$$\text{var}^{\text{TS}^2}\left(\widehat{\sigma}_\theta^{-1}\right) = \sigma_\theta^{-4}\text{var}\left(\widehat{\sigma}_\theta\right) - \sigma_\theta^{-6}\text{var}^2\left(\widehat{\sigma}_\theta\right) \tag{55}$$

$$= \text{var}^{\text{TS}}\left(\widehat{\sigma}_\theta^{-1}\right)\left[1 - \sigma_\theta^{-2}\text{var}\left(\widehat{\sigma}_\theta\right)\right]. \tag{56}$$

For large sample sizes, $\sigma_\theta^2 \gg \text{var}\left(\widehat{\sigma}_\theta\right)$, and the additional second-order term is very small. For example, for all subscales for the 2017 NAEP reading and mathematics assessments, $\sigma_\theta^{-2}\text{var}\left(\widehat{\sigma}_\theta\right) < .001$ and $\text{var}^{\text{TS}}\left(\sigma_\theta^{-1}\right) \approx \text{var}^{\text{TS}^2}\left(\sigma_\theta^{-1}\right)$. Because higher order terms are expected to be even smaller, the first-order approximation should be effective.

Prelinking Variance Component

Theorem 3. *For a type AB estimator, the prelinking variance component can be approximated as*

$$\text{var}^{\text{TS}}\left(\widehat{Y}_{\text{AB,prelink}}\right) = Z^2a^2\text{var}\left(\widehat{\sigma}_\theta\right) + Z\left[2\sigma_X a\text{covar}\left(\widehat{\sigma}_\theta, \widehat{Z}\right) + 2a^2\text{covar}\left(\widehat{\sigma}_\theta, \widehat{\mu}_\theta\right)\right]$$
$$+ \sigma_X^2\text{var}\left(\widehat{Z}\right) + a^2\text{var}\left(\widehat{\mu}_\theta\right) + 2\sigma_X a\text{covar}\left(\widehat{Z}, \widehat{\mu}_\theta\right). \tag{57}$$

Proof. The variance of $\widehat{Y}_{\text{AB,prelink}}$ is approximated by the variance of the first-order Taylor series of Equation 39 at $\left[\widehat{\mu}_\theta, \widehat{\sigma}_\theta, \widehat{Z}\right] = \left[\mu_\theta, \sigma_\theta, Z\right]$:

$$\text{var}\left(\widehat{Y}_{\text{AB,prelink}}\right) \approx \text{var}^{TS}\left(\widehat{Y}_{\text{AB,prelink}}\right) \tag{58}$$

$$= \nabla^T \widehat{Y}_{\text{AB,prelink}}\left(\mu_\theta, \sigma_\theta, Z\right) \cdot \boldsymbol{\Sigma} \cdot \nabla \widehat{Y}_{\text{AB,prelink}}\left(\mu_\theta, \sigma_\theta, Z\right), \tag{59}$$

where

$$\nabla \widehat{Y}_{\text{AB,prelink}}\left(\mu_\theta, \sigma_\theta, Z\right) = \begin{bmatrix} \frac{\delta}{\delta \widehat{\mu}_\theta} \\ \frac{\delta}{\delta \widehat{\sigma}_\theta} \\ \frac{\delta}{\delta \widehat{Z}} \end{bmatrix} \widehat{Y}_{\text{AB,prelink}}\left(\mu_\theta, \sigma_\theta, Z\right) \tag{60}$$

$$= \begin{bmatrix} a & Za & \sigma_X \end{bmatrix}^T \tag{61}$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \text{var}\left(\widehat{\mu}_\theta\right) & \text{covar}\left(\widehat{\mu}_\theta, \widehat{\sigma}_\theta\right) & \text{covar}\left(\widehat{\mu}_\theta, \widehat{Z}\right) \\ \text{covar}\left(\widehat{\sigma}_\theta, \widehat{\mu}_\theta\right) & \text{var}\left(\widehat{\sigma}_\theta\right) & \text{covar}\left(\widehat{\sigma}_\theta, \widehat{Z}\right) \\ \text{covar}\left(\widehat{Z}, \widehat{\mu}_\theta\right) & \text{covar}\left(\widehat{Z}, \widehat{\sigma}_\theta\right) & \text{var}\left(\widehat{Z}\right) \end{bmatrix}. \tag{62}$$

■

Theorem 4. *For a type A estimator, the prelinking variance component can be approximated as*

$$\text{var}^{\text{TS}}_{\text{prelink}}\left(\widehat{Y}_{\text{A}}\right) = \sigma_X^2 \text{var}\left(\widehat{R}\right) + R^2 a^2 \text{var}\left(\widehat{\sigma}_\theta\right) + 2Ra\sigma_X \text{covar}\left(\widehat{\sigma}_\theta, \widehat{R}\right). \tag{63}$$

Proof. The variance of $\widehat{Y}_{\text{A,prelink}}$ is approximated by the variance of the first-order Taylor series of Equation 41 at $\left[\widehat{\sigma}_\theta, \widehat{R}\right] = \left[\sigma_\theta, R\right]$:

$$\text{var}\left(\widehat{Y}_{\text{A,prelink}}\right) \approx \text{var}^{TS}\left(\widehat{Y}_{\text{A,prelink}}\right) \tag{64}$$

$$= \nabla^T \widehat{Y}_{\text{A,prelink}}\left(\sigma_\theta, R\right) \cdot \boldsymbol{\Sigma} \cdot \nabla \widehat{Y}_{\text{A,prelink}}\left(\sigma_\theta, R\right), \tag{65}$$

where

$$\nabla \widehat{Y}_{\text{A,prelink}}\left(\sigma_\theta, R\right) = \begin{bmatrix} \frac{\delta}{\delta \widehat{\sigma}_\theta} \\ \frac{\delta}{\delta \widehat{R}} \end{bmatrix} \widehat{Y}_{\text{A,prelink}}\left(\sigma_\theta, R\right) \tag{66}$$

$$= \begin{bmatrix} Ra & \sigma_X \end{bmatrix}^T \tag{67}$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \text{var}\left(\widehat{\sigma}_\theta\right) & \text{covar}\left(\widehat{\sigma}_\theta, \widehat{R}\right) \\ \text{covar}\left(\widehat{R}, \widehat{\sigma}_\theta\right) & \text{var}\left(\widehat{R}\right) \end{bmatrix}. \tag{68}$$

■

The preceding equations show that the prelinking variance component for type AB estimators depends on the variance of $\widehat{\mu}_\theta$ and the variance of $\widehat{\sigma}_\theta$ and that the prelinking variance component for type A estimators depends on the variance of $\widehat{\sigma}_\theta$. Because both the prelinking and linking components depend on these variances, covariance between the prelinking and linking components is expected. Notably, the prelinking component is positively dependent on $\widehat{\mu}_\theta$ and $\widehat{\sigma}_\theta$, while the linking component is negatively dependent on the same estimators.

Prelinking and Linking Covariance

Theorem 5. *For a type AB estimator, the covariance of the linking and prelinking components can be approximated as*

$$\begin{aligned}\text{covar}^{\text{TS}}_{\text{link,prelink}}\left(\hat{Y}_{\text{AB}}\right) &= Z^2\left[a\text{covar}\left(\hat{\sigma}_X,\hat{\sigma}_\theta\right)-a^2\text{var}\left(\hat{\sigma}_\theta\right)\right]\\ &+Z\left[a\text{covar}\left(\hat{\mu}_X,\hat{\sigma}_\theta\right)+a\text{covar}\left(\hat{\sigma}_X,\hat{\mu}_\theta\right)-2a^2\text{covar}\left(\hat{\mu}_\theta,\hat{\sigma}_\theta\right)+\sigma_X\text{covar}\left(\hat{Z},\hat{\sigma}_X\right)-\sigma_X a\text{covar}\left(\hat{\sigma}_\theta,\hat{Z}\right)\right]\\ &-a^2\text{var}\left(\hat{\mu}_\theta\right)+a\text{covar}\left(\hat{\mu}_X,\hat{\mu}_\theta\right)-\sigma_X a\text{covar}\left(\hat{Z},\hat{\mu}_\theta\right)+\sigma_X\text{covar}\left(\hat{Z},\hat{\mu}_X\right).\end{aligned} \tag{69}$$

Proof. The covariance of $\hat{Y}_{\text{AB,link}}$ and $\hat{Y}_{\text{AB,prelink}}$ is approximated by the covariance of the first-order Taylor series of Equation 38 at $\left[\hat{\mu}_\theta,\hat{\sigma}_\theta,\hat{\mu}_X,\hat{\sigma}_Y\right]=\left[\mu_\theta,\sigma_\theta,\mu_X,\sigma_Y\right]$ and the first-order Taylor series of Equation 39 at $\left[\hat{\mu}_\theta,\hat{\sigma}_\theta,\hat{Z}\right]=\left[\mu_\theta,\sigma_\theta,Z\right]$:

$$\text{covar}\left(\hat{Y}_{\text{AB,link}},\hat{Y}_{\text{AB,prelink}}\right)\approx\text{covar}^{\text{TS}}\left(\hat{Y}_{\text{AB,link}},\hat{Y}_{\text{AB,prelink}}\right) \tag{70}$$

$$=\nabla^T\hat{Y}_{\text{AB,link}}\left(\mu_\theta,\sigma_\theta,\mu_X,\sigma_X\right)\cdot\mathbf{\Sigma}\cdot\nabla\hat{Y}_{\text{AB,prelink}}\left(\mu_\theta,\sigma_\theta,Z\right), \tag{71}$$

where $\nabla\hat{Y}_{\text{AB,link}}\left(\mu_\theta,\sigma_\theta,\mu_X,\sigma_X\right)$ is given by Equation 46 and $\nabla\hat{Y}_{\text{AB,prelink}}\left(\mu_\theta,\sigma_\theta,Z\right)$ is given by Equation 61, and where

$$\mathbf{\Sigma}=\begin{bmatrix}\text{var}\left(\hat{\mu}_\theta\right) & \text{covar}\left(\hat{\mu}_\theta,\hat{\sigma}_\theta\right) & \text{covar}\left(\hat{\mu}_\theta,\hat{Z}\right)\\ \text{covar}\left(\hat{\sigma}_\theta,\hat{\mu}_\theta\right) & \text{var}\left(\hat{\sigma}_\theta\right) & \text{covar}\left(\hat{\sigma}_\theta,\hat{Z}\right)\\ \text{covar}\left(\hat{\mu}_X,\hat{\mu}_\theta\right) & \text{covar}\left(\hat{\mu}_X,\hat{\sigma}_\theta\right) & \text{covar}\left(\hat{\mu}_X,\hat{Z}\right)\\ \text{covar}\left(\hat{\sigma}_X,\hat{\mu}_\theta\right) & \text{covar}\left(\hat{\sigma}_X,\hat{\sigma}_\theta\right) & \text{covar}\left(\hat{\sigma}_X,\hat{Z}\right)\end{bmatrix}. \tag{72}$$

■

Theorem 6. *For a type A estimator, the covariance of the linking and prelinking components can be approximated as*

$$\text{covar}^{\text{TS}}_{\text{link,prelink}}\left(\hat{Y}_{\text{A}}\right)=R^2\left[a\text{covar}\left(\hat{\sigma}_X,\hat{\sigma}_\theta\right)-a^2\text{var}\left(\hat{\sigma}_\theta\right)\right] \tag{73}$$

$$+R\left[\sigma_X\text{covar}\left(\hat{\sigma}_X,\hat{R}\right)-a\sigma_X\text{covar}\left(\hat{\sigma}_\theta,\hat{R}\right)\right]. \tag{74}$$

Proof. The covariance of $\hat{Y}_{\text{A,link}}$ and $\hat{Y}_{\text{A,prelink}}$ is approximated by the covariance of the first-order Taylor series of Equation 40 at $\left[\hat{\sigma}_\theta,\hat{\sigma}_Y\right]=\left[\sigma_\theta,\sigma_Y\right]$ and the first-order Taylor series of Equation 41 at $\left[\hat{\sigma}_\theta,\hat{R}\right]=\left[\sigma_\theta,R\right]$:

$$\text{covar}\left(\hat{Y}_{\text{A,link}},\hat{Y}_{\text{A,prelink}}\right)\approx\text{covar}^{\text{TS}}\left(\hat{Y}_{\text{A,link}},\hat{Y}_{\text{A,prelink}}\right) \tag{75}$$

$$=\nabla^T\hat{Y}_{\text{A,link}}\left(\sigma_\theta,\sigma_X\right)\cdot\mathbf{\Sigma}\cdot\nabla\hat{Y}_{\text{A,prelink}}\left(\sigma_\theta,R\right), \tag{76}$$

where $\nabla\hat{Y}_{\text{A,link}}\left(\sigma_\theta,\sigma_X\right)$ is given by Equation 52 and $\nabla\hat{Y}_{\text{A,prelink}}\left(\sigma_\theta,R\right)$ is given by Equation 67, and where

$$\mathbf{\Sigma}=\begin{bmatrix}\text{var}\left(\hat{\sigma}_\theta\right) & \text{covar}\left(\hat{\sigma}_\theta,\hat{R}\right)\\ \text{covar}\left(\hat{\sigma}_X,\hat{\sigma}_\theta\right) & \text{covar}\left(\hat{\sigma}_X,\hat{R}\right)\end{bmatrix}. \tag{77}$$

■

Notably, the negative dependency of the linking component and the positive dependency of the prelinking component on $\hat{\mu}_\theta$ and $\hat{\sigma}_\theta$ result in the presence of the variance of these estimators in the covariance component with a negative sign.

Total Variance

Theorem 7. *For a type AB estimator, the variance can be approximated as*

$$\begin{aligned}\text{var}\left(\hat{Y}_{\text{AB}}\right)&\approx Z^2\text{var}\left(\hat{\sigma}_X\right)+\sigma_X^2\text{var}\left(\hat{Z}\right)+\text{var}\left(\hat{\mu}_X\right)\\ &+2Z\sigma_X\text{covar}\left(\hat{\sigma}_X,\hat{Z}\right)+2Z\text{covar}\left(\hat{\sigma}_X,\hat{\mu}_X\right)+2\sigma_X\text{covar}\left(\hat{Z},\hat{\mu}_X\right).\end{aligned} \tag{78}$$

Proof. From Equation 16, the variance component decomposition approach takes the approximation that the variance of a type AB estimator can be written as

$$\text{var}\left(\hat{Y}_{\text{AB}}\right) \approx \text{var}_{\text{link}}\left(\hat{Y}_{\text{AB}}\right) + \text{var}_{\text{prelink}}\left(\hat{Y}_{\text{AB}}\right) + \text{covar}_{\text{link,prelink}}\left(\hat{Y}_{\text{AB}}\right). \tag{79}$$

Each variance component may be approximated based on first-order Taylor series,

$$\text{var}\left(\hat{Y}_{\text{AB}}\right) \approx \text{var}^{\text{TS}}_{\text{link}}\left(\hat{Y}_{\text{AB}}\right) + \text{var}^{\text{TS}}_{\text{prelink}}\left(\hat{Y}_{\text{AB}}\right) + \text{covar}^{\text{TS}}_{\text{link,prelink}}\left(\hat{Y}_{\text{AB}}\right). \tag{80}$$

Insert Equations 42, 57, and 69 into Equation 80. ■

Theorem 8. *For a type A estimator, the total variance can be approximated as*

$$\text{var}\left(\hat{Y}_{\text{A}}\right) \approx \sigma_X^2 \text{var}\left(\hat{R}\right) + R^2 \text{var}\left(\hat{\sigma}_X\right) + 2\sigma_X R \text{covar}\left(\hat{\sigma}_X, \hat{R}\right). \tag{81}$$

Proof. From Equation 16, the variance component approach takes the approximation that the variance of a type A estimator can be written as

$$\text{var}\left(\hat{Y}_{\text{A}}\right) \approx \text{var}_{\text{link}}\left(\hat{Y}_{\text{A}}\right) + \text{var}_{\text{prelink}}\left(\hat{Y}_{\text{A}}\right) + \text{covar}_{\text{link,prelink}}\left(\hat{Y}_{\text{A}}\right). \tag{82}$$

Each variance component may be approximated based on first-order Taylor series,

$$\text{var}\left(\hat{Y}_{\text{A}}\right) \approx \text{var}^{\text{TS}}_{\text{link}}\left(\hat{Y}_{\text{A}}\right) + \text{var}^{\text{TS}}_{\text{prelink}}\left(\hat{Y}_{\text{A}}\right) + \text{covar}^{\text{TS}}_{\text{link,prelink}}\left(\hat{Y}_{\text{A}}\right). \tag{83}$$

Insert Equations 48, 63, and 74 into Equation 83. ■

Remark 2. The mutual dependency of the linking and prelinking components on $\hat{\mu}_\theta$ and $\hat{\sigma}_\theta$, symmetrically but in different directions, results in the dependency on the variances of $\hat{\mu}_\theta$ and $\hat{\sigma}_\theta$ falling out when all variance components are combined to obtain the total variance.

Equation 78 shows that the variance of a type AB estimator is a function of the variance of $\hat{\sigma}_X$ proportional to Z^2, the variance of $\hat{Z}$ with the coefficient of σ_X^2 to place the term in the target metric, the variance of $\hat{\mu}_X$, and the corresponding covariances for these terms. Equation 81 shows that the variance of a type A estimator is a function of the variance of $\hat{\sigma}_X$ proportional to R^2 and the variance of $\hat{R}$ with the coefficient of σ_X^2 to place the term in the target metric, and the corresponding covariances for these terms.

Assuming that the linking and prelinking components are independent is very convenient in practice, because the linking variance component can be estimated independently of the conventional variance (i.e., the prelinking component) and simply added to the conventional variance to approximate the variance of $\hat{Y}$ (Mazzeo et al., 2018). However, the equations show the importance of estimating the covariance of the linking and prelinking components when using the variance component decomposition approximation in practice. Assuming that the linking and prelinking covariance term is zero has the impact of accounting for the variance of the source sample estimators of the population mean and standard deviation twice, once in the prelinking component and once in the linking component, when in fact the dependency on the source sample estimators should fall out (see Remark 2). The smaller the sample size of the source sample, the more important the linking and prelinking covariance term is.

Exact Total Variance

In the previous section, I showed that the intuitive and convenient approach of decomposing the variance of an estimator into variance components makes certain assumptions that are highly approximated by some estimator types (e.g., sample means) but may not be approximated well by other estimator types (e.g., achievement levels). Furthermore, accounting for the covariance between the components is important for accurate variance estimation.

Given the inherent approximation in decomposing the variance of estimators posttransformation into linking and prelinking components, considering both sources of variance (i.e., due to variance of $\hat{\theta}$ and due to variances of $\hat{a}$ and $\hat{b}$) together can be beneficial. In the Variance Component Decomposition Approximation section, I derived exact expressions for the components in terms of $\hat{\theta}$, $\hat{a}$, and $\hat{b}$ but Taylor series-based approximations for the components in terms of $\widehat{\mu}_X$, $\widehat{\sigma}_X$, $\widehat{\mu}_\theta$, $\widehat{\sigma}_\theta$, and $\widehat{Z}$ or $\widehat{R}$. Obtaining exact equations for the total variances of type AB and type A estimators in terms of these variables is actually more tractable than obtaining exact equations for the variances and covariances of the components. For this reason, the exact variances of type AB and type A estimators are shown in the following pages.

Type AB Estimators

Theorem 9. *In general, the exact variance of a type AB estimator is*

$$\text{var}\left(\widehat{Y}_{\text{AB}}\right) = \sigma_X^2 \text{var}\left(\widehat{Z}\right) + Z^2 \text{var}\left(\widehat{\sigma}_X\right) + \text{var}\left(\widehat{\mu}_X\right) + \text{var}\left(\widehat{Z}\right)\text{var}\left(\widehat{\sigma}_X\right) + 2\text{covar}\left(\widehat{\sigma}_X \widehat{Z}, \widehat{\mu}_X\right) + \text{covar}\left(\widehat{\sigma}_X^2, \widehat{Z}^2\right) - \text{covar}^2\left(\widehat{\sigma}_X, \widehat{Z}\right) - 2\sigma_X Z \text{covar}\left(\widehat{\sigma}_X, \widehat{Z}\right). \quad (84)$$

Proof. Starting with Equation 10 and inserting Equations 8, 9, and 36,

$$\widehat{Y}_{\text{AB}} = \widehat{a}\widehat{\theta}_{\text{AB}} + \widehat{b} \quad (85)$$

$$= \widehat{\sigma}_X \left(\widehat{\sigma}_\theta^{-1} \widehat{\theta}_{\text{AB}} - \widehat{\mu}_\theta\right) + \widehat{\mu}_X \quad (86)$$

$$= \widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X. \quad (87)$$

It follows that

$$\text{var}\left(\widehat{Y}_{\text{AB}}\right) = \text{var}\left(\widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X\right). \quad (88)$$

Denote $\mathbb{E}$ as the expectation of a random variable. By the definition of a variance, and simplifying,

$$\text{var}\left(\widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X\right) = \mathbb{E}\left[\left(\widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X - \mathbb{E}\left[\widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X\right]\right)^2\right] \quad (89)$$

$$= \mathbb{E}\left[\left(\widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X\right)^2\right] - \mathbb{E}^2\left[\widehat{\sigma}_X \widehat{Z} + \widehat{\mu}_X\right] \quad (90)$$

$$= \mathbb{E}\left[\widehat{\sigma}_X^2 \widehat{Z}^2\right] - \mathbb{E}^2\left[\widehat{\sigma}_X \widehat{Z}\right] - \mathbb{E}^2\left[\widehat{\mu}_X\right] + 2\mathbb{E}\left[\widehat{\sigma}_X \widehat{Z} \widehat{\mu}_X\right] + \mathbb{E}\left[\widehat{\mu}_X^2\right] - 2\mathbb{E}\left[\widehat{\sigma}_X \widehat{Z}\right]\mathbb{E}\left[\widehat{\mu}_X\right] \quad (91)$$

$$= \text{var}\left(\widehat{\sigma}_X \widehat{Z}\right) + \text{var}\left(\widehat{\mu}_X\right) + 2\text{covar}\left(\widehat{\sigma}_X \widehat{Z}, \widehat{\mu}_X\right). \quad (92)$$

For $\text{var}\left(\widehat{\sigma}_X \widehat{Z}\right)$, the variance of a product of random variables may be obtained following Goodman (1960). Again, by the definition of a variance,

$$\text{var}\left(\widehat{\sigma}_X \widehat{Z}\right) = \mathbb{E}\left[\left(\widehat{\sigma}_X \widehat{Z} - \mathbb{E}\left[\widehat{\sigma}_X \widehat{Z}\right]\right)^2\right] \quad (93)$$

$$= \mathbb{E}\left[\widehat{\sigma}_X^2 \widehat{Z}^2\right] - \mathbb{E}^2\left[\widehat{\sigma}_X \widehat{Z}\right]. \quad (94)$$

With the analogous expressions for covariance,

$$\mathbb{E}\left[\widehat{\sigma}_X^2 \widehat{Z}^2\right] = \text{covar}\left(\widehat{\sigma}_X^2, \widehat{Z}^2\right) + \mathbb{E}\left[\widehat{\sigma}_X^2\right]\mathbb{E}\left[\widehat{Z}^2\right] \quad (95)$$

$$\mathbb{E}\left[\widehat{\sigma}_X \widehat{Z}\right] = \text{covar}\left(\widehat{\sigma}_X, \widehat{Z}\right) + \mathbb{E}\left[\widehat{\sigma}_X\right]\mathbb{E}\left[\widehat{Z}\right]. \quad (96)$$

Similarly,

$$\mathbb{E}\left[\widehat{\sigma}_X^2\right] = \text{var}\left(\widehat{\sigma}_X\right) + \mathbb{E}^2\left[\widehat{\sigma}_X\right] \quad (97)$$

$$\mathbb{E}\left[\widehat{Z}^2\right] = \text{var}\left(\widehat{Z}\right) + \mathbb{E}^2\left[\widehat{Z}\right]. \quad (98)$$

Inserting Equations 97 and 98 into Equation 95, and the resulting equation together with Equation 96 into Equation 94, gives the exact variance of $\hat{\sigma}_X \hat{Z}$:

$$\begin{aligned} \text{var}\left(\hat{\sigma}_X \hat{Z}\right) &= E^2\left[\hat{\sigma}_X\right] \text{var}\left(\hat{Z}\right) + E^2\left[\hat{Z}\right] \text{var}\left(\hat{\sigma}_X\right) + \text{var}\left(\hat{Z}\right) \text{var}\left(\hat{\sigma}_X\right) \\ &+ \text{covar}\left(\hat{\sigma}_X^2, \hat{Z}^2\right) - \text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right) - 2E\left[\hat{\sigma}_X\right] E\left[\hat{Z}\right] \text{covar}\left(\hat{\sigma}_X, \hat{Z}\right). \end{aligned} \tag{99}$$

Substituting the estimands σ_X and Z for the expected values of the estimators $\hat{\sigma}_X$ and $\hat{Z}$,

$$\begin{aligned} \text{var}\left(\hat{\sigma}_X \hat{Z}\right) &= \sigma_X^2 \text{var}\left(\hat{Z}\right) + Z^2 \text{var}\left(\hat{\sigma}_X\right) + \text{var}\left(\hat{Z}\right) \text{var}\left(\hat{\sigma}_X\right) \\ &+ \text{covar}\left(\hat{\sigma}_X^2, \hat{Z}^2\right) - \text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right) - 2\sigma_X Z \text{covar}\left(\hat{\sigma}_X, \hat{Z}\right), \end{aligned} \tag{100}$$

and inserting into Equation 92 gives Equation 84. ∎

Remark 3. The exact variance of a type AB estimator (Equation 84) can be compared to the first-order Taylor series-based approximation previously obtained (Equation 78), to understand the approximations made by the Taylor series-based approach. The exact variance has a very small term, $\text{var}\left(\hat{Z}\right) \text{var}\left(\hat{\sigma}_X\right)$, that is not in the Taylor series-based version, but the term is negligible in practice (see Remark 4). Apart from that term, the covariance of $\hat{Z}$ with $\hat{\mu}_X$ and $\hat{\sigma}_X$ is represented differently in the exact and approximate versions. The exact variance of a type AB estimator may be approximated to become the previously obtained Taylor series-based approximation by ignoring the small covariance term, $\text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right)$, and applying first-order Taylor series-based approximation to $\text{covar}\left(\hat{\sigma}_X^2, \hat{Z}^2\right)$ and $\text{covar}\left(\hat{\sigma}_X \hat{Z}, \hat{\mu}_X\right)$.

Theorem 10. *Under multivariate normality of $\hat{\sigma}_X$, $\hat{Z}$, and $\hat{\mu}_X$, the exact variance of a type AB estimator is*

$$\begin{aligned} \text{var}^{\text{MVN}}\left(\hat{Y}_{\text{AB}}\right) &= \sigma_X^2 \text{var}\left(\hat{Z}\right) + Z^2 \text{var}\left(\hat{\sigma}_X\right) + \text{var}\left(\hat{\mu}_X\right) + 2Z\sigma_X \text{covar}\left(\hat{\sigma}_X, \hat{Z}\right) \\ &+ 2Z \text{covar}\left(\hat{\sigma}_X, \hat{\mu}_X\right) + 2\sigma_X \text{covar}\left(\hat{Z}, \hat{\mu}_X\right) + \text{var}\left(\hat{Z}\right) \text{var}\left(\hat{\sigma}_X\right) + \text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right). \end{aligned} \tag{101}$$

Proof. Bohrnstedt and Goldberger (1969) found that the variance of products of random variables may be simplified if the random variables are bivariate normally distributed. If $\hat{\sigma}_X$ and $\hat{Z}$ are bivariate normally distributed, $\text{var}\left(\hat{\sigma}_X \hat{Z}\right)$ simplifies to

$$\begin{aligned} \text{var}\left(\hat{\sigma}_X \hat{Z}\right) &= \sigma_X^2 \text{var}\left(\hat{Z}\right) + Z^2 \text{var}\left(\hat{\sigma}_X\right) + 2\sigma_X Z \text{covar}\left(\hat{\sigma}_X, \hat{Z}\right) \\ &+ \text{var}\left(\hat{Z}\right) \text{var}\left(\hat{\sigma}_X\right) + \text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right). \end{aligned} \tag{102}$$

Bohrnstedt and Goldberger (1969) also noted that the covariance with a product of random variables may be simplified if the random variables are multivariate normally distributed. Applying their result,

$$\begin{aligned} \text{covar}\left(\hat{\sigma}_X \hat{Z}, \hat{\mu}_X\right) &= \mathbb{E}\left(\hat{\sigma}_X\right) \text{covar}\left(\hat{Z}, \hat{\mu}_X\right) + \mathbb{E}\left(\hat{Z}\right) \text{covar}\left(\hat{\sigma}_X, \hat{\mu}_X\right) \\ &+ \mathbb{E}\left[\left(\hat{\sigma}_X - \mathbb{E}\left[\hat{\sigma}_X\right]\right)\left(\hat{Z} - \mathbb{E}\left[\hat{Z}\right]\right)\left(\hat{\mu}_X - \mathbb{E}\left[\hat{\mu}_X\right]\right)\right]. \end{aligned} \tag{103}$$

If $\hat{\sigma}_X$, $\hat{Z}$, and $\hat{\mu}_X$ are multivariate normally distributed, Equation 103 becomes

$$\text{covar}\left(\hat{\sigma}_X \hat{Z}, \hat{\mu}_X\right) = \mathbb{E}\left(\hat{\sigma}_X\right) \text{covar}\left(\hat{Z}, \hat{\mu}_X\right) + \mathbb{E}\left(\hat{Z}\right) \text{covar}\left(\hat{\sigma}_X, \hat{\mu}_X\right). \tag{104}$$

Substituting the estimands σ_X and Z for the expected values of the estimators $\hat{\sigma}_X$ and $\hat{Z}$,

$$\text{covar}\left(\hat{\sigma}_X \hat{Z}, \hat{\mu}_X\right) = \sigma_X \text{covar}\left(\hat{Z}, \hat{\mu}_X\right) + Z \text{covar}\left(\hat{\sigma}_X, \hat{\mu}_X\right), \tag{105}$$

which is equivalent to the covariance obtained from first-order Taylor series approximation. Finally, insert Equations 102 and 105 into Equation 92. ∎

Remark 4. Equation 101 is readily compared to the first-order Taylor series-based approximation previously obtained (Equation 78) by noting that, apart from the final two terms in Equation 101, $\text{var}\left(\hat{Z}\right)\text{var}\left(\hat{\sigma}_X\right)$ and $\text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right)$, the equation is the same as approximate variance based on first-order Taylor series. For large samples, $\sigma_X^2 \gg \text{var}\left(\hat{\sigma}_X\right)$, so $\text{var}\left(\hat{Z}\right)\text{var}\left(\hat{\sigma}_X\right)$ is expected to be small relative to $\sigma_X^2\text{var}\left(\hat{Z}\right)$. For example, for every scale in the 2017 NAEP reading and mathematics assessments, σ_X^2 was estimated to be over 10,000 times larger than the estimate of $\text{var}\left(\hat{\sigma}_X\right)$. Similarly, for large samples, $2\sigma_X Z \gg \text{covar}\left(\hat{\sigma}_X, \hat{Z}\right)$, so $\text{covar}^2\left(\hat{\sigma}_X, \hat{Z}\right)$ is expected to be small relative to $2\sigma_X Z\text{covar}\left(\hat{\sigma}_X, \hat{Z}\right)$. This observation confirms the accuracy of the Taylor series-based approximation.

Type A Estimators

Theorem 11. *In general, the exact variance of a type A estimator is*

$$\begin{aligned}\text{var}\left(\hat{Y}_\text{A}\right) &= \sigma_X^2\text{var}\left(\hat{R}\right) + R^2\text{var}\left(\hat{\sigma}_X\right) + \text{var}\left(\hat{R}\right)\text{var}\left(\hat{\sigma}_X\right)\\ &\quad + \text{covar}\left(\hat{\sigma}_X^2, \hat{R}^2\right) - \text{covar}^2\left(\hat{\sigma}_X, \hat{R}\right) - 2\sigma_X R\text{covar}\left(\hat{\sigma}_X, \hat{R}\right). \end{aligned} \tag{106}$$

Proof. Starting with Equation 11 and inserting Equations 8 and 37,

$$\hat{Y}_\text{A} = \hat{a}\hat{\theta}_\text{A} + \hat{b} \tag{107}$$

$$= \hat{\sigma}_X\left(\hat{\sigma}_\theta^{-1}\hat{\theta}_\text{AB}\right) \tag{108}$$

$$= \hat{\sigma}_X\hat{R}. \tag{109}$$

It follows that

$$\text{var}\left(\hat{Y}_\text{A}\right) = \text{var}\left(\hat{\sigma}_X\hat{R}\right). \tag{110}$$

The proof then follows the same logic used to obtain $\text{var}\left(\hat{\sigma}_X\hat{Z}\right)$ in Theorem 9. ■

Remark 5. When the very small terms $\text{covar}^2\left(\hat{\sigma}_X, \hat{R}\right)$ and $\text{var}\left(\hat{R}\right)\text{var}\left(\hat{\sigma}_X\right)$ are ignored (similar to logic in Remark 4) and first-order Taylor series-based approximation of $\text{covar}\left(\hat{\sigma}_X^2, \hat{R}^2\right)$ at $\hat{\sigma}_X = \sigma_X$ and $\hat{R} = R$ is used, the variance of a type A estimator (Equation 106) becomes the first-order Taylor series approximation previously obtained (Equation 81).

Theorem 12. *Under bivariate normality of $\hat{\sigma}_X$ and $\hat{R}$, the exact variance of a type A estimator is*

$$\begin{aligned}\text{var}^\text{MVN}\left(\hat{Y}_\text{A}\right) &= \sigma_X^2\text{var}\left(\hat{R}\right) + R^2\text{var}\left(\hat{\sigma}_X\right) + 2\sigma_X R\text{covar}\left(\hat{\sigma}_X, \hat{R}\right)\\ &\quad + \text{var}\left(\hat{R}\right)\text{var}\left(\hat{\sigma}_X\right) + \text{covar}^2\left(\hat{\sigma}_X, \hat{R}\right). \end{aligned} \tag{111}$$

Proof. The proof follows the same logic used to obtain $\text{var}\left(\hat{\sigma}_X\hat{Z}\right)$ in Theorem 10. ■

Remark 6. Equation 111 is readily compared to the first-order Taylor series-based approximation previously obtained (Equation 81) by noting that the terms on the first line for Equation 111 give the first-order Taylor series-based approximation, while the terms on the second line do not appear in the Taylor series linearization approximation. The terms neglected in the Taylor series-based approximations of the variance of type A estimators are the same terms that are neglected by the first-order Taylor series-based approximation for type AB estimators, so the discussion for type AB estimators (Remark 4) is relevant here.

Remark 7. When the target and source samples are independent, assuming multivariate normality of the estimators is not necessary. Note that Equation 103 shows that $\text{covar}\left(\hat{\sigma}_X\hat{Z}, \hat{\mu}_X\right) = \mathbb{E}\left(\hat{Z}\right)\text{covar}\left(\hat{\sigma}_X, \hat{\mu}_X\right)$ when the samples are independent. Therefore, when the samples are independent,

$$\text{var}^\text{ind}\left(\hat{Y}_\text{AB}\right) = \sigma_X^2\text{var}\left(\hat{Z}\right) + Z^2\text{var}\left(\hat{\sigma}_X\right) + \text{var}\left(\hat{\mu}_X\right) + \text{var}\left(\hat{Z}\right)\text{var}\left(\hat{\sigma}_X\right) + 2Z\text{covar}\left(\hat{\sigma}_X, \hat{\mu}_X\right) \tag{112}$$

$$\text{var}^{\text{ind}}\left(\hat{Y}_{\text{A}}\right) = \sigma_X^2 \text{var}\left(\hat{R}\right) + R^2 \text{var}\left(\hat{\sigma}_X\right) + \text{var}\left(\hat{R}\right)\text{var}\left(\hat{\sigma}_X\right). \quad (113)$$

Effect of Linking on the Total Variance

Two corollaries may be especially salient to understanding the effect of linking on the total variance.

Corollary 1. *The variance of the estimator of the population mean from the source sample, posttransformation, is equal to the variance of the estimator of the population mean from the target sample,*

$$\text{var}\left(\hat{\mu}_Y\right) = \text{var}\left(\hat{\mu}_X\right). \quad (114)$$

Proof. Insert $Z = 0$ and $\text{var}\left(\hat{Z}\right) = 0$ into Equation 84. ■

Corollary 2. *The variance of the source sample population standard deviation estimator in the target metric is equal to the variance of the target sample population standard deviation estimator*:

$$\text{var}\left(\hat{\sigma}_Y\right) = \text{var}\left(\hat{\sigma}_X\right). \quad (115)$$

Proof. Insert $R = 1$ and $\text{var}\left(\hat{R}\right) = 0$ into Equation 106. ■

The results for the population mean and standard deviation estimators reflect that the transformation has the effect of fixing the source mean and standard deviation estimates to the target mean and standard deviation estimates, respectively.

More generally, the impact of linking on the variance may be understood by comparing the variance of $\hat{Y}$ accounting for variance of $\hat{a}$ and $\hat{b}$ with the variance of $\hat{Y}$ if $\hat{a}$ and $\hat{b}$ were known. If $\hat{a}$ and $\hat{b}$ were known exactly, the total variance would be the prelinking variance component. Therefore the difference between the total variance and the prelinking variance component describes how the linking affects the total variance of $\hat{Y}$.

Theorem 13. *For a type AB estimator, the effect of linking on the variance is approximately*

$$\begin{aligned}\text{var}^{\text{TS}}\left(\hat{Y}_{\text{AB}}\right) - \text{var}^{\text{TS}}\left(\hat{Y}_{\text{AB,prelink}}\right) &= Z^2\left[\text{var}\left(\hat{\sigma}_X\right) - a^2\text{var}\left(\hat{\sigma}_\theta\right)\right] \\ &+ Z\left[2\text{covar}\left(\hat{\mu}_X, \hat{\sigma}_X\right) - 2a^2\text{covar}\left(\hat{\mu}_\theta, \hat{\sigma}_\theta\right) + 2\sigma_X\text{covar}\left(\hat{Z}, \hat{\sigma}_X\right) - 2\sigma_X a\text{covar}\left(\hat{Z}, \hat{\sigma}_\theta\right)\right] \\ &+ \text{var}\left(\hat{\mu}_X\right) - a^2\text{var}\left(\hat{\mu}_\theta\right) + 2\sigma_X\text{covar}\left(\hat{Z}, \hat{\mu}_X\right) - 2\sigma_X a\text{covar}\left(\hat{Z}, \hat{\mu}_\theta\right). \quad (116)\end{aligned}$$

Proof. Equation 116 is obtained as the difference between the Taylor series-based approximations of the variance of $\hat{Y}_{\text{AB}}$ and the variance of $\hat{Y}_{\text{AB,prelink}}$ (Equations 78 and 57, respectively). The Taylor series-based approximations were used to simplify the expression by neglecting small terms, but the exact variance (Equation 84) may be used to obtain the exact difference. ■

Theorem 14. *For a type A estimator, the effect of linking on the variance is approximately*

$$\begin{aligned}\text{var}^{\text{TS}}\left(\hat{Y}_{\text{A}}\right) - \text{var}^{\text{TS}}\left(\hat{Y}_{\text{A,prelink}}\right) &= R^2\left[\text{var}\left(\hat{\sigma}_X\right) - a^2\text{var}\left(\hat{\sigma}_\theta\right)\right] \\ &+ R\left[2\sigma_X\text{covar}\left(\hat{\sigma}_X, \hat{R}\right) - 2a\sigma_X\text{covar}\left(\hat{\sigma}_\theta, \hat{R}\right)\right]. \quad (117)\end{aligned}$$

Proof. Equation 117 is obtained as the difference between the Taylor series-based approximations of the variance of $\hat{Y}_{\text{A}}$ and the variance of $\hat{Y}_{\text{A,prelink}}$ (Equations 81 and 63, respectively). The Taylor series-based approximations were used to simplify the expression by neglecting small terms, but the exact variance (Equation 106) may be used to obtain the exact difference. ■

Equations 116 and 117 show that the mechanism of the linking transformation can be thought of as decomposing the variance of the source sample estimators into (co)variance associated with the source mean, standard deviation, and Z-score estimators and replacing the (co)variance associated with the source mean and standard deviation estimators with (co)variance of the corresponding estimators from the target sample. This corresponds to the transformation of the scores (Equation 7), by which the mean and standard deviation estimates from the source sample are fixed to the corresponding estimates from the target sample, while the source sample Z-scores remain invariant to the transformation.

In practice, the largest terms in Equations 116 and 117 may often be the variance terms. Considering only the variance terms and neglecting the covariance terms, when the effective sample size of the target sample is smaller than the effective sample size of the source sample, the variances of the target sample mean and standard deviation estimators are expected to be larger than the corresponding variances for the source sample estimators, and the estimated variance of type AB and type A estimators are expected to be larger than their prelinking variance components alone. In contrast, if the effective sample size of the target sample is larger, the estimated variances of type AB and type A estimators are expected to be smaller than their prelinking variance components alone.

The result that accounting for variances of the linking coefficients ($\widehat{a}$ and $\widehat{b}$) may reduce the variance of estimators may be initially counterintuitive. However, this can be understood from the perspective that the linking transformation both adds and removes sources of variance (Equations 116 and 117). Alternatively, this may be understood from the fact that the linking and prelinking covariance is expected to be negative (Equations 69 and 74, respectively).

Variances in Practice

Up until this point, I have described the variance of estimators from the source sample posttransformation, $\widehat{Y}$. However, in practice, variances of many other types of estimators are required: Scores are usually not used alone but compared to some other score. A subtle point is that linking affects not only the variance of an estimator but also the correlation of the estimator with other random variables. For this reason, when $\widehat{Y}$ is used with other random variables to calculate some estimator, the effects of linking on the correlations between the random variables must be taken into consideration beyond simply the correct variance of $\widehat{Y}$ that accounts for the linking.

Linking does not affect the correlation of estimators from the target and source samples and estimators from previous assessments, which remain independent, so the variance equations described earlier can be used to estimate differences between assessments. For example, the variance of the difference between performance of a particular subgroup in 2017 NAEP and 2015 NAEP can be estimated simply by assuming that the corresponding samples are independent, after calculating the correct variance of the subgroup in 2017 NAEP that accounts for linking.

Linking does affect the correlation of estimators within the source sample and between the source sample and target samples. This leads to three distinct types of aggregated estimators that are seen in practice and are affected by linking: differences between two estimators within the source sample, differences between two estimators between the source and target samples, and estimators obtained from combining data from the source and target samples. Examples of each of the three from NAEP are provided in the following pages.

Estimators of Differences Based on Two Estimators From the Source Sample

Estimators of differences of two source sample parameters include estimators of differences between demographic groups, such as the difference between the subgroup mean of boys and the subgroup mean of girls. Notably, the point estimate of a difference of two source sample parameters may be obtained with or without treating $\widehat{a}$ and $\widehat{b}$ as known. Effectively, the difference may be interpreted as being either in the original or target metrics. However, while the point estimate will be the same in both approaches, the variance will not be the same.

Theorem 15. *For estimators of differences between two type AB parameters, accounting for variance of $\widehat{a}$ and $\widehat{b}$, the variance can be approximated as*

$$\begin{aligned}\text{var}^{\text{TS}}\left(\widehat{Y}_{\text{AB},1-2}\right) &= \sigma_X^2\left[\text{var}\left(\widehat{Z}_1\right)+\text{var}\left(\widehat{Z}_2\right)-2\text{covar}\left(\widehat{Z}_1,\widehat{Z}_2\right)\right]\\ &\quad+\left(Z_1-Z_2\right)^2\text{var}\left(\widehat{\sigma}_X\right)+2\sigma_X\left(Z_1-Z_2\right)\left[\text{covar}\left(\widehat{\sigma}_X,\widehat{Z}_1\right)-\text{covar}\left(\widehat{\sigma}_X,\widehat{Z}_2\right)\right].\end{aligned} \tag{118}$$

Proof. With Equations 8, 10, and 36,

$$\widehat{Y}_{\text{AB},1-2}=\widehat{Y}_{\text{AB1}}-\widehat{Y}_{\text{AB2}} \tag{119}$$

$$= \hat{a}\left(\hat{\theta}_{\text{AB1}} - \hat{\theta}_{\text{AB2}}\right) \tag{120}$$

$$= \hat{\sigma}_X\left(\hat{Z}_1 - \hat{Z}_2\right). \tag{121}$$

First-order Taylor series was used to simplify the expression by neglecting small terms, but the exact variance of the difference estimator could be obtained with similar logic as used in the Exact Total Variance section. The variance of $\hat{Y}_{\text{AB},1-2}$ is approximated by the variance of the first-order Taylor series of Equation 121 at $\left[\hat{\sigma}_X, \hat{Z}_1, \hat{Z}_2\right] = \left[\sigma_X, Z_1, Z_2\right]$:

$$\text{var}\left(\hat{Y}_{\text{AB},1-2}\right) \approx \text{var}^{\text{TS}}\left(\hat{Y}_{\text{AB},1-2}\right) \tag{122}$$

$$= \nabla^T \hat{Y}_{\text{AB},1-2}\left(\sigma_X, Z_1, Z_2\right) \cdot \mathbf{\Sigma} \cdot \nabla \hat{Y}_{\text{AB},1-2}\left(\sigma_X, Z_1, Z_2\right), \tag{123}$$

where

$$\nabla \hat{Y}_{\text{AB},1-2}\left(\sigma_X, Z_1, Z_2\right) = \begin{bmatrix} \frac{\delta}{\delta\hat{\sigma}_X} \\ \frac{\delta}{\delta\hat{Z}_1} \\ \frac{\delta}{\delta\hat{Z}_2} \end{bmatrix} \hat{Y}_{\text{AB},1-2}\left(\sigma_X, Z_1, Z_2\right) \tag{124}$$

$$= \begin{bmatrix} Z_1 - Z_2 & \sigma_X & \sigma_X \end{bmatrix}^T \tag{125}$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} \text{var}\left(\hat{\sigma}_X\right) & \text{covar}\left(\hat{\sigma}_X, \hat{Z}_1\right) & \text{covar}\left(\hat{\sigma}_X, \hat{Z}_2\right) \\ \text{covar}\left(\hat{Z}_1, \hat{\sigma}_X\right) & \text{var}\left(\hat{Z}_1\right) & \text{covar}\left(\hat{Z}_1, \hat{Z}_2\right) \\ \text{covar}\left(\hat{Z}_2, \hat{\sigma}_X\right) & \text{covar}\left(\hat{Z}_2, \hat{Z}_1\right) & \text{var}\left(\hat{Z}_2\right) \end{bmatrix}. \tag{126}$$

■

Theorem 16. *For estimators of differences of two type AB parameters, without accounting for variance of $\hat{a}$ and $\hat{b}$, the variance can be approximated as*

$$\begin{aligned} \text{var}^{\text{TS}}_{\text{naive}}\left(\hat{Y}_{\text{AB},1\text{-}2}\right) = {} & \sigma_X^2\left[\text{var}\left(\hat{Z}_1\right) + \text{var}\left(\hat{Z}_2\right) - 2\text{covar}\left(\hat{Z}_1, \hat{Z}_2\right)\right] \\ & + \left(Z_1 - Z_2\right)^2 a^2 \text{var}\left(\hat{\sigma}_\theta\right) + 2\sigma_X a\left(Z_1 - Z_2\right)\left[\text{covar}\left(\hat{\sigma}_\theta, \hat{Z}_1\right) - \text{covar}\left(\hat{\sigma}_\theta, \hat{Z}_2\right)\right]. \end{aligned} \tag{127}$$

Proof. The proof proceeds differently than the proof for the previous theorem because $\hat{a}$ is assumed to be known. With Equations 10 and 36,

$$\hat{Y}_{\text{AB},1-2} = \hat{Y}_{\text{AB1}} - \hat{Y}_{\text{AB2}} \tag{128}$$

$$= a\left(\hat{\theta}_1 - \hat{\theta}_2\right) \tag{129}$$

$$= a\hat{\sigma}_\theta\left(\hat{Z}_1 - \hat{Z}_2\right). \tag{130}$$

First-order Taylor series was used to simplify the expression by neglecting small terms, but the exact variance of the difference could be obtained with similar logic as used in the Exact Total Variance section. The variance of $\hat{Y}_1 - \hat{Y}_2$, without accounting for the variance of $\hat{a}$ and $\hat{b}$, is approximated by the variance of the first-order Taylor series of Equation 130 at $\left[\hat{\sigma}_\theta, \hat{Z}_1, \hat{Z}_2\right] = \left[\sigma_\theta, Z_1, Z_2\right]$:

$$\text{var}\left(\hat{Y}_{\text{AB},1-2}\right) \approx \text{var}^{\text{TS}}\left(\hat{Y}_{\text{AB},1-2}\right) \tag{131}$$

$$= \nabla^T \hat{Y}_{\text{AB},1-2}\left(\sigma_\theta, Z_1, Z_2\right) \cdot \mathbf{\Sigma} \cdot \nabla \hat{Y}_{\text{AB},1-2}\left(\sigma_\theta, Z_1, Z_2\right), \tag{132}$$

where

$$\nabla \widehat{Y}_{\mathrm{AB},1-2}\left(\sigma_{\theta}, Z_1, Z_2\right) = \begin{bmatrix} \frac{\delta}{\delta \widehat{\sigma}_{\theta}} \\ \frac{\delta}{\delta \widehat{Z}_1} \\ \frac{\delta}{\delta \widehat{Z}_2} \end{bmatrix} \widehat{Y}_{\mathrm{AB},1-2}\left(\sigma_{\theta}, Z_1, Z_2\right) \tag{133}$$

$$= \left[a\left(Z_1 - Z_2\right) \quad \sigma_X \quad \sigma_X\right]^T \tag{134}$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{var}\left(\widehat{\sigma}_{\theta}\right) & \mathrm{covar}\left(\widehat{\sigma}_{\theta}, \widehat{Z}_1\right) & \mathrm{covar}\left(\widehat{\sigma}_{\theta}, \widehat{Z}_2\right) \\ \mathrm{covar}\left(\widehat{Z}_1, \widehat{\sigma}_{\theta}\right) & \mathrm{var}\left(\widehat{Z}_1\right) & \mathrm{covar}\left(\widehat{Z}_1, \widehat{Z}_2\right) \\ \mathrm{covar}\left(\widehat{Z}_2, \widehat{\sigma}_{\theta}\right) & \mathrm{covar}\left(\widehat{Z}_2, \widehat{Z}_1\right) & \mathrm{var}\left(\widehat{Z}_2\right) \end{bmatrix}. \tag{135}$$

■

Remark 8. The proof for type A estimators proceeds in the same way as for type AB estimators. For corresponding equations for type A estimators, simply replace each instance of $\widehat{Z}$ with $\widehat{R}$ in Equation 118 or Equation 127. The equations for type A and AB estimators have the same form because the difference between type AB estimators and the difference between type A estimators have the same form.

Both equations have similar forms, in which the total variance of the difference can be construed as a sum of a component involving the variance of the standardized score difference and a component associated with the metric and of covariances between the two components.

Most commonly in educational assessment, the score $\widehat{\theta}$ is linearly indeterminate. This is true for item response theory-based scores (de Ayala, 2009). In this case, the transformation can be interpreted as an arbitrary rescaling of the scores, where $\widehat{a}$ and $\widehat{b}$ are treated as unitless and known. The prelinking variance component can be treated as the total variance to give a valid estimate of the difference pretransformation, in the original metric, but arbitrarily rescaled with known and unitless linear coefficients.

However, if the difference is to be interpreted in the target metric, the variance of the linking coefficients should be accounted for. In this case, $\widehat{a}$ and $\widehat{b}$ are not unitless and are not known. While the variance from the Z-scores is identical in both cases, the variance associated with the metric is different.

Although ignoring the linking variance is valid when comparing two estimates from the source sample, ignoring the linking variance is not valid when comparing an estimate from the source sample and an estimate in the target metric, such as an estimate from the target sample or an estimate from the prior assessment that is also in the target metric. A comparison such as a difference between two values with different metrics is invalid. Therefore the source sample estimates must be interpreted in the target metric to compare to the target sample estimates, and the linking variance must be considered.

Estimating Differences Between Target and Source Parameters

To ensure that the linking transformation is appropriate for all subgroups, the differences between the results from the two samples may be evaluated for practical significance. For example, we may wish to evaluate whether student groups based on demographics have the same performance on the previous and new assessments (e.g., the digitally based vs. paper-based 2017 NAEP mode study; Jewsbury et al., 2019). Notably, the linking transformation induces a dependency between the estimators from the source sample and estimators from the target sample, because estimators from both samples depend on the population mean and standard deviation estimators from the target sample.

Denote $\widehat{X}$ and $\widehat{Y}$ as estimators from the target and source samples of the same parameter, respectively, with subscripts AB or A as appropriate, depending on the type of estimator. Furthermore, denote

$$\widehat{Z}_X = \widehat{\sigma}_X^{-1}\left(\widehat{X}_{\mathrm{AB}} - \widehat{\mu}_X\right), \tag{136}$$

$$\widehat{Z}_{\theta} = \widehat{\sigma}_{\theta}^{-1}\left(\widehat{\theta}_{\mathrm{AB}} - \widehat{\mu}_{\theta}\right), \tag{137}$$

$$\hat{R}_X = \hat{\sigma}_X^{-1}\hat{X}_A, \tag{138}$$

$$\hat{R}_\theta = \hat{\sigma}_\theta^{-1}\hat{\theta}_A. \tag{139}$$

Theorem 17. *For a pair of type AB estimators from the two samples, the variance of a difference estimator can be approximated as*

$$\begin{aligned}\text{var}^{\text{TS}}\left(\hat{Y}_{\text{AB}} - \hat{X}_{\text{AB}}\right) = \sigma_X^2\left[\text{var}\left(\hat{Z}_\theta\right) + \text{var}\left(\hat{Z}_X\right) - 2\text{covar}\left(\hat{Z}_X, \hat{Z}_\theta\right)\right] \\ + \left(Z_\theta - Z_X\right)^2 \text{var}\left(\hat{\sigma}_X\right) + 2\sigma_X\left(Z_\theta - Z_X\right)\left[\text{covar}\left(\hat{\sigma}_\theta, \hat{Z}_1\right) - \text{covar}\left(\hat{\sigma}_X, \hat{Z}_X\right)\right].\end{aligned} \tag{140}$$

Proof. Using the defining equation for type AB estimators (Equation 10) and inserting the definitions of $\hat{a}$ and $\hat{b}$ (Equations 5 and 6),

$$\hat{Y}_{\text{AB}} - \hat{X}_{\text{AB}} = \hat{a}\hat{\theta}_{\text{AB}} + \hat{b} - \hat{X}_{\text{AB}} \tag{141}$$

$$= \hat{\sigma}_X\left(\hat{\sigma}_\theta^{-1}\hat{\theta}_{\text{AB}} - \hat{\mu}_\theta\right) + \hat{\mu}_X - \hat{X}_{\text{AB}}. \tag{142}$$

Inserting Equations 136 and 137,

$$\hat{Y}_{\text{AB}} - \hat{X}_{\text{AB}} = \left(\hat{\sigma}_X\hat{Z}_\theta + \hat{\mu}_X\right) - \left(\hat{\sigma}_X\hat{Z}_X + \hat{\mu}_X\right) \tag{143}$$

$$= \hat{\sigma}_X\left(\hat{Z}_\theta - \hat{Z}_X\right). \tag{144}$$

First-order Taylor series was used to simplify the expression by neglecting small terms, but the exact variance of the difference could be obtained with similar logic as used in the Exact Total Variance section. The variance of $\hat{Y}_1 - \hat{Y}_2$ is approximated by the variance of the first-order Taylor series of Equation 144 at $\left[\hat{\sigma}_X, \hat{Z}_1, \hat{Z}_2\right] = \left[\sigma_X, Z_1, Z_2\right]$. The estimator of a difference across the two samples (Equation 144) has the same form as the estimator of a difference within the source sample (Equation 121), so the proof in Theorem 15 applies here. ■

Remark 9. The proof for type A estimators proceeds in the same way as for type AB estimators. For corresponding equations for type A estimators, simply replace each instance of $\hat{Z}$ with $\hat{R}$ in Equation 140. The equations for type A and AB estimators have the same form because the difference between type AB estimators and the difference between type A estimators have the same form.

Comparing Equation 140 to Equation 118 shows that the variance of the estimator of the difference of target and source parameters has the same form as the estimator of the difference between two source parameters. Although the former compares estimates between the target and source samples and the latter compares estimates within the source sample, both the target and source samples have the same mean and standard deviation estimate because of the transformation, resulting in similar equations between and within the two samples.

Because the transformation effectively fixes the source sample mean and standard deviation estimate to the target sample mean and standard deviation estimate, the dependency of the scores from the two samples increases. Even if the test takers from the two samples were sampled independently, the scores from the two samples will be correlated after transformation. Consequently, comparisons between the two samples that properly account for the transformation-induced dependency between the two samples may have relatively high power.

Combining the Target and Source Samples

Posttransformation, scores from the target and source samples are in the same metric. Consequently, the two samples may be combined into one *combined* sample after the source sample scores have been transformed.

Estimators based on the combined sample are expected to be superior to estimators based on the source sample alone, as the combined sample will always be larger than the source sample alone. However, the transformation-induced dependency between the source and target samples means that the benefit from combining the two samples is less than what may be expected based on the dependency of the samples pretransformation, as shown in the following discussion.

The proofs in the present section begin with the combined sample estimator as a function of the source and target sample estimators. As the form of this relationship may not be the same for all type AB or type A estimators, all estimators of the same type cannot be considered together in this section. Instead, only estimators of means are considered. For example, an estimator of a mean from the combined sample can be expressed as

$$\widehat{\mu}_C = \left(\sum_i w_i\right)^{-1} \sum_i w_i \widehat{C}_i \tag{145}$$

$$= \left(\sum_i w_i\right)^{-1} \left(\sum_{i\in\mathcal{Y}} w_i \widehat{Y}_i + \sum_{i\in\mathcal{X}} w_i \widehat{X}_i\right), \tag{146}$$

where $\widehat{C}_i$ is a score for test taker i; $\widehat{X}_i$ is a score for test taker i, who is in the target sample; w_i is the weight for test taker i, $i \in \mathcal{Y}$ denotes that test taker i is in the source sample, and $i \in \mathcal{X}$ denotes that test taker i is in the target sample.

Theorem 18. *If the target and source samples are combined into one sample, the variance of an estimator of a mean based on the combined sample is*

$$\begin{aligned}\mathrm{var}\left(\widehat{\mu}_C\right) &= \sigma_X^2 \mathrm{var}\left(\widehat{Z}_C\right) + Z_C^2 \mathrm{var}\left(\widehat{\sigma}_X\right) + \mathrm{var}\left(\widehat{\mu}_X\right)\\ &+ \mathrm{var}\left(\widehat{Z}_C\right)\mathrm{var}\left(\widehat{\sigma}_X\right) + 2\mathrm{covar}\left(\widehat{\sigma}_X \widehat{Z}_C, \widehat{\mu}_X\right)\\ &+ \mathrm{covar}\left(\widehat{\sigma}_X^2, \widehat{Z}_C^2\right) - \mathrm{covar}^2\left(\widehat{\sigma}_X, \widehat{Z}_C\right) - 2\sigma_X Z_C \mathrm{covar}\left(\widehat{\sigma}_X, \widehat{Z}_C\right),\end{aligned} \tag{147}$$

where

$$\widehat{Z}_C = \left(\sum_i w_i\right)^{-1} \sum_i w_i \widehat{Z}_{Ci} \tag{148}$$

$$Z_C = \mathbb{E}\left[\widehat{Z}_C\right] \tag{149}$$

and where

$$\widehat{Z}_{Ci} = \begin{cases} \widehat{\sigma}_X^{-1}\left(\widehat{X}_i - \widehat{\mu}_X\right), & \text{if } i \in \mathcal{X}, \\ \widehat{\sigma}_\theta^{-1}\left(\widehat{\theta}_i - \widehat{\mu}_\theta\right), & \text{if } i \in \mathcal{Y}. \end{cases} \tag{150}$$

Proof. Defining Z-scores with respect to the source or target sample moment estimators,

$$\widehat{Z}_{\theta i} = \widehat{\sigma}_\theta^{-1}\left(\widehat{\theta}_i - \widehat{\mu}_\theta\right), \quad \text{for} \quad i \in \mathcal{Y}, \tag{151}$$

$$\widehat{Z}_{Xi} = \widehat{\sigma}_X^{-1}\left(\widehat{X}_i - \widehat{\mu}_X\right), \quad \text{for} \quad i \in \mathcal{X}. \tag{152}$$

Substituting the definitions of $\widehat{a}$ and $\widehat{b}$ (Equations 5 and 6, respectively) and Equation 151 into Equation 7 and rearranging Equation 152 gives

$$\widehat{Y}_i = \widehat{\sigma}_X \widehat{Z}_{\theta i} + \widehat{\mu}_X, \quad \text{for} \quad i \in \mathcal{Y}, \tag{153}$$

$$\widehat{X}_i = \widehat{\sigma}_X \widehat{Z}_{Xi} + \widehat{\mu}_X, \quad \text{for} \quad i \in \mathcal{X}. \tag{154}$$

Substituting Equations 153 and 154 into Equation 146 and reexpressing in terms of Equation 150 gives

$$\widehat{\mu}_C = \widehat{\sigma}_X \widehat{Z}_C + \widehat{\mu}_X. \tag{155}$$

As Equation 155 has the same form as Equation 10 for type AB estimators for the target sample alone, the exact variance of $\widehat{\mu}_C$ can be obtained by the same logic as the proof for Theorem 9. ■

In practice, this approach can be applied by exploiting the fact that the source sample Z-scores are invariant to linear transformation. As such, the transformation does not cause any change in the dependency between the Z-scores of the two samples, unlike the nonstandardized scores. If there is no dependency between the samples, there will be no dependency between the Z-scores. If there is sample dependency, such as due to common primary sampling units, the procedures that the testing program normally use to account for this dependency will apply to the Z-scores. In either case, the standard procedures the testing program normally uses to estimate the variance of scores can be used for $\widehat{Z}_C$.

For example, some NAEP studies involved sampling the same schools in the target and source samples, including the 2017 reading and mathematics transition to digitally based assessments (Jewsbury et al., 2019). Jackknife replicate weights were constructed that account for the school-level sampling dependency, which may be appropriately applied to estimate $\text{var}\left(\widehat{Z}_C\right)$. However, the same jackknife replicate weights would not appropriately account for nonstandardized statistics aggregated across the two samples, as the sample dependency induced by the transformation would not be accounted for properly.

Corollary 3. *If the target and source samples are combined into one sample, the variance of the population mean estimator from the combined sample will equal the variance of the population mean estimator from the target sample. Specifically*,

$$\text{var}\left(\widehat{\mu}_C\right) = \text{var}\left(\widehat{\mu}_X\right). \tag{156}$$

Proof. Insert $Z_C = 0$ and $\text{var}\left(\widehat{Z}_C\right) = 0$ into Equation 147. ■

This result, and Equation 147 more generally, shows that there is no advantage to combining the two samples in improving the estimator of the population mean. However, the variance of the estimator of the standardized score in the combined sample, $\widehat{Z}_C$, will have smaller variance than the estimator of the standardized score in the source sample alone, $\widehat{Z}$. For smaller subgroups, the standardized score estimator may contribute the majority of the variance of the subgroup mean estimator. Therefore combining the two samples is expected to have the most benefit to the estimators of means based on smaller subgroups and less benefit to the estimators of means based on very large subgroups.

Implementation With Resampling Methods

The equations presented throughout this report express variances of estimators posttransformation (usually, $\widehat{Y}$) as a linear function of specific variances and covariances (e.g., $\text{var}\left(\widehat{\sigma}_X\right)$), which is important for understanding the nature of the transformation on the variances. While the equations can be used to estimate variances in practice, a more general approach may be used to estimate the variance of any estimator posttransformation.

As I describe in the Variance Component Decomposition Approximation section, the variance decomposition of the total variance into conventional variance and linking variance is inherently approximate. Furthermore, the sum of the variance due to $\widehat{\theta}$ and the variance due to $\widehat{a}$ and $\widehat{b}$ does not closely approximate the variance for type AB and type A estimators, unless the source sample is very large, because the covariance of the two components needs to be considered as well. Therefore procedures that simultaneously estimate the variance due to both sources and naturally account for the covariance of $\widehat{\theta}$ with $\widehat{a}$ and $\widehat{b}$ have a significant practical advantage over procedures that estimate the variance components separately.

Resampling methods, including the bootstrap (Efron & Tibshirani, 1993) and the jackknife (Tukey, 1958), are commonly used for a wide range of applications to estimate variances. For example, NAEP uses a type of jackknife to estimate variances (Johnson & Rust, 1992). While applying a resampling method directly to the scores posttransformation will not account for variance of $\widehat{a}$ and $\widehat{b}$, a relatively simple modification to the resampling method to allow for $\widehat{a}$ and $\widehat{b}$ to vary with $\widehat{\theta}$ in the resamples can be used to directly obtain the total variance of $\widehat{Y}$ from both linking ($\widehat{a}$ and $\widehat{b}$) and prelinking ($\widehat{\theta}$) sources. The use of resampling to obtain the total variance may be referred to as *total resampling*.

Up until the present point in the report, the target and source samples have been assumed to be dependent, with the understanding that the covariance terms between estimates from different samples may simply be set to zero if the samples are independent. However, for the resampling methods considered here, a slightly different approach is recommended based on whether the two samples are independent or dependent.

Dependent Samples

Following the definition of type AB estimators (Equation 10), the total variance of a type AB estimator is

$$\mathrm{var}\left(\hat{Y}_{\mathrm{AB}}\right) = \mathrm{var}\left(\hat{a}\hat{\theta}_{\mathrm{AB}} + \hat{b}\right). \tag{157}$$

Following the definition of type A estimators (Equation 11), the total variance of a type A estimator is

$$\mathrm{var}\left(\hat{Y}_{\mathrm{A}}\right) = \mathrm{var}\left(\hat{a}\hat{\theta}_{\mathrm{A}}\right). \tag{158}$$

The total resampling method can be employed simply by using the posttransformation estimator (Equation 10 for type AB estimators or Equation 11 for type A estimators) in each resample, and otherwise following the standard resampling procedures to obtain the variance of the *total* estimator rather than just one component in the Taylor series of the estimator (see the Variance Component Decomposition Approximation section). Critically, all estimates (i.e., $\hat{a}$, $\hat{b}$, and $\hat{\theta}_{\mathrm{AB}}$ for type AB estimators or $\hat{a}$ and $\hat{\theta}_{\mathrm{A}}$ for type A estimators) must be reestimated in every resample, with $\hat{a}$ and $\hat{b}$ calculated from the mean and standard deviation estimates for that resample.

The total resampling method can be generalized for the case of any estimator, even estimators that do not conform to the type AB or type A definitions. The transformation is applied to the individual scores with Equation 7 within each resample, providing correct variances of individual scores. Correct variances of statistics of interest calculated via standard procedures from the individual scores therefore follow. For example, this procedure can be used to estimate the variance of achievement levels posttransformation, which was otherwise an intractable problem.

B. Liu and Mazzeo (2019) developed a total resampling method by extending the jackknife procedure. They applied this procedure to NAEP data, but the procedure would also apply to other assessments involving jackknife replicate weights.

Independent Samples

While the total resampling method described for dependent samples may be applied to independent samples, there is no need to account for dependency between the target and pretransformation source sample estimators, which are independent. Relatedly, applying the total resampling method described above requires pairing the resamples from the two samples to get multiple estimates of a and b, but the pairing is not meaningful.

Instead, the independency between the samples may be leveraged to obtain the variance contribution of each sample separately, which may then be combined. Following a similar logic to decompose the variance of an estimator posttransformation into linking and prelinking variance components, the variance for the independent samples case, $\mathrm{var}^{\mathrm{ind}}\left(\hat{Y}\right)$, may be approximated by a sum of variance from the target sample, $\mathrm{var}_X\left(\hat{Y}\right)$, and variance from the source sample, $\mathrm{var}_\theta\left(\hat{Y}\right)$, as

$$\mathrm{var}^{\mathrm{ind,RS}}\left(\hat{Y}\right) = \mathrm{var}_X\left(\hat{Y}\right) + \mathrm{var}_\theta\left(\hat{Y}\right) \approx \mathrm{var}^{\mathrm{ind}}\left(\hat{Y}\right), \tag{159}$$

where the superscript ind, RS denotes that the variance is for the independent samples resampling method. In the independent sample case, the covariance between the source and target sample components is zero.

Specifically, $\mathrm{var}_X\left(\hat{Y}\right)$ is the variance of $\hat{Y}$ assuming all source statistics are known, and $\mathrm{var}_\theta\left(\hat{Y}\right)$ is the variance of $\hat{Y}$ assuming all target statistics are known. For type AB estimators and type A estimators, respectively, the variance components may be expressed as

$$\mathrm{var}_\theta\left(\hat{Y}_{\mathrm{AB}}\right) = \mathrm{var}\left(\sigma_X\hat{\sigma}_\theta^{-1}\left(\hat{\theta}_{\mathrm{AB}} - \hat{\mu}_\theta\right) + \mu_X\right), \tag{160}$$

$$\mathrm{var}_X\left(\hat{Y}_{\mathrm{AB}}\right) = \mathrm{var}\left(\hat{\sigma}_X\sigma_\theta^{-1}\left(\theta_{\mathrm{AB}} - \mu_\theta\right) + \hat{\mu}_X\right), \tag{161}$$

$$\mathrm{var}_\theta\left(\hat{Y}_{\mathrm{A}}\right) = \mathrm{var}\left(\sigma_X\hat{\sigma}_\theta^{-1}\hat{\theta}_{\mathrm{A}}\right), \tag{162}$$

$$\mathrm{var}_X\left(\hat{Y}_{\mathrm{A}}\right) = \mathrm{var}\left(\hat{\sigma}_X\sigma_\theta^{-1}\theta_{\mathrm{A}}\right). \tag{163}$$

Theorem 19. *For type AB estimators, the difference between the variance of the total resampling method for independent samples and the exact variance when samples are independent is*

$$\mathrm{var}^{\mathrm{ind,RS}}\left(\hat{Y}_{\mathrm{AB}}\right) - \mathrm{var}^{\mathrm{ind}}\left(\hat{Y}_{\mathrm{AB}}\right) = \mathrm{var}\left(\hat{Z}\right)\mathrm{var}\left(\hat{\sigma}_X\right). \tag{164}$$

Proof. The variance from the source sample can be simplified by treating the target sample estimates as known:

$$\text{var}_\theta\left(\widehat{Y}_{\text{AB}}\right) = \text{var}\left[\sigma_X\widehat{\sigma}_\theta^{-1}\left(\widehat{\theta}_{\text{AB}} - \widehat{\mu}_\theta\right) + \mu_X\right] \tag{165}$$

$$= \text{var}\left(\sigma_X\widehat{Z} + \mu_X\right) \tag{166}$$

$$= \sigma_X^2\text{var}\left(\widehat{Z}\right). \tag{167}$$

Similarly, the variance from the target sample can be simplified by treating the source sample estimates as known:

$$\text{var}_X\left(\widehat{Y}_{\text{AB}}\right) = \text{var}\left[\widehat{\sigma}_X\sigma_\theta^{-1}\left(\theta_{\text{AB}} - \mu_\theta\right) + \widehat{\mu}_X\right] \tag{168}$$

$$= \text{var}\left(\widehat{\sigma}_X Z + \widehat{\mu}_X\right) \tag{169}$$

$$= Z^2\text{var}\left(\widehat{\sigma}_X\right) + \text{var}\left(\widehat{\mu}_X\right) + 2Z\text{covar}\left(\widehat{\sigma}_X, \widehat{\mu}_X\right). \tag{170}$$

Inserting Equations 167 and 170 into Equation 159 gives

$$\text{var}^{\text{ind,RS}}\left(\widehat{Y}_{\text{AB}}\right) = Z^2\text{var}\left(\widehat{\sigma}_X\right) + \text{var}\left(\widehat{\mu}_X\right) + 2Z\text{covar}\left(\widehat{\sigma}_X, \widehat{\mu}_X\right) + \sigma_X^2\text{var}\left(\widehat{Z}\right). \tag{171}$$

The exact variance of a type AB estimator when samples are independent can be obtained simply by setting any covariance of a pretransformed source sample estimator and a target sample estimator to zero in Equation 84, which gives

$$\text{var}^{\text{ind}}\left(\widehat{Y}_{\text{AB}}\right) = \sigma_X^2\text{var}\left(\widehat{Z}\right) + Z^2\text{var}\left(\widehat{\sigma}_X\right) + \text{var}\left(\widehat{\mu}_X\right) + 2Z\text{covar}\left(\widehat{\sigma}_X, \widehat{\mu}_X\right) + \text{var}\left(\widehat{Z}\right)\text{var}\left(\widehat{\sigma}_X\right). \tag{172}$$

Finally, take the difference between Equations 171 and 172. ■

Theorem 20. *For type A estimators, the difference between the variance of the total resampling method for independent samples and the exact variance when samples are independent is*

$$\text{var}^{\text{ind,RS}}\left(\widehat{Y}_{\text{A}}\right) - \text{var}^{\text{ind}}\left(\widehat{Y}_{\text{A}}\right) = \text{var}\left(\widehat{R}\right)\text{var}\left(\widehat{\sigma}_X\right). \tag{173}$$

Proof. The variance from the source sample can be simplified by treating the target sample estimates as known:

$$\text{var}_\theta\left(\widehat{Y}_{\text{A}}\right) = \text{var}\left(\sigma_X\widehat{\sigma}_\theta^{-1}\widehat{\theta}_{\text{A}}\right) \tag{174}$$

$$= \text{var}\left(\sigma_X\widehat{R}\right) \tag{175}$$

$$= \sigma_X^2\text{var}\left(\widehat{R}\right). \tag{176}$$

Similarly, the variance from the target sample can be simplified by treating the source sample estimates as known:

$$\text{var}_X\left(\widehat{Y}_{\text{A}}\right) = \text{var}\left(\widehat{\sigma}_X\sigma_\theta^{-1}\theta_{\text{A}}\right) \tag{177}$$

$$= \text{var}\left(\widehat{\sigma}_X R\right) \tag{178}$$

$$= R^2\text{var}\left(\widehat{\sigma}_X\right). \tag{179}$$

Inserting Equations 176 and 179 into Equation 159 gives

$$\text{var}^{\text{ind,RS}}\left(\widehat{Y}_{\text{A}}\right) = R^2\text{var}\left(\widehat{\sigma}_X\right) + \sigma_X^2\text{var}\left(\widehat{R}\right). \tag{180}$$

The exact variance when samples are independent can be obtained simply by setting any covariance of a pretransformed source sample estimator and a target sample estimator to zero in Equation 106, which gives

$$\text{var}^{\text{ind}}\left(\widehat{Y}_{\text{A}}\right) = \sigma_X^2\text{var}\left(\widehat{R}\right) + R^2\text{var}\left(\widehat{\sigma}_X\right) + \text{var}\left(\widehat{R}\right)\text{var}\left(\widehat{\sigma}_X\right). \tag{181}$$

Finally, take the difference between Equations 180 and 181. ■

Equations 164 and 173 show that the total resampling method for independent samples is not exact for types AB and A estimators. However, the term that is neglected by the total resampling method for independent samples was previously found to be negligible for large samples (Remark 4). Therefore the difference between the total resampling method for independent samples and the exact variance, for types AB and A estimators, is negligible for large samples.

When computational efficiency is an issue, the target and/or source sample components may be estimated with equations or precalculated. Most notably, the target sample component for types AB and A estimators are quadratic functions of Z and R, respectively (see Equations 167 and 176). The coefficients of the quadratic functions may be preestimated, and the function may be used for a given estimate of Z to obtain the target sample component without use of a resampling method in real time, with very little computation.

Differences and Combinations

Earlier theoretical results showed that the linking transformation affected not only the variance of estimators based on the source sample scores posttransformation but the correlations of estimators from the source sample with other estimators from both the target and source samples. This affects the variances of difference estimators within the source sample (e.g., the mean of boys vs. the mean of girls), variances of difference estimators between the target and source samples (e.g., the mean of boys who took the new assessment vs. the mean of boys who took the previous assessment), and variances of estimators from a combined target and source sample (e.g., the mean of boys aggregated across the results from both assessments).

To properly estimate the variance of the differences and combinations described previously, and other estimators not described, the total resampling method can be readily generalized. The same logic applies when the *total* statistic is estimated in every resample. For example, for the difference between two type AB or type A estimators from the source sample,

$$\mathrm{var}\left(\hat{Y}_1 - \hat{Y}_2\right) = \mathrm{var}\left(\hat{a}\left[\hat{\theta}_1 - \hat{\theta}_2\right]\right); \tag{182}$$

for the difference between a type AB estimator from the source sample and an estimator from the target sample,

$$\mathrm{var}\left(\hat{Y} - \hat{X}\right) = \mathrm{var}\left(\hat{a}\hat{\theta} + \hat{b} - \hat{X}\right). \tag{183}$$

Measurement Error

Until this point, the present report has considered only one source of variance. However, many assessments have two distinct sources of variance that can be considered separately. When the scores are not measured directly but rather estimated indirectly with a model such as item response theory, variance of the estimate is present due to the latency of the score (Mislevy, Johnson, & Muraki, 1992; von Davier, Sinharay, Oranje, & Beaton, 2006). In large-scale educational assessments, this variance is defined as *measurement variance*, distinct from *sampling variance*, from the sampling design. The measurement variance, $\mathrm{var}_{\mathrm{meas}}$, is typically assumed to be independent of sampling variance, $\mathrm{var}_{\mathrm{samp}}$ (von Davier et al., 2006); that is,

$$\mathrm{var}\left(\hat{Y}\right) \approx \mathrm{var}_{\mathrm{samp}}\left(\hat{Y}\right) + \mathrm{var}_{\mathrm{meas}}\left(\hat{Y}\right). \tag{184}$$

Each of the sampling and measurement variance components can be obtained with any of the procedures described above, and then combined with Equation 184. For example, the exact variance equation for type A estimators (Equation 106) can be used to obtain the two variance components separately:

$$\begin{aligned}\mathrm{var}_{\mathrm{samp}}\left(\hat{Y}_{\mathrm{A}}\right) = {} & \sigma_X^2 \mathrm{var}_{\mathrm{samp}}\left(\hat{R}\right) + R^2 \mathrm{var}_{\mathrm{samp}}\left(\hat{\sigma}_X\right) + \mathrm{var}_{\mathrm{samp}}\left(\hat{R}\right)\mathrm{var}_{\mathrm{samp}}\left(\hat{\sigma}_X\right) \\ & + \mathrm{covar}_{\mathrm{samp}}\left(\hat{\sigma}_X^2, \hat{R}^2\right) - \mathrm{covar}_{\mathrm{samp}}^2\left(\hat{\sigma}_X, \hat{R}\right) - 2\sigma_X R\,\mathrm{covar}_{\mathrm{samp}}\left(\hat{\sigma}_X, \hat{R}\right)\end{aligned} \tag{185}$$

and

$$\begin{aligned}\mathrm{var}_{\mathrm{meas}}\left(\hat{Y}_{\mathrm{A}}\right) = {} & \sigma_X^2 \mathrm{var}_{\mathrm{meas}}\left(\hat{R}\right) + R^2 \mathrm{var}_{\mathrm{meas}}\left(\hat{\sigma}_X\right) + \mathrm{var}_{\mathrm{meas}}\left(\hat{R}\right)\mathrm{var}_{\mathrm{meas}}\left(\hat{\sigma}_X\right) \\ & + \mathrm{covar}_{\mathrm{meas}}\left(\hat{\sigma}_X^2, \hat{R}^2\right) - \mathrm{covar}_{\mathrm{meas}}^2\left(\hat{\sigma}_X, \hat{R}\right) - 2\sigma_X R\,\mathrm{covar}_{\mathrm{meas}}\left(\hat{\sigma}_X, \hat{R}\right).\end{aligned} \tag{186}$$

Notably, even if the samples are dependent, the models are typically estimated independently. Consequently, applying the dependent sample total resampling method for the sampling variance combined with the independent sample total resampling method for the measurement variance may be most practical. Note that applying a resampling method for the measurement variance would involve sampling from posterior distributions (see B. Liu & Mazzeo, 2019).

Discussion

In the present report, I describe the variances of linearly transformed estimators in common population linking bridge studies. A distinction is made between type AB estimators, such as individual scores, means, and percentiles, in which variances of both $\widehat{a}$ and $\widehat{b}$ may impact the total variance of the estimator, and type A estimators, such as differences or gaps and standard deviations, in which only the variance of $\widehat{a}$ may impact the total variance of the estimator.

For type AB and type A estimators, both exact and approximate equations for the variance components and exact equations for the total variance are derived and discussed. Because linking affects not only the variance of an estimator but also the covariance of the estimator with other estimators from either the target or source samples, equations for the variance of (a) estimated differences between source parameters, (b) estimated differences between source and target parameters, and (c) statistics based on a combined sample obtained by aggregating results from the target and source samples were derived and discussed. Finally, a general method to estimate variances of estimators posttransformation is recommended.

For type AB estimators, the contribution of estimators from the two samples to the Taylor series-based equation for the linking variance component is symmetrical, comprising the sum of the variances of the target and source population mean estimators (for type AB estimators), the sum of the variances of the target and source population standard deviation estimators proportional to the standardized score squared, and covariance terms (see Equations 42 and 48). The mutual dependency of both the linking and prelinking variance components on the variances of the source sample mean and standard deviation estimators falls out when combined with the linking – prelinking covariance, meaning that, surprisingly, uncertainty of source sample estimates after transformation does not depend on uncertainty of the source sample mean and standard deviation estimates (see Equation 84). For type A estimators, the same is true, except the terms with the mean estimators are not present in the equations (see Equation 106).

The findings that the terms involving the source mean and standard deviation fall out in the total variance of estimators posttransformation reflects the fact that the mean estimator from the target sample (for type AB estimators), the standard deviation estimator from the target sample, and the standardized score from the source sample are sufficient for type AB and type A estimators posttransformation. The derivations for the exact variance began with the posttransformed estimator as functions of these variables. This parameterization is especially helpful because the only source sample estimator in the resulting equations is invariant to the transformation, which may be estimated without concern about the effects of the transformation.

When a source sample score is understood as a function of the source sample mean estimate, the source sample standard deviation estimate, and a standardized score estimate (see Equations 36 and 63), the effect of linking is readily understood. Essentially, the effect of linking is to swap the source sample mean and standard deviation estimates with the target sample mean and standard deviation estimates while the standardized score estimate remains unchanged. Correspondingly, the effect of linking on the variance of the individual score is the swapping of the (co)variances of the source sample mean and standard deviation estimators with the (co)variances of the target sample mean and standard deviation estimators (see Equations 116 and 117).

The swapping of (co)variance terms impacts not only the variance of estimators but also the covariance of estimators with other estimators. First, the variance for a comparison of two scores from the source sample posttransformation has the same form as the variance for a comparison of two scores from the target sample (see Equation 118). Second, because the scores from both samples posttransformation have the same mean and standard deviation estimates, the positive dependency between scores from the two samples is increased. With increased positive dependency, tests for differences between source and target parameters become more powerful (see Equation 140). Third, the increased positive dependency between the samples results in less efficiency gain from combining the two samples than may be expected (see Equation 147).

In general, the relative difference between the (co)variances of an estimator posttransformation and an estimator pretransformation depends on the extent to which the variance of the estimator depends on the variances of the population mean and standard deviation estimators. For estimators based on small subgroups, the variance of population mean and standard deviation estimators may be of relatively small importance. Consequently, the effect of linking may be relatively small. In contrast, estimators based on larger subgroups may see larger relative changes when linking is properly accounted for — or when linking is improperly accounted for.

The variance component decomposition approach into linking and prelinking variance components was evaluated. Although the approach is very intuitive, involving the conventional variance (prelinking variance component) and an additional source of variance associated with uncertainty of the transformation coefficients, the approach has some limitations. The approach is inherently approximate but can be shown to arise from two assumptions described in the Variance Component Decomposition Approximation section. While these assumptions appear to be highly approximated for types AB and A estimators, such as means and standard deviations, whether the assumptions are appropriate for other types of estimators, such as achievement level estimators, is less clear. Furthermore, the procedure may substantially overestimate the total variance if the covariance of the two components is not taken into account, unless the source sample is very large.

The total resampling procedures are based on standard resampling procedures, such as the bootstrap (Efron & Tibshirani, 1993) and the jackknife (Tukey, 1958), but modified to simultaneously account for variance of both sources: prelinking variance (i.e., variance due to $\widehat{Y}$) and linking variance (i.e., variance due to $\widehat{a}$ and $\widehat{b}$). By simultaneously accounting for all sources, the approximation inherent in the variance component decomposition is avoided, and the covariances between all sources are naturally accounted for. Furthermore, the procedure is readily extended to properly account for how the transformation impacts the correlation of the estimators with each other and other estimators.

The effect of linking on the variances should be considered in the design of the bridge study. The relative size of the two samples is the major factor in determining whether the linking causes the variance of source sample estimators to increase or decrease, and by how much (see Equations 116 and 117). In particular, the mean and standard deviation of the transformed scores have the same variance as the target sample mean and standard deviation, regardless of whether the source sample scores are used alone or combined with the target sample scores (see Equations 114, 115, and 156). For educational assessments that track changes over time, the target sample should be large enough to obtain satisfactory population mean and standard deviation estimates, and the source sample should be large enough to obtain satisfactory standardized score estimates for subgroups of interest.

Accurate estimation of errors is critical for the validity of inferences made on the basis of the results. Conventional procedures to estimate errors are not valid for scores that have been transformed in a bridge study. The results in this report show that the effects of the linking transformation on the variance of estimators have consequences for understanding the results, designing the bridge study, and selecting or developing a valid error variance estimation procedure.

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Appendix

AB Estimators

Lemma 1. *An individual score is a type AB estimator*.

Proof. Equation 7 has the same form as Equation 10. ■

Lemma 2. *A sample mean is a type AB estimator*.

Proof. The posttransformation sample mean for group g comprising all test takers within the set G is defined as

$$\widehat{\mu}_g = \left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \widehat{Y}_i, \tag{A1}$$

where w_i is the weight for student i. Substituting Equation 7 into Equation A1 and simplifying gives

$$\widehat{\mu}_g = \left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \left[\widehat{a}\widehat{\theta}_i + \widehat{b}\right] \tag{A2}$$

$$= \widehat{b} + \left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \widehat{a}\widehat{\theta}_i \tag{A3}$$

$$= \widehat{b} + \widehat{a}\left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \widehat{\theta}_i \tag{A4}$$

$$= \widehat{b} + \widehat{a}\widehat{\mu}_{g'}, \tag{A5}$$

where $\hat{\mu}_{g'}$ is the pretransformation subgroup mean for group g,

$$\hat{\mu}_{g'} = \left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \hat{\theta}_i. \tag{A6}$$

Equation A5 has the same form as Equation 10. ■

Lemma 3. *A sample percentile can be defined as a type AB estimator*.

Proof. If the estimated value at the *p*th percentile is defined as the mean for all test takers at the *p*th percentile, the *p*th percentile is a group mean. The transformation does not change the rank order of scores, so a given percentile comprises the same test takers both pre- and posttransformation. Therefore Lemma 2 for group means applies to percentiles. ■

Type A Estimators

Lemma 4. *A sample standard deviation is a type A estimator*.

Proof. The posttransformation sample standard deviation for group g comprising all test takers within the set G is defined as

$$\hat{\sigma}_g = \sqrt{\left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \left(\hat{Y}_i - \hat{\mu}_Y\right)^2}. \tag{A7}$$

Substituting Equation 7 and the definition of a mean and simplifying,

$$\hat{\sigma}_g = \sqrt{\left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \left(\hat{a}\hat{\theta}_i + \hat{b} - \sum_i w_i \left(\hat{a}\hat{\theta}_i + \hat{b}\right)\right)^2} \tag{A8}$$

$$= \sqrt{\left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \left(\hat{a}\hat{\theta}_i - \sum_i w_i \hat{a}\hat{\theta}_i\right)^2} \tag{A9}$$

$$= \hat{a}\sqrt{\left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \left(\hat{\theta}_i - \hat{\mu}_\theta\right)^2} \tag{A10}$$

$$= \hat{a}\,\hat{\sigma}_{g'}, \tag{A11}$$

where $\hat{\sigma}_{g'}$ is the pretransformation sample mean for group g,

$$\hat{\sigma}_{g'} = \sqrt{\left(\sum_{i \in G} w_i\right)^{-1} \sum_{i \in G} w_i \left(\hat{\theta}_i - \hat{\mu}_\theta\right)^2}. \tag{A12}$$

Note that $\sum_i w_i = 1$. Equation A11 has the same form as Equation 11. ■

Lemma 5. *A difference between two type AB estimators is a type A estimator*.

Proof. By the definition of a type AB estimator (Equation 10), the difference between two type AB estimators posttransformation is related to the corresponding pretransformed estimator as

$$\hat{Y}_{\text{AB1}} - \hat{Y}_{\text{AB2}} = \hat{a}\hat{\theta}_{\text{AB1}} + \hat{b} - \left(\hat{a}\hat{\theta}_{\text{AB2}} + \hat{b}\right) \tag{A13}$$

$$= \hat{a}\left(\hat{\theta}_{\text{AB1}} - \hat{\theta}_{\text{AB2}}\right). \tag{A14}$$

■

Lemma 6. *A difference between two type A estimators is a type A estimator*.

Proof. By the definition of a type A estimator (Equation 11), the difference between two type A estimators posttransformation is related to the corresponding pretransformation estimator as

$$\widehat{Y}_{A1} - \widehat{Y}_{A2} = \widehat{a}\widehat{\theta}_{A1} - \widehat{a}\widehat{\theta}_{A2} \tag{A15}$$

$$= \widehat{a}\left(\widehat{\theta}_{A1} - \widehat{\theta}_{A2}\right). \tag{A16}$$

■

First-Order Taylor Series–Based Variance Approximation

First-order Taylor series-based variance approximation is a common method for approximating variances and can provide excellent approximations for variances of linear functions (Binder, 1983). Denote $\widehat{\mathbf{\Phi}} = \left[\widehat{\phi}_1, \ldots, \widehat{\phi}_N\right]$ as an N-length vector of estimators and $\mathbf{\Phi} = [\phi_1, \ldots, \phi_N]$ as an N-length vector of the corresponding estimands. In the present report, the Taylor series-based approximation to the variance of a function of estimators, $f\left(\widehat{\mathbf{\Phi}}\right)$, is based on its Taylor series at $\widehat{\mathbf{\Phi}} = \mathbf{\Phi}$. Specifically,

$$f\left(\widehat{\mathbf{\Phi}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\widehat{\mathbf{\Phi}} - \mathbf{\Phi}\right) \cdot \nabla_{\widehat{\mathbf{\Phi}}}\right]^n f(\mathbf{\Phi}), \tag{A17}$$

where

$$\nabla_{\widehat{\mathbf{\Phi}}} = \frac{d}{d\widehat{\mathbf{\Phi}}} = \begin{bmatrix} \frac{\delta}{\delta\widehat{\phi}_1} \\ \vdots \\ \frac{\delta}{\delta\widehat{\phi}_N} \end{bmatrix}. \tag{A18}$$

First-order Taylor series are based only on the zeroth- and first-order terms ($n = 0, 1$); that is,

$$f\left(\widehat{\mathbf{\Phi}}\right) \approx \sum_{n=0}^{1} \frac{1}{n!} \left[\left(\widehat{\mathbf{\Phi}} - \mathbf{\Phi}\right) \cdot \nabla_{\mathbf{\Phi}}\right]^n f(\mathbf{\Phi}) \tag{A19}$$

$$= f(\mathbf{\Phi}) + \left(\widehat{\mathbf{\Phi}} - \mathbf{\Phi}\right) \cdot \nabla_{\mathbf{\Phi}} f(\mathbf{\Phi}). \tag{A20}$$

As first-order Taylor series are simply a sum of random variables, deriving the variance of the first-order Taylor series is straightforward and may be much more tractable than the original function. The first-order Taylor series-based approximation to the variance is the variance of the first-order Taylor series, which in general is

$$\text{var}\left[f\left(\widehat{\mathbf{\Phi}}\right)\right] \approx \text{var}^{\text{TS}}\left[f\left(\widehat{\mathbf{\Phi}}\right)\right] \tag{A21}$$

$$= \text{var}\left[f(\mathbf{\Phi}) + \left(\widehat{\mathbf{\Phi}} - \mathbf{\Phi}\right) \cdot \nabla_{\widehat{\mathbf{\Phi}}} f(\mathbf{\Phi})\right] \tag{A22}$$

$$= \text{var}\left[\widehat{\mathbf{\Phi}} \cdot \nabla_{\widehat{\mathbf{\Phi}}} f(\mathbf{\Phi})\right] \tag{A23}$$

$$= \text{var}\left[\sum_{i}^{N} \frac{\delta f(\mathbf{\Phi})}{\delta\widehat{\phi}_i} \widehat{\phi}_i\right] \tag{A24}$$

$$= \sum_{i}^{N} \sum_{j}^{N} \frac{\delta f(\mathbf{\Phi})}{\delta\widehat{\phi}_i} \frac{\delta f(\mathbf{\Phi})}{\delta\widehat{\phi}_j} \text{covar}\left(\widehat{\phi}_i, \widehat{\phi}_j\right). \tag{A25}$$

Finally, in matrix notation,

$$\text{var}\left[f\left(\widehat{\mathbf{\Phi}}\right)\right] \approx \text{var}^{\text{TS}}\left[f\left(\widehat{\mathbf{\Phi}}\right)\right] = \nabla^T_{\widehat{\mathbf{\Phi}}} f(\mathbf{\Phi}) \cdot \mathbf{\Sigma}_{\widehat{\mathbf{\Phi}}} \cdot \nabla_{\widehat{\mathbf{\Phi}}} f(\mathbf{\Phi}), \tag{A26}$$

where

$$\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\Phi}}} = \begin{bmatrix} \text{covar}\left(\widehat{\phi}_1, \widehat{\phi}_1\right) & \cdots & \text{covar}\left(\widehat{\phi}_1, \widehat{\phi}_N\right) \\ \vdots & \ddots & \vdots \\ \text{covar}\left(\widehat{\phi}_N, \widehat{\phi}_1\right) & \cdots & \text{covar}\left(\widehat{\phi}_N, \widehat{\phi}_N\right) \end{bmatrix}. \tag{A27}$$

First-order Taylor series approximation may also be used to approximate the covariance of two functions. Denote $\widehat{\boldsymbol{\Phi}}_i = \left[\widehat{\phi}_{i1}, \ldots, \widehat{\phi}_{iN_i}\right]$ as an N_i-length vector of estimators and $\boldsymbol{\Phi}_i = \left[\phi_{i1}, \ldots, \phi_{N_i}\right]$ as an N_i-length vector of the corresponding estimands, for $i = 1, 2$. The covariance of two functions, $f_1\left(\widehat{\boldsymbol{\Phi}}_1\right)$ and $f_2\left(\widehat{\boldsymbol{\Phi}}_2\right)$, may be approximated as the covariance of the first-order Taylor series of each of the two functions. Specifically,

$$\text{covar}\left[f_1\left(\widehat{\boldsymbol{\Phi}}_1\right), f_2\left(\widehat{\boldsymbol{\Phi}}_2\right)\right] \approx \text{covar}^{\text{TS}}\left[f_1\left(\widehat{\boldsymbol{\Phi}}_1\right), f_2\left(\widehat{\boldsymbol{\Phi}}_2\right)\right] \tag{A28}$$

$$= \text{covar}\left[f_1\left(\boldsymbol{\Phi}_1\right) + \left(\widehat{\boldsymbol{\Phi}}_1 - \boldsymbol{\Phi}_1\right) \cdot \nabla_{\widehat{\boldsymbol{\Phi}}_1} f\left(\boldsymbol{\Phi}_1\right), f_2\left(\boldsymbol{\Phi}_2\right) + \left(\widehat{\boldsymbol{\Phi}}_2 - \boldsymbol{\Phi}_2\right) \cdot \nabla_{\widehat{\boldsymbol{\Phi}}_2} f\left(\boldsymbol{\Phi}_2\right)\right] \tag{A29}$$

$$= \text{covar}\left[\widehat{\boldsymbol{\Phi}}_1 \cdot \nabla_{\widehat{\boldsymbol{\Phi}}_1} f\left(\boldsymbol{\Phi}_1\right), \widehat{\boldsymbol{\Phi}}_2 \cdot \nabla_{\widehat{\boldsymbol{\Phi}}_2} f\left(\boldsymbol{\Phi}_2\right)\right] \tag{A30}$$

$$= \text{covar}\left[\sum_i^N \frac{\delta f_1\left(\boldsymbol{\Phi}_1\right)}{\delta \widehat{\phi}_{1i}} \widehat{\phi}_{1i}, \sum_i^N \frac{\delta f_2\left(\boldsymbol{\Phi}_2\right)}{\delta \widehat{\phi}_{2i}} \widehat{\phi}_{2i}\right] \tag{A31}$$

$$= \sum_i^N \sum_j^N \frac{\delta f_1\left(\boldsymbol{\Phi}_1\right)}{\delta \widehat{\phi}_{1i}} \frac{\delta f_2\left(\boldsymbol{\Phi}_2\right)}{\delta \widehat{\phi}_{2j}} \text{covar}\left(\widehat{\phi}_{1i}, \widehat{\phi}_{2j}\right). \tag{A32}$$

Finally, in matrix notation,

$$\text{covar}\left[f_1\left(\widehat{\boldsymbol{\Phi}}_1\right), f_2\left(\widehat{\boldsymbol{\Phi}}_2\right)\right] \approx \text{covar}^{\text{TS}}\left[f_1\left(\widehat{\boldsymbol{\Phi}}_1\right), f_2\left(\widehat{\boldsymbol{\Phi}}_2\right)\right] \tag{A33}$$

$$= \nabla^T_{\widehat{\boldsymbol{\Phi}}_1} f\left(\boldsymbol{\Phi}_1\right) \cdot \boldsymbol{\Sigma}_{\widehat{\boldsymbol{\Phi}}_1, \widehat{\boldsymbol{\Phi}}_2} \cdot \nabla_{\widehat{\boldsymbol{\Phi}}_2} f\left(\boldsymbol{\Phi}_2\right), \tag{A34}$$

where

$$\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\Phi}}_1, \widehat{\boldsymbol{\Phi}}_2} = \begin{bmatrix} \text{covar}\left(\widehat{\phi}_{11}, \widehat{\phi}_{21}\right) & \cdots & \text{covar}\left(\widehat{\phi}_{11}, \widehat{\phi}_{2N_2}\right) \\ \vdots & \ddots & \vdots \\ \text{covar}\left(\widehat{\phi}_{1N_1}, \widehat{\phi}_{21}\right) & \cdots & \text{covar}\left(\widehat{\phi}_{1N_1}, \widehat{\phi}_{2N_2}\right) \end{bmatrix}. \tag{A35}$$

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