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RESEARCH REPORT

A Learning Progression for Geometric Transformations

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In this paper, a learning progression for geometric transformations is developed based on research that demonstrates the importance of viewing transformations as functions of the plane. The 5 levels of the progression reflect a student's evolving understanding of transformations as functions and their evolving understanding of the domain of these transformation as functions. The learning progression developed here is designed to be in alignment with the Common Core State Standards in Mathematics (CCSSM), in that a student who is placed at a particular level in the learning progression would have mastered the Common Core standards at the corresponding grade level. The description of the learning progression also includes sample tasks at each level that are intended to target that level of the progression.

Keywords Geometric transformation; learning progression; coordinate plane; mapping; geometric shape; parameter; translation; rotation; reflection; dilation

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A learning progression is "a sequence of successively more complex ways of thinking about an idea that might reasonably follow one another in a student's learning" (Smith, Wiser, Anderson, Krajcik, & Coppola, 2004, p. 5). In the context of the ETS research initiative for the *CBAL*® learning and assessment tool, a learning progression was described as

> a description of qualitative change in a student's level of sophistication for a key concept, process, strategy, practice or habit of mind. Change may occur due to a variety of factors, including maturation and instruction, and each progression is presumed to hold for most, but not all, students. As with all scientific research, the progressions are open to empirical verification and theoretical challenge. (Educational Testing Service, n.d., list item #2)

A learning progression identifies what Clements and Sarama (2004) have called "developmental progression[s] of levels of thinking" (p. 83). Although there will be points of connection between learning progressions in related domains, creating a network of learning paths, it is generally thought that the developmental levels are best understood by recognizing individual learning progressions belonging to individual mathematical domains (Daro, Mosher, & Corcoran, 2011).

Learning progressions in mathematics are based on research in mathematics education and mathematics cognition as well as the demands of mathematical coherence, balancing these concerns when there is a conflict. In turn, learning progressions are then validated through empirical research to verify that the conjectured levels in the progression actually represent states of knowledge and understanding of most students as they study and attempt to master the domain (Confrey & Maloney, 2010; Graf & van Rijn, 2016). A properly created and validated learning progression can then inform teaching and assessment. For example, a formative assessment containing items targeted to levels of a learning progression can provide evidence of a student's level in the progression. The teacher can then use this evidence to customize further assignments and instruction for that student.

The Importance of Geometric Transformations

In this paper, we present a proposed learning progression for geometric transformations. A geometric transformation is a function from the plane to the plane, that is, a function $f : \mathbf{R}^2 \to \mathbf{R}^2$. The transformations that are fundamental to geometry are the rigid motions — translations, reflections, and rotations and compositions of these — which preserve distance and angles, together with dilations, which expand or contract (National Governors Association Center for Best Practices &

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Council of Chief State School Officers [NGA/CCSSO], 2010). A *translation* shifts the plane to the right or left and up or down; for example, the function $f(x, y) = (x + 2, y - 3)$ shifts the plane two units to the right and three units down. A *reflection* produces the mirror image about the line of reflection; for example, the function $f(x, y) = (x, -y)$ reflects the plane about the x-axis. A *rotation* turns the plane around a fixed point (the *center of rotation*) through a fixed angle (the *angle of rotation*). For example, the function $f(x, y) = (-y, x)$ rotates the plane about the origin through an angle of 90°. A rigid motion is any sequence of translations, reflections, and rotations. Finally, a *dilation* expands or contracts the plane with reference to a fixed point. For example, the function $f(x, y) = (cx, cy)$ expands or contracts the plane by a factor of $|c|$ with reference to the origin; the dilation is an expansion if $c > 0$ and is a contraction if $c < 0$.

Our focus in this paper will be on geometric transformations as covered in high school geometry, although we will begin our learning progression in Grade 7 and earlier. The importance of geometric transformations in the high school geometry curriculum has been recognized for some time (Daro et al., 2011; Hollebrands, 2003; Portnoy, Grundmeier, & Graham, 2006; Yanik & Flores, 2009). In 1971, Kort, in his doctoral thesis, reported that students who took a transformation-based geometry course instead of a traditional course had better retention, after 1 year, of concepts such as congruence, similarity, and symmetry and that they were able to transfer what they had learned in the transformation-based geometry course to what they were learning in other courses about relations and functions (as cited in Hollebrands, 2003). Two years later, Peterson (1973) observed that geometric transformations provide a manipulative approach to the teaching of geometry in which students can become physically involved in the material. Jones (2002) observed that the study of transformations

> permits students to develop broad concepts of congruence and similarity and apply them to all figures. . . . Studying transformations can enable students to realize . . . that all parabolas are similar because they can be mapped onto each other, that the graphs of $y = \cos x$ and $y = \sin x$ are congruent, that matrices have powerful geometric applications, and so on. (p. 131)

Hollebrands (2003, p. 55) gave three reasons for including geometric transformations in the high school curriculum:

1. Geometric transformations provide students with opportunities to think in new ways about important mathematical concepts (e.g., functions whose domain and range are $\mathbf{R}^2$).
2. Geometric transformations provide students a context within which they can view mathematics as an interconnected discipline.
3. Geometric transformations provide students with opportunities to engage in higher-level reasoning activities using a variety of representations.

Guven (2012) pointed out that

> transformation geometry encourages students to investigate geometric ideas through an informal and intuitive approach [stressing] sensitivity, conjecturing, transformation and inquisitiveness. Transformations can lead students to the explorations of abstract mathematical concepts of congruence, symmetry, similarity and parallelism, enrich students' geometrical experience, thought and imagination; and thereby enhance their spatial abilities. (p. 366)

According to Yanik (2014), students “may discover patterns, construct generalizations, and develop spatial competencies and critical thinking through studying geometric transformations” (p. 33).

Furthermore, the importance of geometric transformations has increased significantly with the adoption of the Common Core State Standards in Mathematics (CCSSM); CCSSM defines the concepts of symmetry, congruence, and similarity in terms of geometric transformations (NGA/CCSSO, 2010). Symmetry is defined in terms of reflections and rotations; two figures are defined to be congruent if one can be transformed into the other by a sequence of rigid motions (translations, reflections, and rotations); and two figures are defined to be similar if one can be transformed into the other by a sequence of geometric transformations including dilations. Properties of geometric transformations are then used to prove theorems. For example, the familiar criteria using the lengths of sides (S) and the measures of angles (A) to show that two triangles are congruent (i.e., ASA, SAS, and SSS) are established by demonstrating that, under the hypothesized conditions, there exists a sequence of rigid motions transforming the first triangle into the second. For a detailed mathematical treatment of how geometry could be developed according to the CCSSM, see Wu (2013a, 2013b). Because we want our learning progression to support instruction, our progression is designed to be aligned with the CCSSM.

Relative Difficulty of Transformations and Tasks

There is general agreement in the literature concerning the relative difficulty of the various types of geometric transformations. Schultz and Austin (1983) observed that translations seem to be the easiest transformations for students to perform, and reflections are easier when the line of reflection is horizontal or vertical (i.e., parallel to one of the coordinate axes). Turgut, Yenilmez, and Anapa (2014) found that prospective mathematics teachers in their senior year of college had difficulty performing rotations when the center of rotation was outside the figure and that sometimes while rotating a figure they made other alterations to the figure, affecting its scale or orientation. Dilations seem to be regarded as more difficult for students than rigid motion transformations (Gülkilika, Uğurlu, & Yürük, 2015). According to the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), rigid motion transformations are introduced informally in Grades 3–5, whereas dilations are not introduced informally until Grades 6–8.

Xistouri, Pitta-Pantazi, and Gagatsis (2014) identified four types of tasks involving geometric transformations, ranked from easiest to most difficult (within each transformation type):

1. Recognize the image of a transformation from among a list of choices. (Which of the following is the [transformation] of [preimage]?)
2. Identify the transformation that was performed from among a list of choices. (Which of the following transformations was performed on [preimage] to obtain [image]?)
3. Identify the parameters of a given transformation.
4. Construct an image of a transformation.

As we shall see, the relative difficulty of transformation types and of types of tasks within transformation type can be used to place tasks at the appropriate level in our proposed learning progression. These tasks can subsequently be used as the basis of an empirical study to validate or refine the learning progression.

Understanding Geometric Transformations as Functions of the Plane

An important key to students' understanding of geometric transformations is understanding a transformation as a mapping of the plane and not just motion over the plane (Hollebrands, 2003; Yanik & Flores, 2009). According to Hollebrands (2003), students who understand that a geometric transformation is a one-to-one mapping from the plane onto itself have a better understanding of the role of the parameters of a transformation, can anticipate the action of a transformation on an object without carrying out the transformation, are less likely to be confused by fixed points of the transformation, and are better able to view a transformation itself as an object. Portnoy et al. (2006) referred to Dubinsky's theory of processes and objects (Dubinsky & McDonald, 2001) when describing students' transition from viewing transformations operationally to viewing them structurally and observed that advanced undergraduate students without an object understanding of transformations were unable to perform more difficult tasks such as proving that the set of Euclidean transformations forms a group.

This progression from process view to object view is related to Sfard's (1991) framework of three stages of mathematical development: *interiorization*, *condensation*, and *reification*. For Sfard, interiorization corresponds to the stage at which one has a process view; for example, at this stage, a geometric transformation is viewed as motion across the plane. Condensation corresponds to the stage at which one begins to have a high-level view of processes; the student might begin to view a geometric transformation as a mapping of objects in the plane. Finally, reification refers to the stage at which the student begins to have an object view, viewing a geometric transformation as a function from the plane to the plane.

Understanding geometric transformations as functions may be difficult at first for students because these functions are different from the functions they have experienced previously. Students have experienced from their algebra classes only real-valued functions of a real variable, functions $f:\mathbf{R}\to\mathbf{R}$. Such a function can be visualized by its graph, the set of all points $(x, y)\in\mathbf{R}^2$ with the property that $y=f(x)$. But a geometric transformation T is a function $T:\mathbf{R}^2\to\mathbf{R}^2$ that assigns to each point $(x, y)\in\mathbf{R}^2$ the point $T(x, y)\in\mathbf{R}^2$. The graph of such a function is a subset of $\mathbf{R}^4$, the set of all points

$(x, y, z, w) \in \mathbf{R}^4$ with the property that $T(x, y) = (z, w)$. Such a set certainly cannot be visualized in the same way that the graph of a real-valued real function can be visualized.

Along similar lines, Portnoy et al. (2006) developed a set of *curriculum modules* in the areas of geometry and linear algebra. The modules were designed to help undergraduate students make connections between geometry, linear algebra, and abstract algebra and were piloted in a junior-level geometry course required of preservice mathematics teachers. Portnoy et al. found that most of the students in the study had a process view of geometric transformations rather than an object view; as a result, they had difficulty writing transformation-based proofs of geometric theorems. Yanik and Flores (2009) found similar results in their study of four education students: two senior undergraduates majoring in elementary education and two students in a master's program in mathematics education. Mhlolo and Schafer's (2013) study of 11th graders in South Africa concluded that most had a motion idea of geometric transformations and not a mapping idea.

Hollebrands (2003) found that a critical requirement for students to understand geometric transformations as mappings is for them to have a proper conception of the domain of the transformation. She found that students generally progress through three levels of understanding:

1. The domain is the set of labeled points on the preimage.
2. The domain is the set of all points on the preimage.
3. The domain is the set of all points on the plane.

The first and second of these levels are consistent with the view of a geometric transformation as a mapping of an object to an object, whereas students in the third level view a transformation as a mapping of the plane.

Students must also understand the role of the parameters of a geometric transformation (Hollebrands, 2003). They must understand what parameters are, how parameters differ from variables (even though they may vary), and how changing the parameters affects the transformation. Students also need to know which properties of geometric objects are preserved by different types of geometric transformations. In particular, students who do not understand geometric transformations as functions whose domain is the plane often have difficulty reasoning about or identifying fixed points (Hollebrands, 2003).

In summary, it is important for students to see that transformations are functions of the plane. This will not only help them to better understand the transformations themselves (e.g., the action of a transformation on an object or the role of the parameters of the transformation), but it will also help them to gain a better understanding of the concept of function. An understanding of the concept of function can help students understand transformations, and an understanding of transformations can help students understand the concept of function.

Common Misconceptions About Geometric Transformations

In writing a learning progression, it is important to make note of common misconceptions and what levels of the progression these misconceptions might indicate. Yanik (2014) noted that some students have the misconception that circular figures cannot be translated, as there are no corners to act as "handles." Other students have the misconception that noncircular figures cannot be rotated because rotation changes the figure (a square is changed to a diamond, for example). Along those lines, students may believe that similar (or even congruent) figures that are oriented differently (such as a square and a diamond, which is the square rotated 45°) are not the same figure (Seago, Nikula, Matassa, & Jacobs, 2012). In our proposed learning progression, we have included a section at each level that lists misconceptions or errors that may indicate that a student should be placed at that level.

Existing Learning Progressions for Geometric Transformations

In our search of the literature, we found two existing learning progressions on geometric transformations, neither of which was entirely adequate for our needs.

The Learning Progression of Xistouri, Pitta-Pantazi, and Gagatsis

Xistouri et al. (2014) studied the performance of 166 students from Grades 4, 5, and 6 on a test containing items of each of the four types of transformation tasks mentioned earlier (recognize the image of a transformation, identify the transformation that was performed, identify the parameters of a given transformation, and construct the image of a transformation) paired with each of the three rigid motion transformations. For translations and reflections, there were at least three tasks for each task type: one focused on horizontal motion, one on vertical motion, and one on diagonal motion. Based on their results, they identified five levels of understanding regarding rigid motion transformations and assigned each combination of task type and transformation to a level. The descriptors they provided for the levels are not always useful, but their labels for the levels are aligned with Hollebrands's (2003) research and provided a framework for our proposed learning progression.

Level 1. Holistic Image

Students at this level have a naïve idea of translations; they "seem to perceptually conceive simple relations of up-down and left-right within the figure, that exist in the real world, but without understanding [either] the properties of the transformation [or] of the geometrical figures represented" (p. 157). Students do not conceive of shapes as separate objects but as part of the whole image.

Level 2. Motion of an Object

Students at this level are able to visualize a geometric transformation (translations, rotation, reflections, and dilations) as the motion of an object from its preimage location to its image location. Students separate the object from the underlying plane.

Level 3. Mapping of an Object

At this level, students are beginning to view a geometric transformation as a mapping from the preimage object to the image object. They may be thinking of the domain of the mapping as the set of labeled vertices (A maps to A', B maps to B', etc.). They begin to think about the properties of transformations, and they can use these properties in simple situations to construct the images of transformations.

Level 4. Mapping of the Plane

Students at this level understand that a geometric transformation is a one-to-one mapping from the plane onto itself.

Level 5. Self-Regulated Mapping of the Plane

It is not clear what Xistouri et al. (2014) meant by this label. Students at this level have a greater facility with geometric transformations and seem "to have some flexibility in manipulating and controlling [their] mental images and can flexibly change between figural units of visualization and visual strategies" (p. 159).

Based on their results, Xistouri et al. (2014) assigned each combination of task type and transformation in their study to a level in their learning progression (see Table 1). Their assignment is consistent with Schultz and Austin's (1983) observation that translations are the easiest of the transformations for students to understand, with reflections and rotations being more difficult. It also suggests that students may proceed through the levels at a different pace for different transformations: A student may have a Level 3 understanding of translations but only a Level 1 understanding of reflection or rotation. This idea needs to be pursued with further research.

Although the research of Xistouri et al. (2014) was focused on elementary students, the levels of their progression are consistent with Hollebrands's (2003) research on high school geometry students' understanding of transformations, especially the Level 4 understanding of transformations as mappings of the plane. These levels were useful in establishing the framework for our learning progression.

Soon's Learning Progression

Soon (1989) developed a somewhat different learning progression for geometric transformations, based on a learning progression for geometry in general developed in the 1950s by Dina and Pierre van Hiele. The van Hiele model, as summarized by Guven (2012), is shown in Table 2.

Table 1 Tasks and Transformation Type by Level of the Xistouri et al. (n.d.) Learning Progression

Task and transformation type	Translate	Reflect	Rotate
Recognize the image of a transformation	Level 1	Level 3	Level 3
Recognize the transformation that was performed	Level 2	Level 3	Level 3
Identify the parameters of a given transformation	Level 3	Level 4	Level 4
Construct the image under a transformation	Level 3	Level 5	Level 5

Note. Adapted from "Primary School Students' Structure and Levels of Abilities in Transformational Geometry" by X. Xistouri, D. Pitta-Pantazi, & A. Gagatsis, 2014, *Revista Latinoamericana de Investigacion en Matematica Educativa, 17*(4), 149–164. Copyright 2014 by ReLime.

Table 2 Van Hiele Learning Progression for Geometry

Level	Stage	Definition
Level 1	Recognition	The student recognizes geometric figures by their global appearance, identifies names of figures, but does not explicitly identify their properties.
Level 2	Analysis	The student analyses figures in terms of their components and properties, discovers properties and rules of a class of shapes empirically, but does not explicitly interrelate figures or properties.
Level 3	Predeductive	The student logically interrelates previously discovered properties and rules by giving or following informal arguments.
Level 4	Deductive	The student proves theorems deductively, develops sequences of statements to deduce one statement from another, but does not yet recognize the need for rigor.
Level 5	Rigor	The student establishes theorems in different axiomatic systems and analyses and compares these systems.

Note. Adapted from "Using dynamic geometry software to improve eighth grade students' understanding of transformation geometry," by B. Guyven, 2012, *Australasian Journal of Educational Technology*, *28*(2), p. 370. Copyright 2012 by the *Australasian Journal of Educational Technology*.

Table 3, adapted from Table 3 in Guven (2012, pp. 370–371), gave descriptors of the different levels of the Soon (1989) progression. It seems clear that even though both learning progressions have five levels, the levels in the Soon progression are more advanced than the same levels in the Xistouri et al. (2014) progression. In particular, Level 1 of the Soon progression corresponds to Level 2 of the Xistouri et al. progression, and Level 2 of the Soon progression seems to correspond to Level 3 of the Xistouri et al. progression, though Soon's Level 2 descriptor "relates transformations using coordinates" seems to be closer to Xistouri et al.'s Level 4. Some of Soon's Level 3 descriptors (e.g., "performs composition of simple transformations") correspond to Xistouri et al.'s Level 4, whereas other Soon Level 3 descriptors (e.g., "represents transformations using coordinates and matrices)" correspond to the Xistouri et al. Level 5. Soon's Level 4 is clearly high school, whereas Level 5 is post high school.

Nonetheless, some of the descriptors in the Soon (1989) progression have influenced descriptors in our own proposed progression. For example, Soon's Level 2 descriptor "relates transformations using coordinates" corresponds to our Level 4 descriptor "Students can represent a geometric transformation by equations involving the coordinates," and Soon's Level 4 descriptor "gives geometric proofs using transformations" corresponds to our Level 4 descriptor "Students can use geometric transformations to prove the usual congruence and similarity theorems about triangles."

The Learning Progression Implicit in the Common Core State Standards

There is also a learning progression implicit in the Common Core State Standards. According to the authors, the standards are based on "research-based learning progressions detailing what is known today about how student's mathematical knowledge, skill, and understanding develop over time" (NGA/CCSSO, 2010, p. 4), although the CCSSM does not cite specific learning progressions. But the standards for Grades K through 7 provide a careful and gradual introduction to ideas that will later coalesce around the notions of geometric transformations, congruence, and similarity. For example, in kindergarten, students are asked to recognize that a shape that has been rotated is still the same shape. A cluster of standards extending from Grade 1 to Grade 3 is entitled "Reason with shapes and their attributes." These standards ask

Table 3 Levels of Soon's Learning Progression for Transformations

Levels	Characteristics: The student …
Level 1	● identifies transformations by the changes in the figure in simple drawings of figures and images and in pictures of everyday applications. ● identifies transformations by performing actual motions. ● names or labels transformations using standard and nonstandard names and labels appropriately. ● solves problems by operating on changes of figures or motion rather than using properties of the changes.
Level 2	● uses the properties of changes to draw the preimage or image of a given transformation. ● discovers properties of changes to figures resulting from a specific transformation. ● uses appropriate vocabulary for the properties of transformations. ● is able to locate the axis of reflection, the center of rotation, the translation vector, and the center of enlargement. ● relates transformations using coordinates. ● solves problems using known properties of transformations.
Level 3	● performs composition of simple transformations. ● describes changes to states (preimage, image) after composite transformations. ● represents transformations using coordinates and matrices. ● interrelates the properties of changes to a figure resulting from transformations. ● given initial and final states, can name a single transformation. ● given initial and final states, can decompose and recombine a transformation as a composition of simple transformations.
Level 4	● gives geometric proofs using transformations. ● gives proofs using coordinates and matrices.
Level 5	● understands associativity and commutivity and the notions of identity and inverse transformation in the context of compositions of transformations. ● identifies groups of transformations and proves or disproves that sets of transformations form group structures.

Note. Adapted from "Using dynamic geometry software to improve eighth grade students' understanding of transformation geometry," by B. Guyven, 2012, *Australasian Journal of Educational Technology*, *28*(2), Table 3, p. 370, 371. Copyright 2012 by the *Australasian Journal of Educational Technology*.

students to identify attributes of shapes and combine shapes to form other shapes. The first of these activities prepares the way for a later discussion of the properties of shapes that are invariant under geometric transformations, and the second will sometimes involve, implicitly, translations and rotations before the shapes can be combined. In Grade 4, students are asked to recognize a line of symmetry, clearly preparing the way for a discussion of reflections. In Grade 6, students draw polygons in the coordinate plane, preparing the way for tasks in Grade 7 that involve scale drawings, which in turn prepare the way for a discussion of dilations in Grade 8.

In Grade 8, students have an informal understanding of transformations, at least at Xistouri et al.'s (2014) Level 2, viewing a transformation as the motion of a figure over the plane. Students also understand that two figures are congruent if one can be moved to the other by a sequence of rigid motion transformations, and two figures are similar if one can be moved to another by a sequence of rigid motion transformations and dilations. Finally, in high school, students understand transformations as functions of the plane and are able to give precise definitions of translations, reflections, rotations, and dilations. Students understand compositions of transformations, and they understand that two figures are similar if one is mapped to the other by a composition of transformations and two figures are congruent if one is mapped to the other by a composition of rigid motion transformations. Students are able to use these definitions to prove theorems about congruence and similarity.

Because we have aligned our proposed learning progression with the CCSSM, many of the descriptors in our progression are adapted from the Common Core standards.

A Learning Progression for the Concept of Function

Because students must ultimately view geometric transformations as functions, we also looked at the learning progression for the concept of function developed by Graf et al. (2018). This learning progression has six levels, of which the first five are of interest to us here (the sixth represents a professional mathematician's level of understanding).

Level 1. Quantity Perception

Students at this level have not yet been introduced to the formal notion of function, but they have an intuitive understanding of patterns in one dimension, recursive rules (e.g., keep adding 3), and proportional reasoning.

Level 2. Function as a Formula

Students at this level have a process view of functions, viewing a function as a computational formula. They may not understand the link between different computational rules that yield the same function or different representations of the same function.

Level 3. Function as a Rule

At this level, students are beginning to think of a function as defined in the CCSSM: "a function is a rule that assigns to each input exactly one output" (NGA/CCSSO, 2010, p. 55), understanding that a rule need not be given by a formula. Students understand that different but equivalent representations correspond to the same function.

Level 4. Full Covariational Relationship

At this level, students understand functions as "a dependency relationship between quantities" (Graf et al., 2018, p. 64). They understand domain and range as the sets of inputs to, and outputs from, a function, and they are paying attention to properties of functions.

Level 5. Object Perception

Students at this level have an object view of a function. Students can use function notation and can pay attention to global properties of functions as well as local properties.

Although some aspects of this learning progression are not applicable to geometric transformations (e.g., the covariational approach to functions), other aspects were useful in creating our learning progression. In particular, we have adapted Level 5 as Level 5 of our progression, in which students achieve an object view of transformations (in the sense of APOS theory[1]) and begin to view functions globally.

A Learning Progression for Geometric Transformations Aligned with the Common Core

In writing our learning progression, we began with the basic structure of the Xistouri et al. (2014) progression, added some of the descriptors from Soon's (1989) progression and some common misconceptions from the literature, and brought the progression into alignment with the CCSSM and Graf et al.'s (2018) Concept of Function Learning Progression. We also included at each level samples of the types of tasks that can provide evidence of thinking that is characteristic of that level. Our learning progression has five levels; see Table 4. In keeping with ETS style, the table begins with Level 5 and moves down to Level 1. Table 4 also indicates how the levels of our learning progression are aligned with the CCSSM and, where appropriate, how the specific tasks align with specific standards in the CCSSM. In some cases, the task is a quote of, or an extract from, the relevant standard. When this is the case, a reference to the specific standard is provided.

Students at Level 1 of the progression will have an intuitive understanding of simple transformations, such as translations and rotations, but they will not yet have a formal understanding of geometric transformations. This level of understanding corresponds to Grade 7 in CCSSM: "In preparation for work on congruence and similarity in Grade 8

Table 4 Learning Progression for Geometric Transformations

Level	What students can do	Misconceptions	Tasks appropriate for this level
Level 5: Object view of a transformation Advanced or post high school	• Students at this level have a good technical facility with transformations and understand transformations as functions of the plane. • Students can represent a geometric transformation as a matrix operation; for example, a rotation of 90° counterclockwise about the origin corresponds to the matrix operation $f(\mathbf{X}) = \mathbf{AX}$ where $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.	• Even though students will have an object view of transformations, in the sense of APOS theory, they will not be familiar with concepts from abstract algebra, such as the concept of a group. They will not have had experience viewing transformations as elements of a group, nor will they be able to prove or disprove that sets of transformations form groups.	• Represent a geometric transformation as a matrix operation; for example, a rotation of 90° counterclockwise about the origin corresponds to the matrix operation $f(\mathbf{X}) = \mathbf{AX}$ where $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. • Express a transformation as a composition of simpler transformations and express the composed transformation as a matrix operation where the matrix is the product of the matrices associated with the simpler transformations. For example, determine the matrix for a rotation about the point $(1, 2)$ by expressing that rotation as a composition of a rotation about the origin with translations.
Level 4: Mapping of the plane High school	• Students understand the precise definitions of translations, reflections, rotations, and dilations as functions from the plane to the plane. • Students should be able to identify the parameters of a given transformation (e.g., the axis of reflection, the number of degrees of rotation, and the dilation constant). • Students can use geometric transformations to prove the usual congruence and similarity theorems about triangles, including the SAS, ASA, and SSS criteria for congruence of triangles and the AA criterion for similarity of triangles. • Students can represent a geometric transformation by equations involving the coordinates.	• Students may not understand the relationship between transformations and matrices.	• Identify the parameters of a given transformation. • Given a geometric figure and a transformation, draw the image of the figure under the transformation. (G-CO-5) • Use geometric descriptions of transformations to predict the effect of a given transformation on a given Figure. (G-CO-6) • Identify a transformation as a function $f : \mathbf{R}^2 \to \mathbf{R}^2$ in terms of coordinates: $f(x, y) = \ldots$. (G-CO-2)

Table 4 Continued

Level	What students can do	Misconceptions	Tasks appropriate for this level
Level 3: Mapping of an object Transition to high school	• Students begin to view a geometric transformation as a mapping from one shape to another. • Students at this level could be Grade 8 students with a strong understanding of geometric transformations, or they could be high school students who are still grappling with the notion of a geometric transformation as a function from the plane onto itself. Alternatively, some students may skip this level entirely and progress from Level 2 directly to Level 4.	• Students may think that the domain of the transformation is the set of labeled points on the shape rather than the entire shape (or the entire plane). • Students may misunderstand the roles of the parameters of a transformation. • Students may not be able to anticipate the action of a transformation on an object without actually carrying out the transformation. • Students may be confused by fixed points of a transformation. • Students may have a process view of a transformation rather than an object view.	• Recognize the image of a transformation from among a list of choices. (Which of the following is the [transformation] of [preimage]?) • Identify the transformation that was performed from among a list of choices. (Which of the following transformations was performed on [preimage] to obtain [image]?)
Level 2: Motion of an object Grade 8	*Rigid motions and congruence:* • Students have an intuitive understanding of translations, reflections, and rotations as geometric transformations that move a shape on top of the plane to another location. • Students have an intuitive understanding of properties of shapes that are preserved by rigid motions. • Students can perform a sequence of rigid motions. • Students understand that two figures are congruent if one can be moved to the other by a sequence of rigid motions. *Dilations and similarity:* • Students have an intuitive understanding of dilations. • Students understand that two figures are similar if one can be moved to the other by a sequence of geometric transformations, including rigid motions and dilations.	• Students at this level may think of a transformation as motion of a shape over the plane and not as a function from the plane to the plane or from a subset of the plane to the plane.	• Identify properties of shapes that are preserved by rigid motions. (8.G.1) • Given two congruent shapes, describe a sequence of rigid motions that moves one shape to the other. (8.G 2) • Describe the effect of a geometric transformation on the labeled points in a shape using coordinates. (8.G.3) • Given two similar shapes, describe a sequence of geometric transformations that moves one shape to the other. (8.G.4) • Give informal proofs of theorems. (8.G.5)

Table 4 Continued

Level	What students can do	Misconceptions	Tasks appropriate for this level
	Proofs of basic theorems: • Students should be able to give an informal proof, based on transformations, of the angle-angle criterion for similar triangles (AA). • Students should be able to use AA to prove that the slope of a nonvertical line does not depend on the choice of points used to compute it. • Students should be able to use AA to prove that the sum of the angles of a triangle equals 180°. • Students should be able to use AA to prove the Pythagorean theorem.		
Level 1: Intuitive understanding Grade 7	• Students at this level have had some experience moving shapes across the plane, implicitly applying translations and rotations but without formalizing the notion of transformation. • Students have an intuitive understanding of the properties of shapes that are preserved when shapes are moved across the plane, including lines of symmetry. • Students have had some experience producing scale drawings of figures.	• Students at this level do not yet have a formal understanding of geometric transformations. • Students may exhibit certain common misconceptions, such as the belief that the orientation of an object is a property that distinguishes a shape; for example, such students may think that a diamond is not a square, even though a diamond is just a square that has been rotated 45°. • On the other hand, students may believe that a circle cannot be rotated because it appears to be unchanged when rotated.	• Compute the actual perimeter and area of shapes (e.g., triangles, rectangles) that are drawn to scale. (7.G.1) • Reproduce a scale drawing at a different scale. (7.G.1) • Draw (or attempt to draw) triangles with given side lengths or given angle measures. Indicate whether the given lengths or measures determine a unique triangle, more than one triangle, or no triangle. (7.G.2) • Solve problems that ask for the unknown measure of an angle in a figure, given other information about the Figure. (7.G.5)

[a]For tasks that are adapted from CCSSM standards, references are given to the specific standard from which the task is taken. For example, this reference is to Standard 5 in the congruence domain of high school geometry.

[Grade 7 students] reason about relationships among two-dimensional figures using . . . informal geometric constructions" (NGA/CCSSO, 2010, p. 46). Students at this level may have certain common misconceptions about transformations, such as the misconception that the orientation of an object determines its shape. For example, students may think that if a square with edges parallel to the coordinate axes is rotated 45°, then it is no longer a square because its edges are no longer parallel to the coordinate axes.

Students at Level 2 will have an informal understanding of all four types of geometric transformations—translations, reflections, rotations, and dilations—but will regard transformations as moving an object from one location in the plane to another. They will not yet regard a transformation as a function. Students will understand that two shapes are congruent if there is a sequence of rigid motions (translations, reflections, and rotations) that moves one shape to the other, and they will understand that two shapes are similar if there is a sequence of transformations, including dilations, that moves one shape to the other. Students will be able to give an informal transformation-based proof of the angle-angle criterion for two triangles to be similar and will be able to use the angle-angle criterion to prove (or explain) three fundamental results: (a) the slope of a line does not depend on the two points chosen to calculate the line, (b) the sum of the angles in a triangle equals 180°, and (c) the Pythagorean theorem. This level of the learning progression is consistent with the Common Core standards for Grade 8 (NGA/CCSSO, 2010, p. 52).

At Level 3, students will be able to think of a transformation as a mapping from one shape to another, although they may view the domain of the mapping as just the set of labeled points in the shape. They may also have difficulty understanding the roles of the parameters of a transformation, and they may be confused by the fixed points of a transformation. Although they may have difficulty anticipating the action of a transformation on a shape, they should be able to identify the image of a transformation from a list of choices or identify the transformation that was performed. Students at this level could be Grade 8 students with a strong understanding of geometric transformations, or they could be high school students who are still struggling with the notion of a geometric transformation as a mapping of the plane to itself.

Level 4 students will understand the precise definitions of translation, rotation, reflection, and dilation; will be able to identify the parameters of a given transformation; will be able to give transformation-based proofs of the standard congruence theorems for triangles; and will be able to represent a transformation $f(x, y) = (x', y')$ with equations that give x' and y' in terms of x and y. This level of understanding is consistent with the Common Core standards for high school geometry (NGA/CCSSO, 2010, p. 74).

Finally, Level 5 students have an object view of transformations, in the sense of APOS theory (Dubinsky & McDonald, 2001). They can represent transformations as matrix operations; for example, they understand that counterclockwise rotation of 90° about the origin corresponds to the matrix operation $f(\mathbf{X}) = \mathbf{AX}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

They can express a transformation as a composition of simpler transformations and can express the composed transformation as a matrix operation where the matrix is the product of the matrices associated with the simpler transformations. For example, they can determine the matrix for a rotation about the point (1, 2) by expressing that rotation as a composition of a rotation about the origin with translations. Placement at this level corresponds to a more sophisticated view of functions in general; a student who places at level 5 on the geometric transformations learning progression would also place at level 5 on the concept of functions learning progression. a student at this level would be an advanced high school student or, perhaps, a beginning college student.

Further Research

The learning progression that we have developed here is a proposed progression. As stated earlier, learning progressions must be validated through empirical research to verify that the conjectured levels in the progression represent the states of understanding of most students as they progress through the subject. Further research on the geometric transformations learning progression could include administering tasks targeted to specific levels of the learning progression in a cognitive laboratory and on a larger scale to a sample of students. The outcomes of the cognitive laboratory study and the large-scale scores could determine if the conjectured levels of understanding are actually present in the student responses (Graf & van Rijn, 2016).

Summary

The learning progression for geometric transformations that we have described in Table 3 is based on research that demonstrates the importance of viewing transformations as functions of the plane — functions from $\mathbf{R}^2$ to $\mathbf{R}^2$. The five levels of the progression reflect a student's evolving understanding of transformations as functions and their evolving understanding of the domain of a transformation as function. The learning progression is designed to be in alignment with the CCSSM in that a student who is placed at a particular level in the learning progression would have mastered the Common Core standards at the corresponding grade level. The description of the learning progression also includes sample tasks at each level that are intended to target that level of the progression.

It is hoped that, after the learning progression has been empirically validated, it will be of use to assessment specialists in designing assessments that can identify the level of a student's understanding. It may also be useful for curriculum planners as they design coursework for a transformation-based geometry course. Finally, it could also be useful for teachers as they plan lessons that meet the differing needs of individual students.

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Note

1 APOS theory (Dubinsky & McDonald, 2001) hypothesizes four interrelated levels of student understanding of mathematics: an *action* level, a *process* level, an *object* level, and a *schema* level. An action is a repeatable operation that transforms an object and that requires, either explicitly or from memory, step-by-step instructions on how to perform the operation; an action becomes a process when the student internalizes it and can perform the action mentally without external stimuli; a process becomes an object when it is perceived as an entity upon which other actions and processes can be made; and finally, a schema is "an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework … that may be brought to bear upon a problem" (p. 276).

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