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8TH GRADE STUDENTS' UNDERSTANDING OF THE DEFINITION OF RECTANGLE[1]

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Abstract: This is a descriptive study which attempts to answer how well 8th grade students define a rectangle. The subjects for this study were 93 of 8th grade students. The students were administered a written open ended test on the definition of a rectangle which was adapted from Ahuja (1996). Responses of the students were coded by each researcher independently and common expressions were grouped. Then, new codes were created and the final codes were generated by the researchers. Findings of the study revealed that none of the students could provide sufficient and necessary conditions for their descriptions and definitions. Besides, it was seen that the majority of the students used the terminology incorrectly. While trying to define the rectangle, most of the students listed the properties of the rectangle and only a few students were able to construct a definition. Students mostly gave visual examples. From these findings, it can be concluded that students were unsuccessful in defining the rectangle and they were at Van Hiele Level 1.

Key words: geometry, Van Hiele levels, definition, rectangle, 8th grade students

1. Introduction

The definition of a concept is a form of words which specifies the concept (Tall and Vinner, 1981). According to Herbst, Gonzales, and Macke (2005, 17) "Mathematical definitions can be described logically as the statement of the necessary and sufficient conditions that an object must meet to be labeled by a certain word or expression". A student can learn a definition by rote or she/he can learn it more meaningfully by relating with the concept defined (Tall and Vinner, 1981). As "*definitions are the basic building blocks of mathematical thinking* (Çakıroğlu, 2013, p. 1), they are fundamental in mathematics education and they improve at school years (Zavlavski and Shir 2005). According to Alonso (2009), the correctness of the definitions promotes the development of science. Definitions show the intellectual progresses by making separations of different features and properties which corrects the transitions connecting with the other definitions and with reasoning which leads to mathematical proofs. Concepts should be defined precisely and clearly to express a mathematical system or idea. Defining concepts in many different ways can cause controversial situations; so some basic elements should be considered while making definitions such as considering the hierarchical concept structure, describing an existing phenomenon, equivalence of different definitions for the same concept, appropriateness with the axiomatic structure, providing sufficient and necessary conditions and being economical (Çakıroğlu, 2013).

Classification is as important as definition in mathematics education (Fujita and Jones 2007). Poincare, (1914, 452, in: Fujita and Jones, 2007, 3) says that: "The definition will not be understood until you have shown not only the object defined, but the neighboring objects from which it has to be distinguished, until you have made it possible to grasp the difference, and have added explicitly your reason for saying this or that in stating the definition". Concepts in mathematics are built on each other

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and this leads to a hierarchical structure and this hierarchical structure is the nature of mathematics (Çakıroğlu, 2013). De Villiers (1994, 12) says that "The classification of any set of concepts does not take place independently of the process of defining". The hierarchy in definitions requires consideration of the hierarchical relationships between these concepts. Considering the definition of "A rectangle in which all angle measurements and side lengths are equal is called a square, the "rectangle" is more general concept class used in defining the square (Çakıroğlu, 2013). In mathematics education, the hierarchical definitions are important in terms of learning processes since it helps students to see the relationships between concepts (Çakıroğlu, 2013). According to Currie and Pegg (1998: 177), "Class inclusion is the ability to have an overview of possible relationships that exist among figures".

In this study, definition and class inclusion concepts are examined in terms of Van Hiele (1986) Theory. According to the van Hiele geometric development model, learners move through five hierarchical levels of understanding named as: visual level (level one), the descriptive level (level two), the theoretical level (level three), formal logic (level four) and the nature of logical laws (level five) (Van Hiele, 1986). As this model proposed, at level 1, learners perceive a figure as a whole shape and do not perceive its parts. They can identify names, compare and operate on geometric figures according to their appearance. At the 2nd level, learners operate on certain objects, namely classes of figures and discover properties for these classes, but they cannot logically order the properties and figures. At level 3, properties are logically ordered; one property precedes or follows from another property. But at this level, the meaning of deduction, that is, the role of axioms, definitions, theorems, and their converses are not understood. At level 4, learners prove theorems deductively and at level 5 different axiomatic systems can be understood by the learners. De Villiers (1998) explained the levels operationally particularly for the concepts of the definition. As he explained, at level 1 visual definitions can be generated, for example a rectangle is a quadrilateral that looks like this (showing or drawing one) or describes it in terms of visual characteristics (two long and two short sides). At level 2, learners can produce uneconomical definitions; mainly they give many properties which are redundant to define the concept. For example, for a learner at this level, the definition of a rectangle can be "as a quadrilateral with opposite sides parallel and equal, four right angles, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc." When the learners reached the level 3, they can generate economical and correct definitions, as "a rectangle is a quadrilateral with an axis of symmetry through each pair of opposite sides".

1. 1. Related Literature

Many studies focused on definitions and/or classifications of quadrilaterals made by students (Aktaş and Aktaş, 2012; Bernabeu, Moreno and Llinares, 2018; Erez and Yerushalmy 2006; Furinghetti and Paola, 2002; Herbst and others, 2005; Monaghan, 2010; Okazaki and Fujita, 2007; Ulusoy, 2015; Ulusoy and Çakıroğlu, 2017; Yavuzsoy-Köse, Yılmaz, Yeşil, and Yıldırım, 2019). These studies investigated understandings and abilities with regard to hierarchical classifications or inclusion relations of quadrilaterals.

Ulusoy and Çakıroğlu (2007) aimed to determine the ways in which 7th grade students distinguish parallelograms and to reveal the errors they have made in this process. Students were asked to define parallelogram, construct three different parallelograms and to identify parallelograms among figures. Students' responses in identifying parallelograms were grouped into four categories as hierarchical examples, partial-hierarchical examples, non-hierarchical examples, and non-examples. Results showed that only limited number of students using a hierarchical approach could distinguish parallelograms without error and in a complete way. Students with high achievement level using partial-hierarchical examples distinguished parallelograms. Students in this group considered the rhombus as a parallelogram but square and rectangle as not being a parallelogram. The results also showed that students at all achievement levels using non-hierarchical approach did not consider square, rectangle and rhombus as parallelogram

Ulusoy (2015) aimed to reveal a meta classification for middle school students' identification of quadrilaterals in terms of trapezoid. She studied with thirteen 7th grade students and used a trapezoid identificationtest with semi structured interviews. The results showed that most of the higher level students selected trapezoids based on exclusive relations of quadrilaterals and they did not think that parallelogram, rhombus, rectangle and square are also a trapezoid.

Yavuzsoy-Köse and others (2019) investigated how students interpreted verbal definitions given for quadrilaterals and reasonings in the process of this interpretation. It was also aimed to investigate the easiest definitions for students to understand, and to determine which mathematical terms they did not understand in the given definitions. Clinical interviews were conducted and the data were analysed via thematic analysis method. According to the results, the more accurately identified definitions were: the exclusive definition and inclusive definition (based on sides) of a parallelogram, the exclusive definitions of a rectangle and a rhombus, the definition of a square based on diagonals. The vast majority of the students who accurately interpreted the inclusive definitions for a parallelogram based on side and angle properties suggested that these definitions also apply to a rectangle, a square and a rhombus. The the inclusive definitions of students were such as "a quadrilateral with two pairs of parallel sides" and "a quadrilateral with opposite angles equal" to be sufficient. The students with definitions of a rectangle and rhombus primarily interpreted the definitions based on side and angle properties accurately but few students were able to interpret the definitions based on diagonal and symmetry properties. The vast majority of the students who misinterpreted the definition "a quadrilateral with sides symmetrical to the perpendicular bisector" in particular identified the definition as a square and they took into account only the quadrilateral it covered. This result was interpretated as that the students established an inverse hierarchical relationship between a rectangle and a square. Some of the students were not able to deduce what the necessary and sufficient condition of a rectangle was based on the given rectangle definition "a quadrilateral with three right angles" and they could not decide whether the definition described a rectangle or a square. Some students, who established an inverse hierarchical relationship, did not see a square as a special type of a rhombus and a special type of a rectangle.

Herbst and others (2005) similar to Yavuzsoy-Köse and others (2019), focused on the economy or necessity and sufficiency of the definitions of some quadrilaterals. Herbst and others (2005) asked 9th grade students to think of the minimal conditions that a shape had to satisfy in order to be a special quadrilateral (square, rectangle, and rhombus). When they were asked to define a rectangle, it was found that only 12% of the definitions of the students stated necessary and sufficient conditions. The half of the rest provided too much information and the other half provided insufficient explanations.

Aktaş and Aktaş (2012) and Okazaki and Fujita (2007) aimed to reveal middle school students' understandings of inclusion relations between quadrilaterals and have reached some conclusions about their understanding of the hierarchical relations of the rectangle. Aktaş and Aktaş (2012) investigated 8th grade students' identification of quadrilaterals and understanding hierarchical classifications of them when diagonals intersect each other in different situations. They found that students could identificate the quadrilaterals by making some measurements easily while following the steps in worksheets, but they couldn't see the hierarchical classification of them at desired level. They reported that students were unsuccessful in seeing that square was a special type of rhombus or rectangle. They could only see a few relations among quadrilaterals related to their hierarchical classification only by the questionings of their teachers. Okazaki and Fujita (2007) investigated inclusion relations between quadrilaterals in terms of the prototype phenomena and the common cognitive paths. In order to determine what mental/personal images of quadrilaterals the participants had, they were asked to choose images of parallelograms, rectangles and rhombuses from various quadrilaterals; whether mathematical statements concerning parallelograms, rectangles and rhombuses were true or false. Most of the students identified rhombuses as parallelograms and many of them failed to see rectangles as a special type of parallelograms, a square as a special type of a rectangle and a rhombus.

Bernabeu and others (2018) aimed to identify the factors that have triggered or inhibited the capacity to identificate different properties of quadrilaterals and how these properties were related with the classifications of them. The results showed that students did not use different attributes of quadrilaterals such as diagonals, symmetry axes, parallelism and length of sides in the same way and this influenced the way children recognize, represent and classify quadrilaterals. This showed that the conceptual understanding of different types of quadrilaterals were gradual and depended on the attributes used.

Monaghan (2000), explored the language used in children's written work as a means of assessing their mathematical understanding by focusing on their conceptualizations of polygon and investigated their attempts to differentiate between these polygons. He found that children were unlikely to develop their higher order conceptualizations (class inclusions) unless they were forced into a Vygotskyan conflict. They were in confusion in distinguishing mental images with concepts and the differences between some quadrilaterals.

Erez and Yerushalmy (2006) and Furinghetti and Paola (2002) investigated students' understandings of classifying quadrilaterals with dynamic geometry environments. Erez and Yerushalmy (2006) investigated the attributes and definitions of geometric figures with using a dragging tool (with dynamic software) with 5th grade students. Their goal was to investigate the influence of different conditions of geometric knowledge. They claimed that students had difficulties in turning the quadrilaterals into another shape with the tool (e.g. a rectangle into a square, etc.). They mentioned that it was hard for students to change the concept image of quadrilaterals and students had problems in their formal thinking. Furinghetti and Paola (2002) investigated 10th grade students' constructions and classifications of quadrilaterals using Cabri Geometry software. They looked for the consistence of the certain definitions. The results showed that the students didn't use the potentialities of the tool in a rational way in constructing and classifying quadrilaterals.

When the related literature is examined, it was seen that three studies (Ulusoy, 2015, Ulusoy and Çakıroğlu, 2017, Yavuzsoy-Köse and others, 2019) focused on a particular quadrilateral (parallelogram or trapezoid) and analyzed the responses of students based on a particular classification. Two studies (Aktaş and Aktaş, 2012 and Okazaki and Fujita, 2007) focused on inclusion relations of quadrilaterals without focus on a particular quadrilateral, but had findings about inclusion relationships with rectangles. Two studies (Herbst and others, 2005, Yavuzsoy-Köse and others, 2019) discussed the economy or necessity and sufficiency of the definitions of some quadrilaterals (square, trapezium, rectangle, etc.), but they didn't focus on a particular quadrilateral. Another study (Monaghan, 2000) attempted to investigate higher order conceptualizations (class inclusions) of students in classifying quadrilaterals. The findings of research showed that students had difficulty in classifying quadrilaterals. Students couldn't provide necessary and sufficient (Herbst and others, 2005, Yavuzsoy-Köse and others, 2019) explanations in their definitions of some quadrilaterals (square, trapezium, rectangle, etc.); as being unsuccessful in seeing that square was a special type of rectangle (Aktaş and Aktaş, 2012; Okazaki and Fujita, 2007); or as treating non-examples of parallelogram as examples of that and exhibited overgeneralization errors (Ulusoy and Çakıroğlu, 2017). Besides, the students didn't use the potentialities of dynamic geometry environments in a rational way in constructing and classifying quadrilaterals (Furinghetti and Paola, 2002) and had difficulties in turning the quadrilaterals into another shape with the tools (e.g. a rectangle into a square, etc.) (Erez and Yerushalmy, 2006). Students were unlikely to develop their higher order conceptualizations (class inclusions) unless they were forced into a Vygotskyan conflict (Monaghan, 2000). Students didn't use different attributes of quadrilaterals in the same way and this influenced the way students classify quadrilaterals. Thus, conceptual understanding of different types of quadrilaterals were gradual and depended on the attributes used (Bernabeu and others, 2018).

1. 2. Aim and Significance

This study attempts to answer of the question: "How well 8th grade students define a rectangle?" Definitions produced by 8th graders may serve as a guide for studying their understanding of the meaning of definition, equivalent definitions, deduction and class inclusion between rectangle and parallelogram. Besides when the definitions they generated are examined, we can understand their Van Hiele geometric thinking levels. The focus was the concept of rectangle, since it is one of the most familiar quadrilaterals in the school mathematics curriculum. The students faced with this concept since their preschool days. When we look at the related literature, there are many studies (Aktaş and Aktaş, 2012, Bernabeu, Moreno and Llinares, 2018; De Villiers, 1998; Erez and Yerushalmy 2006; Furinghetti and Paola, 2002, Herbst and others, 2005, Monaghan, 2010; Okazaki and Fujita, 2007, Ulusoy, 2015, Ulusoy and Çakıroğlu, 2017, Yavuzsoy-Köse and others, 2019) investigated hierarchical classifications or inclusion relations among quadrilaterals and mentioned about rectangle with other quadrilaterals but there were no studies which looked for the inclusion relationships by focusing on rectangle while doing this with analyzing them in terms of Van Hiele geometric levels. This study is important with focusing on the thinking levels of students while investigating their definitions in terms of necessity and sufficiency. So, it is thought that this study will contribute to the related literature.

2. Method

This is a descriptive study examining 8th graders understanding of definition with the focus of rectangle. Survey design was used to examine the definitions of the students.

2. 1. Participants

The subjects for this study were 93 of 8th grade students (36 females and 57 males) at a public school in west part of Turkey, central district.

2. 2. Instrument

The students were administered a written open ended test on the definition of a rectangle. The aim of the questions was to understand how and at what Van Hiele level they define the rectangle, whether they understand class inclusion, definitions

The test was adapted from the written geometry test used by Ahuja (1996). It included 4 open ended questions as:

1. Giving a description of a rectangle as if telling a friend over the telephone,
2. Writing down the properties that are enough to describe a rectangle,
3. Writing down the mathematical definition of the rectangle,
4. Writing another definition of the rectangle if any.

In the first question; the students were asked to describe the rectangle as if telling over the telephone (not a smart phone). The idea was to explain someone what a rectangle was in words without having a visual interaction. In this way, students were expected to give details as much as possible just by using words. In the second question; students were asked to provide properties that were enough to describe a rectangle. In the third question; it was aimed to reveal the knowledge of the students about the definitions of rectangle. And finally, in the last question, the aim was to learn if the students were able to compose another definition.

In order to get content related validity evidence, 5 faculty members (from mathematics education department) and 2 mathematic teachers were asked to evaluate the questions. According to their comments the test was reviewed. Then the pilot study was conducted with 15 of 8th grade students before the main study.

2. 3. Procedure

The test was conducted with students in one-hour lesson period. Students were asked to write whatever they thought and after one-hour period, their papers were collected.

2. 4. Data Analysis

Students' responses were coded by each researcher individually and common expressions were grouped. Then, new codes were generated from these expressions independently and the final codes were generated by the researchers. Besides, the definitions of the students were analyzed according to their necessity and sufficiency similarly with Zazkis and Leikin's (2008) study.

3. Findings

In the first section (3.1) responses of the students to the first question were presented in detail. The explanations of the students while describing rectangle as if telling a friend over the telephone were revealed in terms of how they defined the rectangle (identification format). Students gave examples, made drawings, etc. Then, the explanations of the students regarding different components of the rectangle were shown. The responses of the students were mentioned regarding the language they used and regarding sufficiency and necessity for the first question. In the next section (3.2), responses of the students to the second question were examined in detail. The explanations of the students while writing down the properties that are enough to describe a rectangle were first analyzed in terms of how they defined the rectangle. The other findings were presented similarly with the Section 3.1. In Section 3.3, responses of students about properties and description (for the first and the second question) were presented together in terms of the identification formats (students gave examples, made drawings, etc.) and necessity and sufficiency. In Section 3.4; responses of the students to the third and fourth questions (definition of rectangle) were revealed. Their responses were detailed regarding the language they used, sufficiency and necessity and the identification formats (listing properties, making drawings, etc.)

3. 1. Students' Descriptions of a Rectangle over the Phone

While describing a rectangle; students gave examples, made class inclusions, drawings, visual explanations and explained how to find the area of a rectangle. The responses of the students are shown in Table 1.

Table 1. *The responses to the first question*

Responses to the first question		**n**
Giving examples		21
Making class inclusions connections with	Total (class inclusions)	19
	Quadrilaterals	13
	Parallelogram	4
	Polygon	2
Making drawings		12
Making visual explanations		11
Explaining how to find the area		2

As seen in Table 1, the most frequently used responses were "giving examples" and "making class inclusions". The students mostly preferred to give examples such as "the white board at school", "the flag", "the window", "the picture of Atatürk", etc. While making class inclusions, the students mostly made connections with quadrilaterals. Students might have thought of giving examples around their environment while trying to describe the rectangle on the phone. The responses of the students were different from each other. Total number of the students who did not give examples, make class inclusions, make drawings and visual explanations were 44. The responses (explanations) of these students were grouped with the other 49 students in regarding different components of the rectangle (Table 2). The students gave multiple responses at the same time (e.g. the student 3 made drawings and also he/she made connections with a regular quadrilateral; the student 16 made visual explanations as well as he/she made drawings). The explanations of the students were about different components of the rectangle. The explanations regarding these components are shown in Table 2.

Table 2. *The explanations of the students regarding different components of the rectangle*

	The explanations of the students	**n (%)**
Explaining different components of the rectangle	Sides	23 (28)
	Sides, vertices and angles	12 (15)
	Sides and vertices	11 (14)
	Sides and angles	8 (9)
	Sides, vertices, angles and class inclusions	5 (6)
	Sides, angles and diagonals	5 (6)

	Sides and angles and class inclusions	4 (5)
	Sides and class inclusion	3 (4)
	Sides, vertices and class inclusions	2 (3)
	Sides,angles,diagonals and class inclusions	2 (3)
	Vertices	2 (3)
	Angles and class inclusions	1 (1)
	Sides, angles and bisectors	1 (1)
	Sides,vertices, angles,bisectors and class inclusions	1 (1)
	Angles and vertices	1 (1)
	Total	81 (100)

12 of the students weren't grouped in explaining regarding different components of the rectangle. Within the remaining 81 responses, the majority of the responses were on the sides of the rectangle in describing it as if telling a friend over the telephone. The other components which the students mostly talked about were "sides and vertices" and "sides, vertices and angles". It can be said that students preferred to describe a rectangle in terms of its sides mostly.

When the responses of the students were examined regarding the language they used, it was seen that none of the students used a correct mathematical language. 11 of the students used partially correct language, the use of mathematical language of 27 students was ambiguous while 55 of the students used incorrect language.

"Ambiguous" answers here means that the responses of the students weren't put into any categories of language using. In other words, in ambiguous answers, the language used by the students weren't understood clearly by the researcher.

So, it can be mentioned that most of the students used an incorrect language and there was nobody who used a correct language.

When the explanations of the students were examined in detail, it was seen that none of the students' responses included sufficient and necessary conditions. 28 students stated conditions which were more than necessary like: "It has four sides. The lengths of opposite edges and all angles are equal. All of the edges cuts each other vertically" or" It is a regular quadrilateral. The lengths of opposite edges are equal. One of the angle is 90° and etc. Responses of 16 students included insufficient conditions like: "The long and short edges of the rectangle are equal to each other". Responses of 47 students included insufficient and unnecessary conditions like": It has four sides. The opposite sides are equal. The angles are 90°. In short they are right angles. The lengths of diagonals are equal". The explanations of the students regarding sufficiency and necessity are shown in Table 3.

Table 3. *Explanations and definitions of the students regarding sufficiency and necessity*

Explanations of the students	n(%)
Sufficient and necessary conditions	0 (0)
Insufficient conditions	16 (18)
Sufficient but unnecessary conditions	28 (30)
Insufficient and unnecessary conditions	47 (52)
Total	91 (100)

According to Table 3, it can be said that, the majority of the students' explanations and responses included insufficient and unnecessary conditions and less than half of the students' responses included sufficient explanations and unnecessary explanations at the same time. 2 students (Students 55 and Student 84) weren't put in any category about sufficiency and necessity. The reason was the ambiguity of their responses.

3. 2. The Properties That Students Wrote to Describe a Rectangle

While writing down the properties that were enough to describe a rectangle; students gave examples, made class inclusions, drawings, visual explanations and explained how to find the area of a rectangle. The responses of the students are shown in Table 4.

Table 4. *The responses to the second question*

Responses to the second question		**n**
Giving examples		8
Making class inclusions connections with	Total (class inclusions)	11
	Quadrilaterals	10
	Parallelogram	1
Making drawings		8
Making visual explanations		5
Explaining how to find the area		1

As seen in Table 4, the responses most frequently used were "making class inclusions", making drawings" and "giving examples". The students mostly preferred to give examples such as "window", "door", "the refrigerator", "banknote", "photocopying paper", etc. While making class inclusions, the students mostly (10 of 11) made connections with quadrilaterals.

The responses of the students were different from each other. 65 of the students did not give any examples, make class inclusions, drawings, visual explanations and etc. These students made explanations about the components of the rectangle and these explanations were put together with the remaining 28 students (who gave examples, made drawings, etc.) into the groupings which are shown in Table 5.

The explanations of the students were about different components of the rectangle. The responses of 11 students weren't be able to be grouped in the explanations regarding different components of the rectangle. 5 of these students were the ones who had already made visual explanations and gave examples; 6 of these students weren't grouped in the responses of giving examples, making drawings, etc. In other words, 6 students weren't grouped in any to the second question. The explanations regarding components are shown in Table 5.

Table 5. *The Explanations of the students regarding different components of the rectangle*

	The explanations of the students	n (%)
Explaining different components of the Rectangle	Sides	23 (28)
	Sides and angles	19 (23)
	Sides and vertices and angles,	11 (13)
	Sides and vertices,	7 (9)
	Sides and vertices angles and class inclusions,	5 (6)
	Sides and angles and class inclusions,	4 (5)
	Sides and angles and diagonals	3 (4)
	Angles	3 (4)
	Vertices	2 (2)
	Angles and vertices,	2 (2)
	Sides and class inclusion,	1 (1)
	Sides and angles and diagonals,	1 (1)
	Sides and angles and vertices and diagonals.	1 (1)
	Class inclusion	1 (1)
	Total	83 (100)

As seen in Table 5, the majority of the components students talked about were the sides of the rectangle. The other components which the students mostly talked about were "sides and angles" and "sides, vertices and angles".

When the responses of the students were examined regarding the language they used, it was seen that only 2 of the students used a correct mathematical language. 5 of the students used the mathematical language partially correct, the use of mathematical language of 25 students was ambiguous and 61 of the students used incorrect mathematical language, So, it can be mentioned that most of the students used incorrect language and only a few students succeeded in using an accurate language.

When the explanations of the students were examined in terms of their necessity and sufficiency, it was seen that none of the students provide sufficient and necessary conditions. The explanations of the students are shown in Table 6.

Table 6. *Explanations and definitions of the students regarding sufficiency and necessity*

	Explanations of the students n (%)
Sufficient and necessary conditions	0 (0)
Insufficient conditions	30 (33)
Sufficient but unnecessary conditions	20 (22)
Insufficient and unnecessary conditions	40 (45)
Total	90 (100)

As seen in Table 6, the responses of 90 students were put into the groupings in terms of their necessity and sufficiency. The responses of the remaining 3 students weren't grouped according to their necessity and sufficiency. Two of them (Student 63 and Student 82) wrote "I don't know", the other (Student 77) wrote "I describe by making drawings".

70 of the students had explanations including insufficient conditions and 40 of these conditions were insufficient and unnecessary. So, it can be mentioned that the majority of the students made explanations including insufficient conditions. It was seen that the students who described a rectangle in terms of its sides had already "insufficient" and "insufficient and unnecessary" explanations. For example, a student (student 4) wrote "the opposite sides are equal and they are longer than the other opposite sides. The other opposite sides are equal and shorter than the others". This student described the rectangle in terms of its sides and made his/her explanations in insufficient and unnecessary conditions. Another student (student 18) wrote" It is enough to say it has two long and two short sides and the opposite sides are equal". This student described the rectangle according to its sides and his/her explanations were insufficient.

3.3. The Evaluation of Students' Responses about Properties and Descriptions of Rectangle Together

The responses of the students for the first and second questions were examined in common (Table 7).

Table 7. *The responses to the first and the second question*

<table>
<tr><th colspan="2">Responses to the first and the second question</th><th>n (first question)</th><th>n (second question)</th></tr>
<tr><td colspan="2">Giving examples</td><td>21</td><td>8</td></tr>
<tr><td rowspan="4">Making class inclusions connections with</td><td>Total (class inclusions)</td><td>19</td><td>11</td></tr>
<tr><td>Quadrilaterals</td><td>13</td><td>10</td></tr>
<tr><td>Parallelogram</td><td>4</td><td>1</td></tr>
<tr><td>Polygon</td><td>2</td><td>-</td></tr>
<tr><td colspan="2">Making drawings</td><td>12</td><td>8</td></tr>
<tr><td colspan="2">Making visual explanations</td><td>11</td><td>5</td></tr>
<tr><td colspan="2">Explaining how to find the area</td><td>2</td><td>1</td></tr>
</table>

As seen in Table 7, all of the responses of the students to the first question were higher in number when compared with the second question. This can be because the students might have a tendency to talk more in describing as telling on telephone (question 1) than writing the properties that were enough to describe a rectangle (question 2). They might have thought to write or say less in in a sufficient way than making explanations on the phone.

Students gave more examples in giving a description as if telling on telephone rather than giving examples in writing down the properties that were enough to describe a rectangle. So, it can be said that the students were not able to describe a rectangle in terms of its properties while visualizing and imagining. Besides, students made more drawings, visual explanations in the first question vis-à-vis the second question. The reason for this may be the tendency to talk less in the second question while trying to write the properties that were enough to describe a rectangle.

When the explanations of the students were examined comparatively in the first and second question in terms of sufficiency and necessity (as seen in Table 8), it was seen that the most frequently seen was the insufficient and unnecessary explanations of students. In other words, students mostly had insufficient and unnecessary explanations for the first and second questions.

Table 8. *Explanations and definitions of the students regarding sufficiency and necessity for the first and the second question*

	Explanations of the students First question n (%)	Explanations of the students Second question n (%)
Sufficient and necessary conditions	0 (0)	0 (0)
Insufficient conditions	16 (18)	30 (33)
Sufficient but unnecessary conditions	28 (30)	20 (22)
Insufficient and unnecessary conditions	47 (52)	40 (45)
Total	91 (100)	90 (100)

The students were higher in number in insufficient and unnecessary explanations in the first question than the second one. The reason for this may be the tendency of the students to say more to tell on the phone. But, it was seen that, nearly half of the students (40 of 90) made insufficient and unnecessary explanations in the second question. So, it can be mentioned that students did not have enough experience on this subject (necessary and sufficient conditions) so, they couldn't understand the meaning of "the properties that were enough to describe a rectangle.

The language students used were examined for the first and second question together. As seen in Table 8, students mostly used the language incorrectly and couldn't be able to use a correct language in the first and second question.

3. 4. Students' Definitions of Rectangle

When the students were asked for constructing the definition of a rectangle, 17 of the students constructed definitions and among these definitions, none of them ensured sufficient and necessary conditions, 5 of them included insufficient, 6 of the definitions included sufficient but unnecessary conditions and 6 of them holded insufficient and unnecessary conditions

While trying to define the rectangle, most of the students (52 of 93) listed the properties of the rectangle and many students (41 of 93) made drawings and some of them (20 of 93) used symbols (10 of the students only wrote the symbol or showed the symbol on the or showed the symbol on the or showed the symbol on the figure; 7 of the students used the symbols in the formulas and 3 of the students wrote the formulas of perimeter-area of the rectangle). The responses of the students are shown in Table 9.

Table 9. *The responses to the third question.*

Responses	(n)
Listing properties	52
Making drawings	41
Constructing definitions	17
Using symbols	20

So, it can be said that when the students were asked to make a definition, most of the students were not able to construct a definition and only a few students were able to construct a definition. Most of them preferred to list the properties of a rectangle instead of defining it.

The students gave multiple responses. For example, while some students (2 of 93) listed the properties as well as they constructed definitions, some of them (25 of 93) were contented with only listing the properties. Some students (24 of 93) listed the properties of a rectangle without constructing a definition, making drawings or using symbols. Some of the students (13 of 93) preferred to use listing, making drawings and using symbols at the same time while some of them done with making drawings only. Some of the students preferred (27 of 93) both preferred to construct definitions and to list the properties.

Among the students who constructed a definition, only one of them could be able to use a correct mathematical language. 6 of the students used the mathematical language partially correct and the use of mathematical language of 3 students was ambiguous while 7 of the students used an incorrect mathematical language. When the definition of the only student who used a correct language was examined in detail, it was seen that the definition included sufficient but unnecessary conditions. He made a drawing of a rectangle with four right angles and he wrote: "It is a figure which has four sides and which the lengths of the sides of it are equal to each other". So, it can be said that the only student who could be able to use the language in a right way couldn't be able to construct a definition in a right way. He couldn't define the rectangle in sufficient and necessary words. 5 of the students defined the rectangle including insufficient conditions while 6 of them tried to define the rectangle including Insufficient and unnecessary conditions. 6 of them made defnitions including sufficient but unnecessary conditions

When the students were asked to write another definition of the rectangle if any; 61 of the students said "Yes", 21 of the students said "No", 1 of them said "Maybe", 3 of the students didn't give an answer, 7 of the students gave ambiguous answers. Here are two examples of an ambiguous answer: "Student 1: "Circle and circumference are different from ellipse because they have edges and vertices" Student 3: "Everybody has different definitions".

3 of the students said "No" but gave different types of answers. The student 44 who said "No" gave an ambiguous answer: "We can't use a different definition to explain the rectangle. But we can we can rotate the rectangle and construct a cylinder". When the students who said "yes" were examined in detail, it was seen that some of the students said "Yes" but gave no answer, the others said "Yes" and gave different types of answers. One of the students (student 77) said "Yes" and gave an ambiguous answer: "We can draw this rectangle laterally or trapezoid". The different responses of the students who said "Yes" are shown in Table 10.

Table 10. *Different responses of the students (for the fourth question) who said "yes" in generating a new definition.*

Responses		n
Making definitions	Defining with the same words	4
	Defining with different words	5
Making no explanation		15
Giving examples about a rectangle (blackboard, table, book, etc.)		19
Adding properties to their old definitions (while 2 of the 13 explained the properties which they have written the listed properties)		13
Explaining how the rectangle was generated		3
Explaining with drawings		10
Associating with upper class inclusion while explaining defining		6

When the responses of the students were examined, it was seen that only a few students attempted to construct a definition and among these definitions only 5 of them were the ones with new words. As seen in Table 10, it was seen that students who said "Yes" gave multiple responses. For example, student 4 explained how the rectangle was generated, also he explained with drawings. Student 18 associated with upper class inclusion, explained with drawings and generated a new definition. Here is the response of the student 18:” rectangle is a corrected form of a parallelogram, a rectangle is an enlarged from the width form of a square (with drawings).

When the definitions of the students were examined, it was seen that none of the definitions of the students ensured sufficient and necessary conditions, 4 of the definitions of the students included insufficient, 1 of the definitions of the students included sufficient but unnecessary, 4 of the definitions of the students included insufficient and unnecessary conditions. It can be said that, only 1 student could construct a definition in a sufficient way but his definition had unnecessary expressions.

4. Conclusion

When we look at the responses to the questions as a whole, it was seen that no student could make an explanation or a definition providing sufficient and necessary conditions. Therefore, it can be said that students had difficulty in making sufficient and necessary explanations while defining or describing a rectangle. These finding is consistent with Herbst and others’ (2005) and Yavuzsoy-Köse and others’ (2019) findings, the studies found that very few students could provide definitions providing economical /necessary and sufficient conditions. As the students in the study of Herbst and others’ (2005) were high school students, it can be said that while providing economical or necessary and sufficient conditions for definitions are hard for high school students, it is not surprising that it is hard for middle school students. The challenge of providing economical or necessary and sufficient conditions for definitions was also true for middle school students (Yavuzsoy-Köse and others, 2019). It was found that some middle school students could not understand what the necessary and sufficient condition of a rectangle was (based on a given rectangle definition “a quadrilateral with three right angles”. These students couldn’t determine the definition if it belonged to a square or a rectangle. It can be said that the results are consistent with this study.

In the first and second questions, no student could use a correct mathematical terminology. In the second and third question, only a few students could use the correct terminology. So, when we look at the responses of all questions as a whole, it can be mentioned that the majority of the students used incorrect terminology.

In the first and second questions, students mostly described a rectangle in terms of its sides and explanations of these students had already “insufficient” and “insufficient and unnecessary” conditions so it can be said that students mostly preferred to use an insufficient component to describe a rectangle.

While trying to define the rectangle, most of the students (52 of 93) listed the properties of the rectangle and only a few students were able to construct a definition. So, it can be said that students were unsuccessful in defining the rectangle. Among the students who were able to construct a definition, only one of them could be able to use a correct mathematical language. When the definition of the only student who used a correct language was examined in detail, it was seen that the definition included sufficient but unnecessary characters. All of the students in the study were unable to define a rectangle in sufficient and necessary words. In other words, it can be concluded that students weren’t successful in constructing a necessary and sufficient conditions for a definition. When the students were asked to write another definition of the rectangle if any; 66 of the students said “Yes” but only 5 of them could define the rectangle with different words. Only 1 student could construct a definition in a sufficient way but his definition was the same as the third question and had unnecessary expressions. So, it can be said that students were unsuccessful in constructing a new definition in necessary and sufficient conditions.

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