Improving Our Practice as Mathematics Teacher Educators through Teaching Research

Su Liang, Raquel Vallines Mira, Priya V. Prasad, & Cody L. Patterson

University of Texas at San Antonio

Received 9 November 2018; Accepted 1 February 2019

Four mathematics teacher educators from a large, minority-serving university formed a teaching research group in Fall 2016. The goal for this project is to establish a repeated cycle of improving our mathematics content course for pre-service teachers and to contribute a shared knowledge base which rests on foundation of well-defined learning goals in mathematics courses for elementary pre-service teachers. Guide by the Continuous Improvement framework (Berk & Hiebert, 2009), we utilized a data-driven approach to improving teaching, as well as embedding a discussion of classroom implementation into an investigation of an innovation (or, in our case, a mathematical task). In this paper, we present an example of iterative task design for the topic of geometric similarity, we hope to share this as a model of professional development for mathematics teacher educators that highlights the benefits to our students and to ourselves.

INTRODUCTION

Due to the lack of a well-defined knowledge base for teaching elementary pre-service teachers (PSTs), mathematics teacher educators (MTEs) are often left to make their own judgments about the proper scope and sequence of topics for mathematics content courses for PSTs (National Research Council, 1996; Center for Research in Mathematics and Science Education, 2010). Berk and Hiebert (2009) proposed the Continuous Improvement model for "systematically improving the mathematics preparation of elementary teachers, one lesson at a time" (p. 337). This process, in addition to helping the field build a knowledge base for elementary teacher mathematics education, also allows MTEs to become reflective practitioners of teacher education (Thanheiser et al, 2016). Moreover, most university-based MTEs, whether their preparation was in mathematics or mathematics education, did not have the opportunity to study and develop the practice of mathematics teacher education (Nicol, 1997; Heaton, 2000; Crespo & Speer, 2004). This issue has not been ddressed until recently. Thus, MTEs can feel unprepared and unsupported to take on the work of pre-service teacher preparation, especially in a new context. The project described in this paper grew out of a similar experience: all the authors of this paper, while experienced in (and in some cases, scholars of) teacher education, struggled without support to understand the challenges of pre-service teacher education in a new institution. By adapting the Continuous Improvement framework (Berk & Hiebert, 2009), we implemented a form of lesson study that helped us develop our understandings of elementary PSTs' thinking and improve the curriculum of our elementary pre-service teacher courses.

In this paper, specifically, we illustrate part of the project by describing an example of several iterations of a single mathematics lesson over serval semesters. One goal in presenting this example to share our experience with other teacher educators by showing how conducting the Continuous Improvement process helped us develop our knowledge of PSTs' mathematical thinking. A second goal is to reflect on how the process helps us, as mathematics teacher educators, to develop tasks that support PSTs' learning.

The lesson we describe comes from the second course in a sequence of two mathematics content courses for pre-service elementary teachers. The second course focuses on algebra, geometry, statistics and probability. In particular, the lesson we will discuss in this paper is similarity. We chose similarity as one of the topics to be one of our research lessons because existing research documented that middle school students in the U.S. struggle with similarity (Masters, 2010; Seago et al, 2013). Starting Fall 2016, the teaching cycles went as follows:

Revision	Timeline
Initial design	Implemented Fall 2016, 2 sections, Implemented Spring 2017, 1 section;
First revision	Fall 2017, 2 sections
Second revision	Implemented Spring 2018, 3 sections

THE TEACHING RESEARCH GROUP

Four mathematics teacher educators from a large, minority-serving university formed a teaching research (Liang, 2013) group in Fall 2016. The goal for this project is to establish a repeated cycle of improving our mathematics content course for pre-service teachers and to contribute a shared knowledge base which rests on foundation of well-defined learning goals in mathematics courses for elementary pre-service teachers (PSTs). Cai and his colleagues (2017a, 2017b) stressed the importance of a data-driven approach to improving teaching, as well as embedding a discussion of classroom implementation into any investigation of an innovation (or, in our case, a mathematical task). The process we followed requires this integration, since all decisions about the effectiveness of the task have to take pre- and post- assessment results and implementation into account. In presenting this example of iterative task design, we hope to share this as a model of professional development for MTEs that highlights the benefits to our students and to ourselves.

THE CONTINUOUS IMPROVEMENT FRAMEWORK

This ongoing teaching research project was guided by the Continuous Improvement framework (Berk & Hiebert, 2009), following a process of repeated cycles of planning, classroom implementation, analysis, revision. Connections between teaching and learning are hypothesized to motivate each cycle of the process. As a teaching research group, in line with the Continuous Improvement (CI) framework (Berk & Hiebert), we implemented the following cycle: (1) design a lesson that targets a particular student misconception or deepens understanding of a particular mathematical idea, (2) develop hypotheses about anticipated student responses to the tasks provided in the lesson, (3) collect data in the form of student work, responses to formative assessments, and recordings of classroom discourse, and analyze these data sources for evidence of the desired student learning outcomes, and (4) record this information and use it to revise the lesson for use in subsequent semesters.

Our phases of implementation of the task design cycle included the following activities, using the phases defined by (Liljedahl, Chernoff, & Zazkis, 2007):

- *Predictive analysis* – Course instructors reflected on their prior experience of teaching the focus concept and PTs' likely responses. All research group members investigate the literature on student learning of the concept. We collectively develop a set of learning goals and the initial version of the task. Instructors administer pre-assessments.
- *Trial* – Course instructors implement the task by facilitating small-group discussions and collaborative work, with the rest of the research group observing. Classes are videotaped, small group interactions are recorded, and field notes are taken. Often, instructors may reflect on the facilitation of the lesson between a class they observed and a class they are about to teach, leading to refined instructional choices during their own facilitation. After the task, instructors administer post-assessments.
- *Reflective analysis* – Observers and instructors meet to discuss observations of the lesson, focusing especially on what was observed during PTs' small group work.
- *Adjustment* – Based on the observations and reflections, we collectively revise the task.

Multiple sections of the course are offered each semester and each lesson plan was implemented by 1-3 of the four members of the research group.

RESEARCH QUESTION

This study was conducted in pre-service teachers' mathematics classes to answer the following research questions:

1. To what extent would the continuous teaching process inform Mathematics Teacher Educators (MTEs) of pre-service teachers' (PSTs) understanding about similarity?

2. To what extent would the continuous teaching process help MTEs to develop tasks that support PSTs' learning about similarity?

METHODS

This project was implemented on the second course in a sequence of two mathematics content courses for pre-service elementary teachers. The second course focuses on algebra, geometry, statistics and probability. We began the cycle described with meeting once a week for 1-2 hours per meeting. Before each meeting, team members reviewed the literature about children's and PST's understanding of similarity, searched for or designed preliminary attempts at hand-on in-class activities, and thought about pedagogical issues that were likely to emerge. During the meetings, our activities included reflecting on previous experiences teaching similarity to PSTs as three members of the team were experienced instructors of these courses, defining and learning goals and hypothetical learning trajectory (Clements & Sarama, 2009), developing tasks and pre/post-assessments, recording anticipated student responses, and implementing/analyzing pre-assessment results. When designing the lessons, we consulted the learning goals of the Common Core State Standards for Mathematics (NGACBP & CCSSO) on these topics. After designing the first iteration of the lesson, the members of the research team who teach the course implemented the lesson plan in their classrooms, while the rest of the team observed class, took field notes, and recorded whole-class discussions. In addition, student work and discussions in small groups were also recorded by LiveScribe pens. Data was collected for the research team to analyze and findings will be used to revise/refine the lesson for improvement.

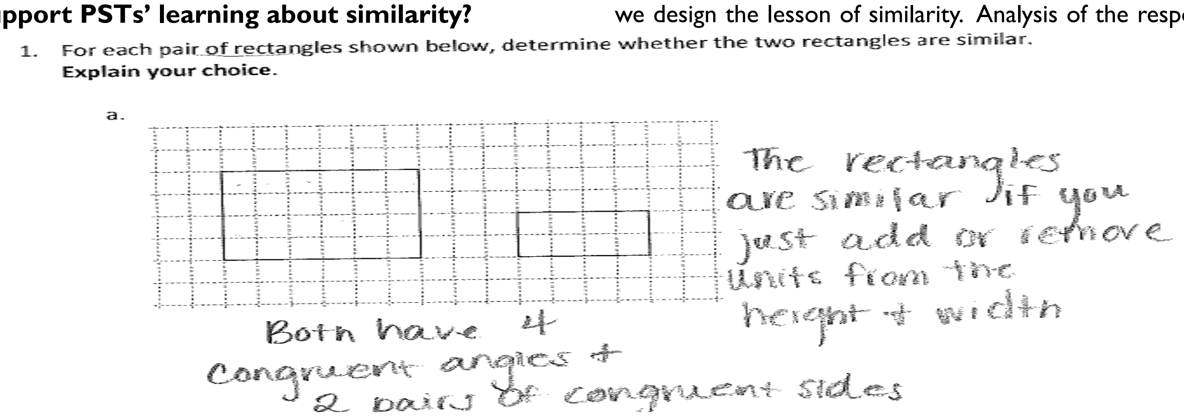
Data collection included pre- and post- assessment responses, video tapes of two classes, audio tapes of the teaching research group meetings, meeting notes of discussions at the teaching research group meetings, field notes of class observations, PSTs' recordings of their group work and discussions by LiveScribe pens, and PSTs' work. Data were coded and categorized to generalize emerging patterns. Taking two examples, we coded this type of PST's responses as Additive Thinking (see below). We coded another type of PST's responses as using appearance of shapes (see next page).

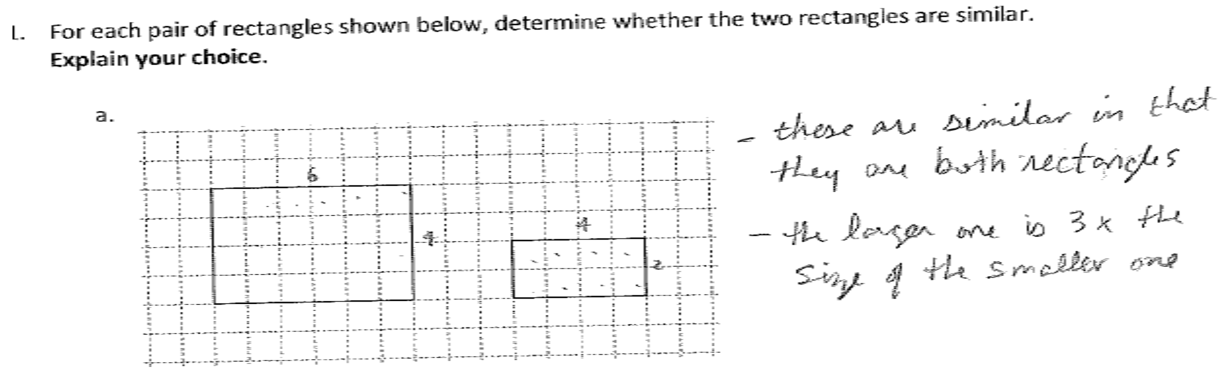
Videos were watched repeatedly to verify information and confirm accurate interpretation. Constant comparisons (Corbin & Strauss, 2008) and triangulation (Patton, 2002) were utilized to synthesize the data.

Similarity Lesson Design and Development

Lesson Design in Fall 2016

Intending to know how our PSTs understand the concept of similarity, we conducted a pre-assessment in the two sections before we design the lesson of similarity. Analysis of the responses to




I. For each pair of rectangles shown below, determine whether the two rectangles are similar. Explain your choice.

a.

the pre-assessment revealed that most of the pre-service teachers in the two classes didn't show understanding of the concept of similarity. Only one student (out of 58) demonstrated thorough understanding of similarity with appropriate mathematical reasoning. Five students (out of 58) showed understanding of similarity but did not show appropriate mathematical reasoning. Fifty-two students did not show evidence of understanding the concept of similarity and their responses can be categorized as: using appearance of shapes (polygons with the same number of sides and interior angles are similar) (14 out of 58); comparing the areas/circumferences of two polygons (12 out of 58); thinking additively (7 out 58); guessing/no answers (19 out of 58). None of the students used terms such as *corresponding angles* or *corresponding sides* when reasoning. The pre-assessment results provided the evidence for us to understand PST's preconceptions (Morrison & Lederman, 2003) about similarity.

Based on the pre-assessment results, we defined our lesson goals as follows:

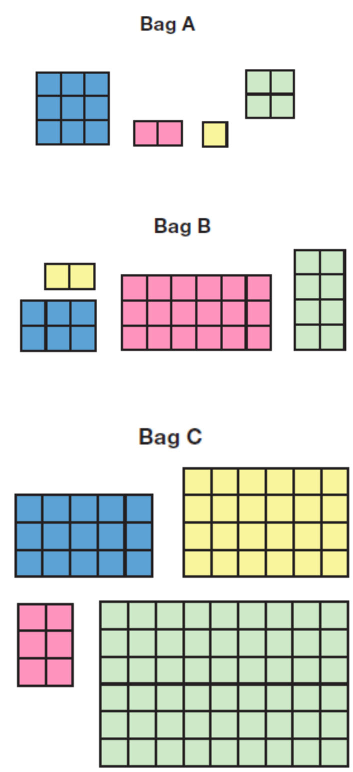
1. Students should be able to determine whether two figures are similar, and justify this determination.
2. Students should be able to use proportionality and scale factors to determine measurement of similar figures.

Looking for tasks that could facilitate our success in reaching the goals, we searched the existing research in the literature. Some existing research indicated that traditional approaches taught similarity from a statictical and measurement point of view, which led to confusion and misconceptions. Researchers have reached a consensus that similarity should be taught from the perspective of geometric transformations (Lappan & Even, 1988; Seago, Driscoll, & Jacobs, 2010, & Seago et al, 2013). Aiming to give students the opportunity to observe dynamic transformations, we decided to use a GeoGebra demonstration to introduce the concepts of dilation and similarity. Our lesson started discussing how a quadrilateral was dilated (enlarging or shrinking) to form a new similar quadrilateral and engaged students in negotiating a definition for similarity based on what they observed from the dynamic transformations. In the introductory part of the lesson, we tried to show students: 1) a dilation of a geometric object is a scaling of a geometric object. It preserves the angles of a polygon, as well as allowing the side lengths of the polygon to be in the same proportion; 2) two geometric objects are similar if one object can be obtained from another after a series of rigid transformations or dilations; in other words, two polygons are similar if corresponding pairs of angles are congruent, and corresponding pairs of sides are in constant proportion; and 3) the constant proportion is called a scale factor.

After the introductory part, a small-group activity followed to engage students in problem solving situations that explore the concept of similarity. The problems were purposefully selected in order to help students better understand similarity and its application through the process of solving problems collaboratively.

The group activity in Fall 2016 included the following problems:

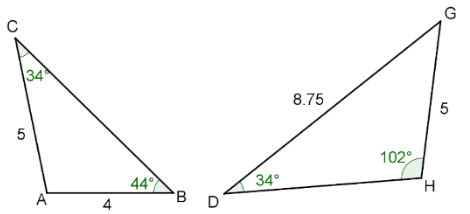
1. The Sorting Rectangle Problem (Seago et al, 2013) For each "bag" (or collection) of rectangles given below, determine which one doesn't belong and why.

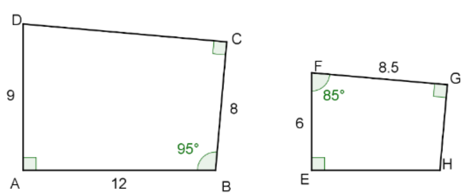


2. Triangle ABC is a right triangle whose legs have measures AB = 6 cm, AC = 4 cm, and an angle with the measure $\angle B = 34°$. Another right triangle DEF has two legs with measures DE = 9 cm, DF = 6 cm, and an angle with measure $\angle F = 56°$. Are these two right triangles similar? Justify your answer.

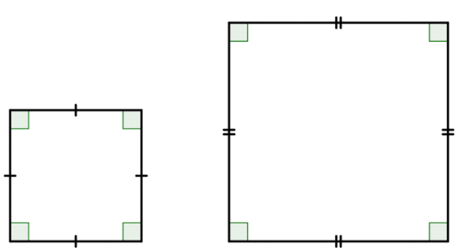
3. Maps are representations that are geometrically similar to the actual layout of a city. In a city map (scale 1:9000), the lengths of Main Street and Broadway on the map are 16 cm and 10 cm respectively.
   a. What are the actual lengths of Main Street and Broadway in meters?
   b. What is the ratio of the lengths of Main Street and Broadway on the map? What is the ratio of the actual lengths of Main Street and Broadway? Why do you think this is true?
   c. Suppose on the map, Euclid Street is $1/5^{th}$ the length of Main Street. What is the ratio of the actual lengths of Euclid Street and Main Street? Justify your answer

4. Each pair of figures given below are similar to each other. Find the measures of the missing angles and side lengths.





5. Consider the two figures shown below. Are these two figures similar? Why or why not?

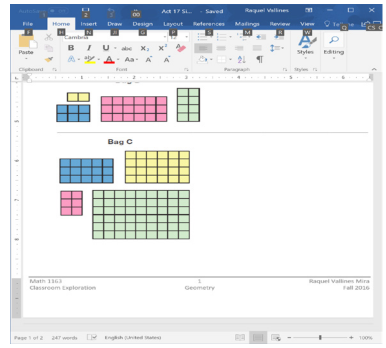


Two faculty in our teaching research group implemented the similarity lesson described above in Fall 2016 and Spring 2017. Based on the data collected from this cycles of teaching, we revised the lesson plan for next cycle of teaching in Fall 2017. Specifically, we revised the tasks for the similarity lesson based on: 1) Analysis of pre- and post-assessments from Fall 2016 and Spring 2017; 2) Observations of students' work during implementations of the lesson in Fall 2016 and Spring 2017; and 3) Discussions among members of the Continuous Improvement team**.** This led to changes in some of the tasks in the lesson, and removal of other tasks.

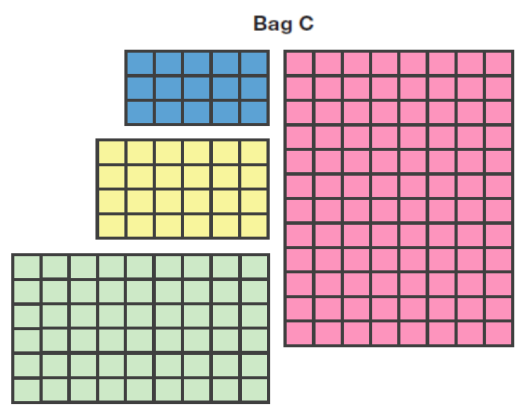
The Revised Lesson Design in Fall 2017

For the Sorting Rectangle Problem, we revised the bag C as follows:

The Original



The Revised

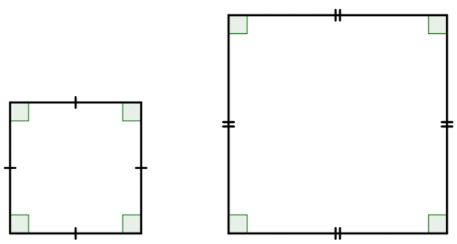


In the original version of Bag C, there are integer scale factors (2 or 3) either between the yellow rectangle and the pink rectangle or between the green rectangle and the pink rectangle. Intending to provide opportunities for PSTs to recognize that it is common that two similar polygons can have non-integer scale factor, we revised the rectangles in the bag C. In the revised task, a non-integer scale factor (3/2 or 3/4) must be recognized in order to demonstrate that the green rectangle is similar to either the yellow or pink rectangle. This task was featured to lead PSTs to visualize that similarity cannot be determined by tiling the figures except in some special cases, and to work toward flexible thinking about similarity in terms of dilations rather than tiling.

We eliminated the "Two Squares" task which is the number 5 in the group activity (see below), considering some students may over generalize that polygons are similar if corresponding pairs of interior angles are congruent or a similar polygon can be obtained by increasing the same amount to each side length. These were two common misconceptions about similarity based on the assessment results.

https://doi.org/10.20429/ijsotl.2019.130212

1. Consider the two figures shown below. Are these two figures similar? Why or why not?



We also eliminated the "Right Triangles" task, which was number 2 in the group activity (see below), or two reasons. First, this task didn't target the conceptions of similarity we wanted to develop in this lesson: two figures are similar if one can be obtained from the other by rigid motions and dilations. Second, our observations of student group discussions found that the task was more conveniently solved using the SAS triangle similarity theorem, which diverted students thinking to investigate the Pythagorean Theorem.

> Triangle ABC is a right triangle whose legs have measures AB = 6 cm, AC = 4 cm, and the measure of angle B is 34°. Another right triangle DEF has two legs with measures DE = 9 cm, DF = 6 cm, and the measure of angle F is 56°. Are these two right triangles similar? Justify your answer.

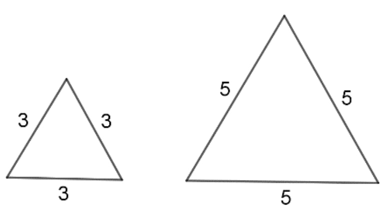
Again, two faculty in the teaching research team implemented the revised lesson discussed above in two sections of the course in Fall 2017. Based on analysis of data collected from this teaching cycle and discussions among faculty members in the teaching research group, the lesson was revised again for the next cycle of teaching in Spring 2018.

The Second Revised Lesson Design in Spring 2018

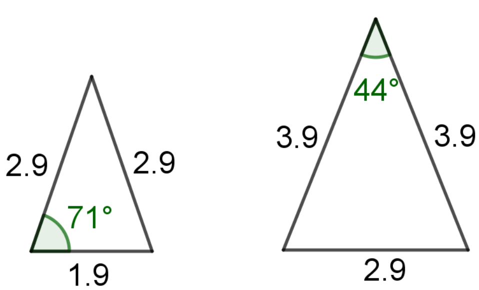
We added two problems to the revised task used in Fall 2017, in order to provide PSTs with an opportunity to see variations of similarity between triangles and between trapezoids. Following the first Problem of Sorting Rectangles, the two problems are:

2. Are the following triangles similar? Explain your reasoning.

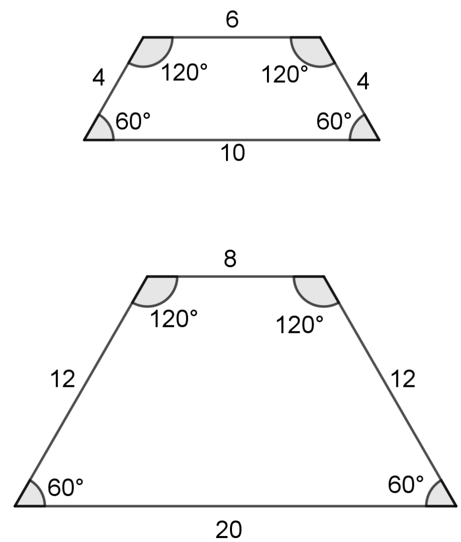
a.



b.



3. Explain why the two given trapezoids are not similar.



Can you change the lengths of some of the sides, **without changing the angles**, to make these two trapezoids similar?

These two problems were utilized to address over-generalizing that a similar polygon can be obtained by adding the same amount to each side length (additive thinking) or that polygons are similar if corresponding pairs of interior angles are congruent.

Problem 2a has two equilateral triangles. Based on our previous observations, students may perceive that each side of the small equilateral triangle increases by 2 and then generalize that two polygons are similar if each side length of one polygon increases by the same amount comparing to each side length of another polygon. Problem 2b was developed to address this possible misconception. In Problem 2b, the isosceles triangles are not similar, even though each side length of the bigger triangle increases by the same amount compared to that of the smaller triangle, because the two triangles' corresponding interior angles are not congruent. The problem provides a good example that leads students to examine their thinking in case they over-generalize.

Problem 3 gives two trapezoids which are not similar. Although the two trapezoids have congruent corresponding interior angles, their corresponding sides are not proportional or in other words their corresponding sides don't have the same scale factor. This problem was developed to show students that polygons are not necessary similar if corresponding pairs of interior angles are congruent.

FINDINGS AND DISCUSSION

As teacher educators, through the ongoing process of integrating our research into teaching practice, we have gained new knowledge about teaching PSTs similarity, which includes PSTs' preconceptions about similarity and a Hypothetical Learning Trajectory for similarity that can be a guide for MTEs to develop effective similarity lesson plans.

PSTs' Preconception about Similarity

Analysis of the 123 pre-assessment responses of four sections in total from Fall 2016, Fall 2017, and Spring 2018, indicates that only about 15% of the PSTs (18 out of 123) correctly answered the pre-assessment questions with appropriate reasoning and about 85% of the PSTs did not show evidence of their understanding of the concept of similarity. The pre-assessment responses were categorized as:

- Additive Thinking

  Explaining that the two shapes are similar because their side lengths increased the same amount.

- By Appearance of shapes

  Explaining that the two shapes are similar because they have the same shape (eg. They are both rectangles).

- By area/perimeters

  Determining if the two shapes are similar or not similar based on their areas/perimeters.

- Inappropriate Explanation

  Having right answers with inappropriate reasoning.

- Not Aware of Rotation

  Not realizing that a polygon can look different after rotation.

- Not Sure/No Answer

  Showing no evidence of understanding similarity.

- Proportional Reasoning

  Reasoning by scaling or Length-Width Ratio.

The following table shows the distribution for each category.

Category	%
By Area/Perimeter	24%
Proportional Reason	16%
Not Aware of Rotation	14%
Inappropriate Explanation	13%
Additive Thinking	12%
By Appearance	12%
Not Sure/No Answer	9%

As indicated in the table, almost one fourth of these PSTs compared the areas or perimeters of two rectangles to determine their similarity; only 16% of them used proportional reasoning; 14% of them didn't recognize that a polygon can look different after rotation; 13% of them provided right answers with inappropriate reasoning (e.g., the two rectangles are similar because their sizes are doubled);12% of them determined similarity of two polygons by their appearance; another 12 % of them thought additively that two polygons were similar if their side lengths increased by the same amount; 9% of them either did not show evidence of understanding similarity or left the questions unanswered.

Additionally, we find that the participating PSTs did not know the special terms for similarity such as *corresponding sides, corresponding angles*, *dilation*, and *scale factor.* Only one student used the term *corresponding sides* and another student used *dilation* when reasoning to support their answers; none used the terms *corresponding angles* or *scale factor*. They were not able to justify their answers with clear and appropriate reasoning because of a lack of knowledge of the special terms for similarity.

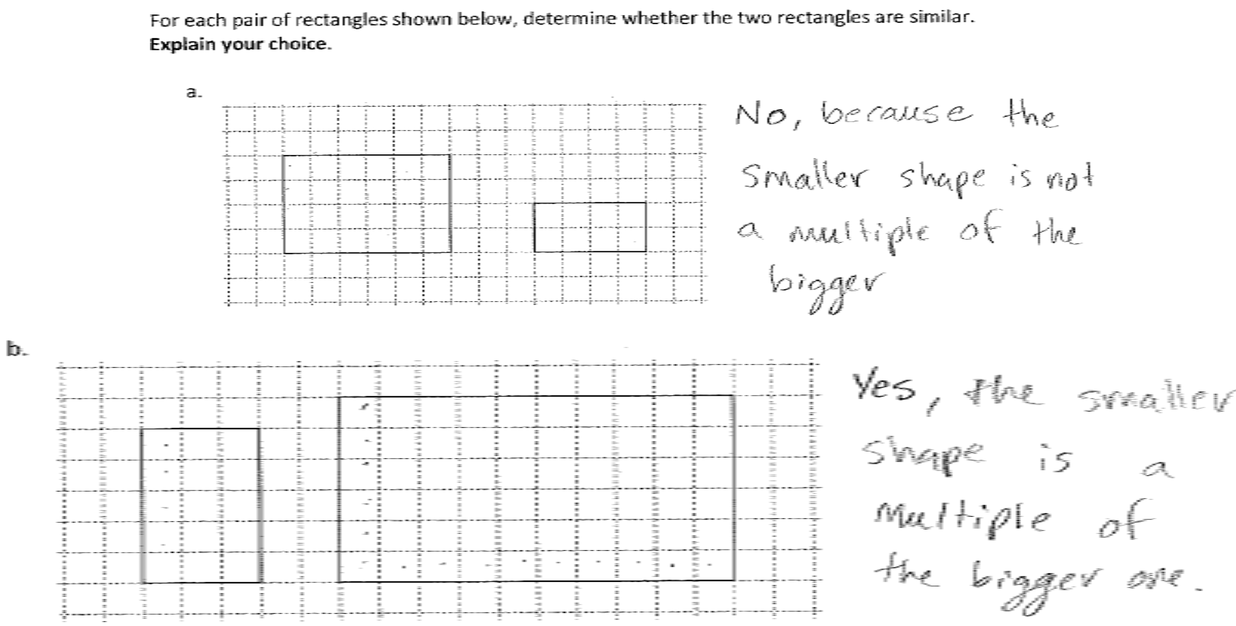
Hypothetical Learning Trajectory for Similarity

Based on the pre-assessment responses, almost all of our participating PSTs were not familiar with vocabulary of similarity terms. Hansen and his colleagues (2014) argued that teachers' vocabulary of geometry terms is crucial for students to understand the concepts in geometry and lack of knowing the vocabulary has caused a variety of mistakes. Because of not knowing the terms, most of PSTs were not able to explain their answers clearly in an appropriate way and many of them relied on appearance to determine the similarity of two shapes instead of considering the properties. For example, in the preassessment a PST explained her answers as seen on the following page:

According to the revised Bloom's Cognitive Taxonomy (Anderson & Krathwohl, 2001), knowledge is classified as different types such as terminology, facts, sequences, classifications, generalizations, theories and structures, etc. The first level of knowledge cognization is factual knowledge including terminology and facts. Students must first know terminology and facts and then be able to further explore and comprehend the interrelationships among the involved factors. In the case of learning similarity, after students know the terminology and facts and are able to understand the interrelationships among the corresponding angles and corresponding sides of different shapes, they then are able to understand properties of figures and use these properties to determine similarity or solve similarity related problems. Guided by the revised Bloom's Cognitive Taxonomy, we created a diagram below that hypothesizes similarity learning trajectory that may guide MTEs to develop more effective lesson plan for PSTs. This hypothetical learning trajectory for similarity will be tested by our continuous teaching research in the near future.

CONCLUSION

Our Continuous Improvement team members have been gaining knowledge of teaching PSTs from conducting successive revisions of the lessons. Three faculties in this team actually taught the course at different semesters (Fall and Spring 2016) or the same semester (Spring 2017). We agree with Cai and his colleagues (2017) that teaching practice was an integral part of research. As teacher educators, we learned and accumulated our knowledge for teaching PSTs through integrating our research into our teaching practice. The collaborative revision process reinforces our own Knowledge of Content for PSTs and Specialized Content Knowledge (Ball, Thames, & Phelps, 2008), which leads to better class activities and tasks. Through the continuous process of data collection and data analysis in our PSTs course, we gained better understanding of PSTs' thinking and extended the shared knowledge base for teaching similarity. As Cai and his colleagues (2017a, b) proposed, we should use data to improve teaching/learning and to build a knowledge base for teaching (2018). Continuously integrating research into teaching practice will constantly help updating and refining a knowledge base for teaching which in turn will improve teaching/learning experience.


For each pair of rectangles shown below, determine whether the two rectangles are similar.
Explain your choice.

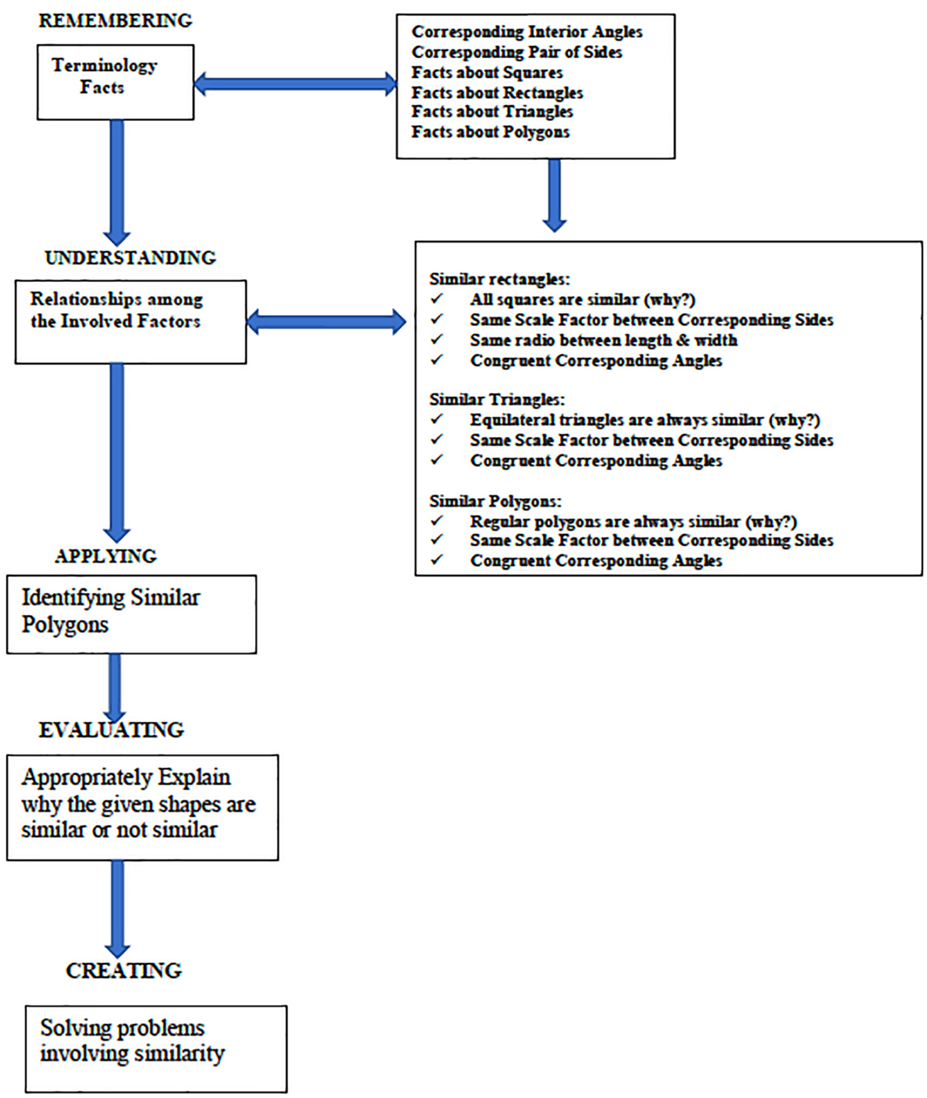
a.

No, because the smaller shape is not a multiple of the bigger

b.

Yes, the smaller shape is a multiple of the bigger one.

A Diagram of Hypothetical Learning Trajectory for Similarity



REFERENCES

Anderson, L.W. and Krathwohl, D. R., et al (Eds..) (2001) *A Taxonomy for Learning, Teaching, and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives*. Allyn & Bacon. Boston, MA (Pearson Education Group)

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of Teacher Education, 59*(5), 389-407.

Beckmann, S. (2014). *Mathematics for Elementary Teachers with Activities*. Pearson.

Berk, D., Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers, one lesson at a time. Teachers and Teaching: Theory an Practice, 15(3), 337-356

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., & Hiebert, J. (2017c). Making classroom implementation an integral part of research. *Journal for Research in Mathematics Education, 48*(4), 342–347. doi:10.5951/jresematheduc.48.4.0342.

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., Hiebert, J. (2018). Using data to understand and improve students' learning: empowering teachers and researchers through building and using a knowledge base. *Journal for Research in Mathematics Education, 49*(4), 362–372. doi:10.5951/jresematheduc.49.4.0362.

Center for Research in Mathematics and Science Education. (2010). *Breaking the Cycle: An International Comparison of U.S. Mathematics Teacher Preparation.* East Lansing: Michigan State University.

Clements, D. H. & Sarama, J. (2009). Learning trajectories in mathematics education. *Mathematic Thinking and Learning*, 6(2), 81-89.

Corbin, J., & Strauss, A. (2008). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory*. Thousand Oaks, CA: Sage.

Crespo, S., & Speer, N. (2004). Learning to teach preservice mathematics teachers: The role of a doctoral course. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 1011-1019). Toronto, Canada.

Hansen, A., Drews, D., & Dudgeon, J. (2014). *Children's Errors in Mathematics*. London: Sage Publications Ltd.

Heaton, R. M. (1994). *Teaching Mathematics to the New Standards: Relearning the Dance*. NY: Teachers College Press.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.

Lappan, G., & Even, R. (1988). Similarity in the middle grades. *Arithmetic Teacher, 35*(9), 32–35. Reston, VA: National Council of Teachers of Mathematics.

Liang, S., Glaz, G., DeFranco, T., Vinsonhaler, C., Grenier, R., Cardetii, F. (2013). An examination of the preparation and practice of grades 7-12 mathematics teachers from the Shandong Province in China. *Journal of Mathematics Teacher Education, 16*(2), 149-160.

Liljedahl, P., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in task design: A tale of one task. *Journal of Mathematics Teacher Education, 10*(4-6), 239-249.

Masters, J. (2010). *Diagnostic Geometry Assessment Project: Item Characteristics*. Chestnut Hill, MA: Technology and Assessment Study Collaborative, Lynch School of Education, Boston College.

Morrison, J. A., & Lederman, N. G. (2003). Science teachers' diagnosis and understanding of students'preconceptions. *Science Education 87*, 849-867.

National Governors Association Center for Best Practices (NGACBP) & Council of Chief State School Officers (CCSSO). (2010). *Common Core State Standards for Mathematics*. Washington D.C.: National Governors Association Center for Best Practices & Council of Chief State School Officers.

National Research Council (1996). *The Preparation of Teachers of Mathematics: Considerations and Challenges: A Letter Report*. Washington, DC: The National Academies Press. doi.org/10.17226/10055.

Nicol, C. (1997). *Learning to teach prospective teachers to teach mathematics*. Unpublished doctoral dissertation. University of British Columbia, Vancouver, B.C.

Patton, M. Q. (2002). *Qualitative Research & Evaluation Methods (3rd ed)*. Thousand Oaks: Sage.

Seago, N., Driscoll, M., & Jacobs, J. (2010). Transforming middle school geometry – designing professional development materials that support the teaching and learning of similarity. *Middle Grade Journal, 5*(4), 199-211.

Seago, N., Jacobs, J., Driscoll, M., Nikula, J., Matassa, M., & Callahan, P. (2013). Developing teachers' knowledge of a transformations-based approach to geometric similarity. *Mathematics Teacher Educator, 2*(1).

Thanheiser, E., Olanoff, D., Hillen, A., Feldman, Z., Tobias, J. M., & Welder, R. M. (2016). *Journal of Mathematics Teacher Education, 19* (2-3), 123-148.