Building conceptual knowledge of fraction operations among pre-service teachers: Effect of a representation-based teaching approach within a teacher education program

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A deep understanding of fraction concepts and operations is necessary if pre-service teachers (PSTs) are to present the concepts in multiple forms to learners. Such an understanding needs to be grounded in rich conceptual knowledge. In the present study, we explore the development of this understanding by supporting a cohort of 103 PSTs, who had previously demonstrated poor conceptual understanding of fraction concepts and operations, with a Representational Reasoning in Teaching and Learning (RRTL) approach aimed at strengthening their conceptual knowledge. A comparison of pre- and post-test results indicated that participants showed a significant improvement in shifting the balance of their fraction knowledge to the conceptual end of the procedural-conceptual spectrum. Insights into how this approach assisted in developing PSTs' conceptual understanding were explored through interviews with four participants and an analysis of their pre- and post-test responses. We suggest that the use of teaching strategies such as RRTL are necessary in order to assist PSTs develop strong conceptual knowledge of fractions.

Keywords fractions · pre-service teachers · teacher education · procedural knowledge · conceptual knowledge

Introduction

It is widely recognised that the quality of teachers' knowledge affects the quality of their teaching in mathematics, which in turn impacts students' mathematical outcomes (Ball, Thames, & Phelps, 2008; Bobis, Higgins, Cavanagh & Roche, 2012). The important relationship between teachers' content knowledge, their practice, and student learning outcomes justifies further research into the nature and development of that knowledge, in both prospective and practising teachers of mathematics (Lloyd, 2014). Against this background, studies have identified pre-service teachers (hereafter PSTs) as having inadequate knowledge for teaching numeracy, particularly of fractions, both during and beyond their teacher education programmes (Harvey, 2012; Jansen & Hohensee, 2016; Olanoff, Lo & Tobias, 2014). In furthering research in this area, Potari (2014) and Ma (2010) have argued the need to focus on the nature and quality of content knowledge of mathematics.

In the domain of fractions, there has been a steady stream of research aiming to unpack the nature of fractions, operations, and teaching of fraction concepts. Mack (1998, 2001) analysed the foundational knowledge underpinning fractions, potential misconceptions, and aspects of the concepts that teachers need to attend in their practice. In extending this work, Lamon (2012) and Webel, Krupa, and McManus (2016) provided important guidelines for teaching fractions and operations. In particular, Lamon investigated the advantages of using multiple representations during instruction and argued that exposing children to different representations or models of fractions enhances their understanding of embedded concepts and associated computations. In a similar vein, results of a recent study by Son and Lee (2016) suggested that future research needs to examine the effect of instructions that assist PSTs' understanding of multiple representations of fractions and operations. Son and Lee noted the relatively limited number of such studies within the context of teacher education programs. In the present study, our aim is to address both the above issues of representations and context by examining the effect of a teaching approach (Representational Reasoning in Teaching and Learning, RRTL) on PSTs' understanding of fraction operations and computations.

We examine PSTs' understanding from the perspective of conceptual and procedural knowledge. Our study is premised on the assumption that: (a) conceptual and procedural knowledge provide windows into the growth of PSTs' fraction understanding; and (b) developments in conceptual and procedural knowledge can be supported through the construction of multiple representations of fraction operations and computations.

Research Question

The following research question guided our inquiry into the effectiveness of RRTL in supporting the development of PSTs' conceptual and procedural knowledge:

> What are the differences in PSTs' conceptual and procedural knowledge of fraction multiplication and subtraction before and after the introduction of the RRTL approach?

Theoretical considerations

Conceptual and Procedural Knowledge

It has been acknowledged that PSTs' understanding and teaching of fractions and fraction operations is problematic (Alenazi, 2016; Jung, 2016) and that there is a need to examine and support the growth of their conceptual knowledge in this important area of primary mathematics (Newton, 2008). In a previous study, Chinnappan and Forrester (2014) found that PSTs tend to exhibit weaker conceptual knowledge of fraction operations in comparison to their procedural knowledge. Conceptual knowledge concerns the understanding of salient aspects of mathematics concepts and their connections, while procedural knowledge involves the effective use of rules and routines such as algorithmic calculations. We suggest that both knowledge strands are as essential as they are symbiotic (Rittle-Johnson, Schneider, & Star, 2015).

For the purpose of this study, procedural knowledge necessary to successfully complete fraction operation tasks involves the use of algorithms to achieve a correct solution. Conceptual knowledge involves several foundational fraction ideas, including an understanding that fractions as numbers are distinct from fractions as parts of things (Gould, 2013), the part-whole/partitioning sub-construct and the operator sub-construct (Kieren, 1976). The part-whole/partitioning sub-construct requires knowledge of equivalent wholes of fractions, equivalent parts of wholes, and an understanding of partitioning and re-partitioning. The operator sub-construct involves an understanding that the multiplication of fractions involves finding a part of a part of a whole.

Representations

The use of representations to examine conceptual/procedural nature of teacher knowledge is the basis of the conceptual framework that guided the design and development of this study. Shulman (1986) proposed that teachers' pedagogical content knowledge is evidenced in their knowledge of "the most useful forms of representations … the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Hattie (2003) found that teachers with a wider repertoire of representations could more effectively capitalise on events occurring in the classroom to support learning, make better predictions about students' knowledge evidenced in their representations, and were more able to determine the types of errors students might make. Likewise, the employment of representations is also an indicator of the rigor, richness, and quality of a teacher's mathematical instruction (Hill et al., 2008). More recently, Jacobson and Izsak (2015) found that teachers' knowledge of visual models of multiplication and division fraction problems are important mediators for motivation and mathematical practice. Thus, we contend that examining the fraction representations produced by practising and prospective teachers can provide powerful insight into their knowledge of content and pedagogy.

While representations are essential to the work of teaching, they also facilitate students' own learning. When students construct and use multiple external representations to explain relationships among concepts, they demonstrate and develop robust mathematical understandings (Acevedo Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009; Behr, Harel, & Post, 1992; Lesh, 1981). This process is captured most eloquently by Barmby, Harries, Higgins, and Suggate (2009) in their Representational-Reasoning model of understanding. Using this model, understanding is seen as a network of internal representations, linked through explicit reasoning. Through experiences with multiple external representations, learners are able to strengthen links between their mental representations. In elucidating understanding, it is important that we distinguish between internal and external representations. Internal representations are defined by Pape and Tchoshanov (2001) as "abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience" (p. 119). However, in interpreting students' understandings, we do not have direct access to their internal representations. Barmby et al. (2009) proposed that learners' understandings can be observed and assessed through their demonstration of connections between external representations.

The need for, and the effectiveness of using, representations to support deeper conceptual knowledge of mathematics is evident in the literature. In order to develop strong understanding of a concept, PSTs need to be able to represent it externally in many different ways, such as visual, verbal, textual, and symbolic modes (Graeber, 1999). Equally, PSTs need to be able to construct and justify connections within and between the representations in order to deepen their understandings. This was the rationale behind the development and implementation of our RRTL approach.

Development of RRTL approach

The Representational Reasoning Teaching and Learning (RRTL) approach was developed subsequent to the identification of PSTs' considerable difficulties in understanding fraction concepts and operations, as evidenced in their first mathematics content and pedagogy subject. The principles that guided the development of RRTL were based on Barmby et al.'s (2009) Representational Reasoning framework. This framework proposes that robust mathematical understanding is evident when learners can construct, utilise, explain, justify, and make connections between and within multiple representations of a mathematical idea. Central to Representational Reasoning are the representations themselves, and the reasoning that connects the representations. Thus, the first consideration for developing RRTL was the identification of representations of fraction concepts. We use the term 'fraction concepts' to refer to: (a) conceptual building blocks underlying the division of a whole into equal parts; and (b) the construction of relations among fractions. Such understandings include equivalence, part-whole relationship, universal wholes, and operating on fractions. The representation of the above two dimensions of fraction concepts was key to our efforts to identify the range of representation. Our second consideration was supporting PSTs' reasoning about links among these representations, concepts, and operations. Regional models were the primary representation in the RRTL approach and were used for demonstrating concepts. Number lines and discrete models were also employed. A fraction such as $\frac{3}{4}$ can be represented alternatively with a region model as three pieces of a rectangle equally partitioned into four pieces, a location on a number line three quarter-length distances from zero, or through a discrete model as a selection of objects that maintains the proportional relationship of three-quarters of the whole collection. Further, operations such as $\frac{3}{4} \times \frac{1}{2}$ can be represented as the area of a rectangle with a length of $\frac{3}{4}$ and width of $\frac{1}{2}$, as a distance that is three-quarters the length of $\frac{1}{2}$ as marked on a number line, or three-quarters of half a collection of objects. Such representations were employed to support PSTs in developing robust reasoning about the links between the representations, fraction concepts and operations. To assist PSTs in constructing procedural and conceptual knowledge of fraction concepts and operations, discussions of PSTs' representations, and explanations, including the misconceptions evident, were scaffolded. These discussions also addressed possible difficulties children experience with fraction concepts, operations, and algorithms and how these might be redressed.

Methodology

Research Design and Plan

Action research aims to improve teaching and learning by finding solutions to identified problems through cycles of planning, implementing, and evaluating change during the course of practice (Hine, 2013). This study evolved from the need to address an important issue in the preparation of a cohort of primary school teachers, namely, their weak understandings of the fraction concepts and operations necessary for primary school teaching. In identifying this problem, an action research approach was considered most appropriate to design, implement, and evaluate an instructional approach that would improve PSTs' learning, with an emphasis on the practical significance of findings for our PST education program. This paper reports on one cycle of action research which utilised pre- and post-instruction evaluation of PSTs' procedural and conceptual knowledge.

Phase 1 of the study involved the analysis of the examination results of a cohort of PSTs, following their first mathematics content and pedagogy subject (Subject 1). Subject 1 aimed to support PSTs' conceptual and procedural understandings of fraction concepts and operations by providing interactive PowerPoint presentations depicting foundational fraction concepts and operations and their representations. Phase 2 involved the delivery of the final mathematics content and pedagogy subject (Subject 2) utilising the RRTL approach, which aimed to strengthen the conceptual knowledge that was covered in Subject 1. This phase also involved an analysis of examination results from Subject 2, and interview data from four participating PSTs interrogating their pre- and post-test responses (details below).

Participants

The participants comprised a cohort of 103 pre-service primary teachers who were enrolled in a BEd degree at an Australian university. The participants had a range of prior-to-university experiences and educational backgrounds, many having completed their final year of school the year prior to university entry. There was also a range of prior mathematics education experiences, most participants having completed studies in mathematics in their final year of high school. Participants who had not completed sufficient mathematics subjects completed two additional mathematics content subjects as part of their degrees.

Pre-service teachers were required to complete two core mathematics content and pedagogy subjects (Subject 1 and Subject 2) that were held in the first and third years of their study respectively. All PSTs experienced RRTL instruction during the course of studying Subject 2 prior to the administration of the post-test.

Data sources

There were two sources of data for this study. The quantitative data were based on PSTs' responses to examination tasks that were conducted at the end of Subject 1 and Subject 2. The end of session examinations results for the Subjects 1 and 2 provided the pre- and post-tests scores respectively. Qualitative data were collected from four of these pre-service primary teachers. Two of these PSTs, Jenny and Matt, had not completed mathematics subjects in their final year of high school, while Ashley and Tamara had. These teachers were chosen to illustrate the conceptual knowledge of PSTs with differing prior-to university experiences, as well as varying levels of development of this knowledge between the teachers' first and third years of study. These four participants' responses can be seen in Figures 3 to 18. Pseudonyms have been used in this report in order to protect their anonymity.

Implementation of RRTL approach

Phase 1 of this project involved analysing the procedural and conceptual knowledge evident in PSTs' examination responses to two fraction operation questions (at the end of Subject 1) which required them to provide a calculation and representation of the concepts involved in subtracting and multiplying fractions (see Table 1). Lecturer-designed animated PowerPoint presentations demonstrating fraction concepts and operations were shown in the two-hour fractions lecture to scaffold understanding of the concepts and procedures. The animated presentations were also provided on the e-learning site for revision. Our goal was to support this cohort to successfully utilise fraction algorithms with a conceptual understanding of concepts such as equivalence and the operations. Students had engaged with tasks similar to those examined throughout the subject.

Phase 2 involved the implementation of the RRTL approach in Subject 2 undertaken by the same cohort of PSTs in their third year. At the end of this course, PSTs responded to two fraction operation questions that were similar to those used in the Subject 1 examination (see Table 1).

The key feature of RRTL implementation was scaffolding PSTs' understanding of fractions and fraction operations through: (a) the provision of animated representations to demonstrate important concepts; (b) the construction of their own representations; (c) the examination of links between and within representations; and (d) analysis of their own and children's representations and misconceptions. While a range of fraction models were introduced including area, linear, ratio, and discrete models, the regional model was utilised explicitly in the PowerPoint animations and tutorial activities. These focused on representing fraction operations as these are considered conducive to highlighting fraction multiplication (Barmby, et al., 2009) and can be used to illustrate the subtraction of fractions through partitioning. While some students experimented with the linear and discrete models in their own representations, most students used regional models. The RRTL approach sought to develop a conceptually rich and robust understanding of fractions by engaging PSTs in generating and explaining multiple representations of essential fraction concepts and operations.

The PowerPoint animations were used in face-to-face lectures. PSTs were also provided with online support materials for personal study. During the tutorial sessions held subsequent to lectures, discussions of PSTs' representations and explanations, including the misconceptions evident in these representations and explanations, were scaffolded. These discussions also addressed children's misconceptions of fraction concepts, operations, and algorithms and how these may inform pedagogical decisions.

We also challenged PSTs to respond to questions about how to use algorithms and discover why they worked. There were ample opportunities for PSTs to reflect on and discuss questions that were raised in the lecture sessions during their weekly tutorial sessions. The RRTL was implemented in a 13-week period.

Pre- and post-test tasks

Both the pre- and post-test included one multiplication and one subtraction of fractions task. The tasks in the first-year examination (1a and 1b; see Table 1) were similar to those in the third year (2a and 2b).

Table 1
Pre- and Post-test tasks

Year	Subtraction Tasks	Multiplication Tasks
First-year tasks (Pre-test):	1a) $1\frac{2}{5}-\frac{5}{6}$	1b) $\frac{1}{4}\times\frac{2}{3}$
Third-year tasks (Post-test):	2a) $1\frac{3}{8}-\frac{3}{4}$	2b) $\frac{1}{3}\times\frac{3}{5}$

In both examinations, PSTs were asked to complete the calculations and draw a model for the multiplication and subtraction tasks to reflect each fraction operation. Both the first- and third-year examinations contained a contextless subtraction operation with a mixed numeral, and a multiplication with a unit fraction (where the numerator is equal to one) and non-unit fraction. The similarity of tasks across the examinations allowed the comparison of the PSTs' knowledge across years. As mentioned earlier, first- and third-year examinations were conducted at the completion of Subject 1 and Subject 2 respectively. In developing these tasks we were guided by three strategies. Firstly, we wanted to ensure that the tasks required conceptual understanding of fraction concepts and operations in order to draw appropriate representations. Secondly, the tasks had to be sensitive to procedural and conceptual dimensions of teacher knowledge and eliminate distractions such as context. Thirdly, the tasks needed to be sufficiently pliable so that we could observe and measure changes in PSTs' knowledge along the two dimensions.

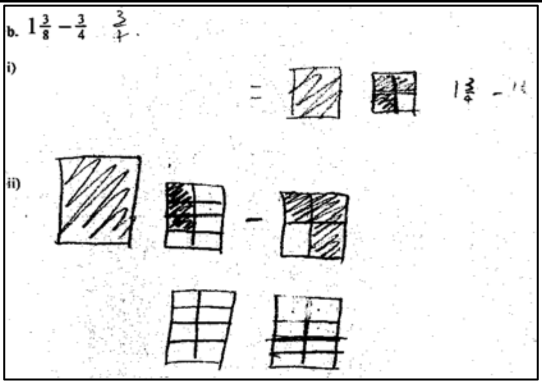
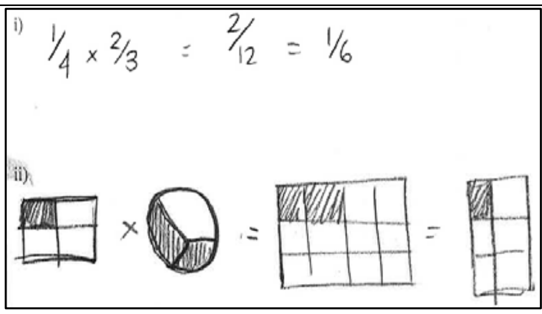
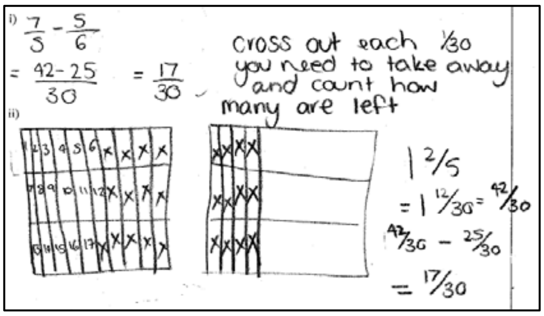
Coding scheme

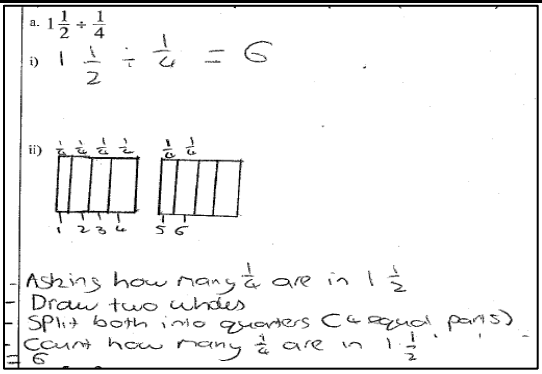
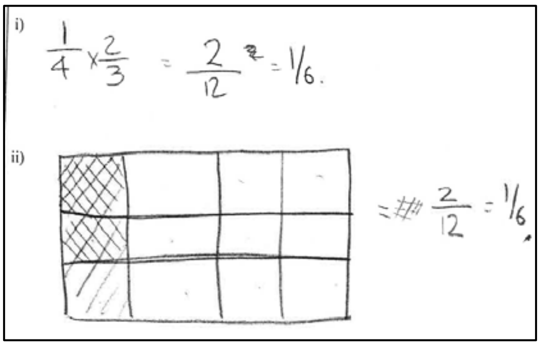
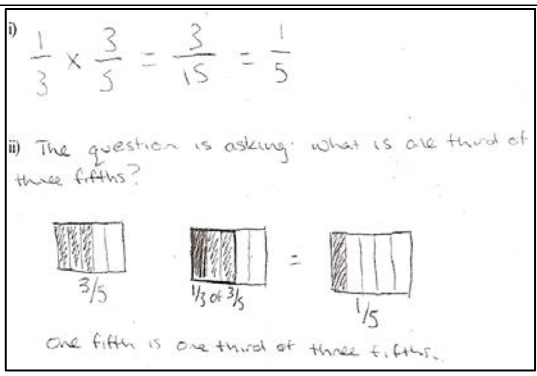
Participants' responses to each of the above four tasks were analysed in terms of the evidence of conceptual and procedural knowledge and coded using the 6 codes detailed in Table 2.

Inter-rater reliability analysis

In order to determine the reliability of the coding scheme, the extent to which two coders agreed was assessed when participants' responses to the multiplication problem were independently coded. The two researchers coded ten participants' responses. The inter-coder reliability determining coding consistency was found to be Kappa = 0.91 ($p<0.001$), 95% CI (0.67, 1.08), indicating substantive agreement (Landis & Koch, 1977) in the way the participants' responses were coded by each researcher. Potential areas of disagreement were analysed which helped us to improve the distance between the codes, thereby reducing areas of ambiguity.

Table 2
Coding categories with illustrations

<table>
<tr><th>Code and description</th><th>Illustration of codes</th></tr>
<tr><td>Code 0: No evidence of Procedural Knowledge (PK) or Conceptual Knowledge (CK).
No or incorrect algorithm, illustration and/or explanation.
Example of code 0: subtraction</td><td>
</td></tr>
<tr><td>Code 1: PK only.
Correct algorithm. Incorrect or no illustration and/or explanation.
Example of code 1: Multiplication</td><td>
</td></tr>
<tr><td>Code 2: PK and some CK.
Correct algorithm. Illustration and/or explanation demonstrates basic conceptual understanding of a fraction concept. Does not demonstrate an understanding of the operation process.
Example of code 2: Subtraction</td><td>
</td></tr>
</table>

Code 3: CK only. No algorithm or incorrect algorithm. Conceptual illustration and/or explanation which achieved a correct solution. Example of code 3: Division*	
Code 4: PK and CK. Correct algorithm, illustration and/or explanation demonstrating procedural and conceptual knowledge. Conceptual understanding of the operation demonstrated, though not of every part of the process. Example of code 4: Multiplication	
Code 5: PK and strong CK. Algorithm, illustration and/or explanation demonstrate procedural and strong conceptual knowledge of the fraction concepts and operation. Example of code 5: Multiplication	

*No student responses to multiplication or subtraction tasks fit with this code. This code was only utilised in division tasks, as reported in Chinnappan and Forrester (2014).

Data analyses

Before the analyses of the data were conducted, the data were tested for normality using the Kolmogorov-Smirnov (KS) test. For all fraction tasks (1a, 1b, 2a, and 2b) the significance was less than 0.000, and failed the normality test, thus, the data are non-normal. Additionally, the data were ordinal. Hence, it was determined that the most appropriate presentation of quantitative results was through the median scores, range, frequencies, and percentages. Non-parametric measures of the significance of difference were also required.

Median scores, frequencies, percentages, and the range of the codes given to the fraction problems were used to compare the state of the cohort's knowledge in the first and third year of studies.

The qualitative data were collected after the PSTs had completed their final examination for the second core mathematics content and pedagogy subject. The data were drawn from semi-structured interviews which served to elicit richer insights into the four teachers' conceptual knowledge of the subtraction and multiplication of fractions, as well as to further investigate the thought processes behind each of the examination tasks (see Table 1). These interviews were then thematically coded based on the conceptual knowledge that PSTs demonstrated in terms of key understandings of fraction concepts (listed under RRTL).

Results

Quantitative Data Analysis

The aim of this study was to investigate the developing state of PSTs' mathematical knowledge for teaching fractions. Quantitative data were collected from participants' responses to two sets of fraction tasks (see Table 1 above), and coded according to the type and levels of knowledge demonstrated (see Table 2 above). Quantitative data will now be presented to highlight the differences across the cohort, with the qualitative data presented subsequently to illustrate some individual cases of change in conceptual knowledge.

Coded responses to first-year fraction task

The quantitative results of the study are presented in Table 3.

Subtraction

A substantial increase in PSTs' ability to demonstrate either procedural or conceptual knowledge was evident across the first and third years with 16 participants (15.5%) in their first year showing no evidence of knowledge, compared with only 5 people (4.9%) in their third year (see Figure 1). There was a large decrease of responses that showed procedural knowledge but no conceptual knowledge when comparing the PSTs' first year (n=56, 54.4%) to their third (n=4, 3.9%). Large increases can be observed in both the "PK and CK" (n= 12, 11.7% to n=53, 51.5%) and "PK and strong CK" categories (n=0, 0.0% to n=19, 18.4%). As with multiplication, the PSTs' procedural and conceptual knowledge improved considerably between their first (*Median* = 1, *Range* = 4) and third year (*Median* = 4, *Range*= 5).

Table 3
Frequency of Coded Responses to Fraction Tasks

Codes	Subtraction Tasks		Multiplication Tasks	
	1st Year	3rd Year	1st Year	3rd Year
0 - No PK or CK	16	5	20	7
1 - PK but no CK	56	4	59	3
2 - PK and some CK	19	22	10	19
3 - CK only	0	0	0	0
4 - PK and CK	12	53	11	45
5 - PK and strong CK	0	19	3	29
Total	103	103	103	103

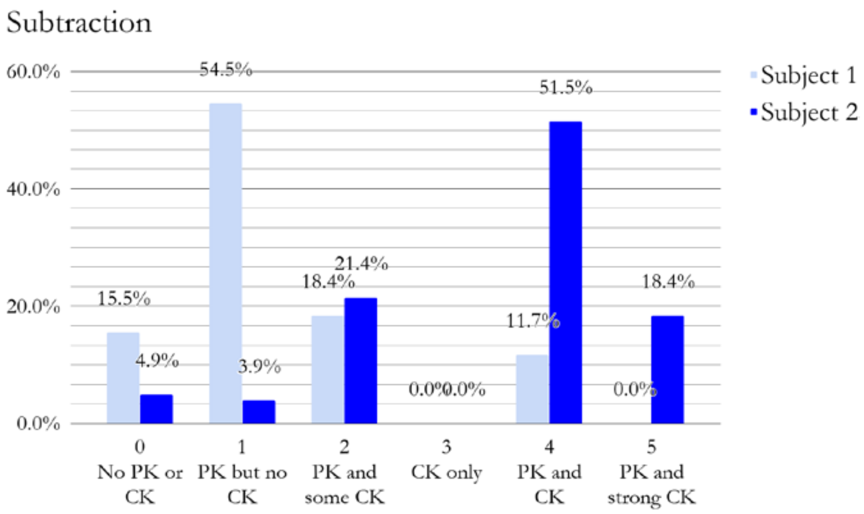


Figure 1: Comparison between subtraction scores in the PSTs' first- and third-year examinations

In general, the responses to the subtraction tasks from their first year to their third demonstrated increased conceptual knowledge. This suggests that the PSTs were able to move from showing either no evidence of any knowledge or only procedural knowledge to demonstrating stronger conceptual knowledge in the subtraction tasks. This is likely a consequence of RRTL experience as no other fraction instruction was implemented consistently across the cohort. A Wilcoxon Signed Ranks Test revealed the mean of the rank for their third-year responses was 48.05, compared with 19.50 for their first which is a significant improvement of their scores (z = -8.095, p <0.001, r = 0.80).

Multiplication

The comparison between the PSTs' first- and third-year responses to the multiplication tasks shows similar trends to those found in the subtraction tasks. Twenty (19.4%) participants in the first year did not show evidence of either conceptual or procedural knowledge, compared with only seven people (6.8%) in their third year (see Figure 2). There was an even greater decrease in the frequency of responses that showed "Some PK but no CK" which dropped from 57.3% of participants (n= 59) to only 2.9% (n=3) in the third year. The number of people who were able to show that they had more than some PK and CK increased from 14 people (13.6%) in the first year, to 74 people (71.9%) in their third year. Similarly, more PSTs demonstrated strong conceptual and procedural knowledge in their third year (n= 29, 28.2%) than in their first (n= 3, 2.9%).

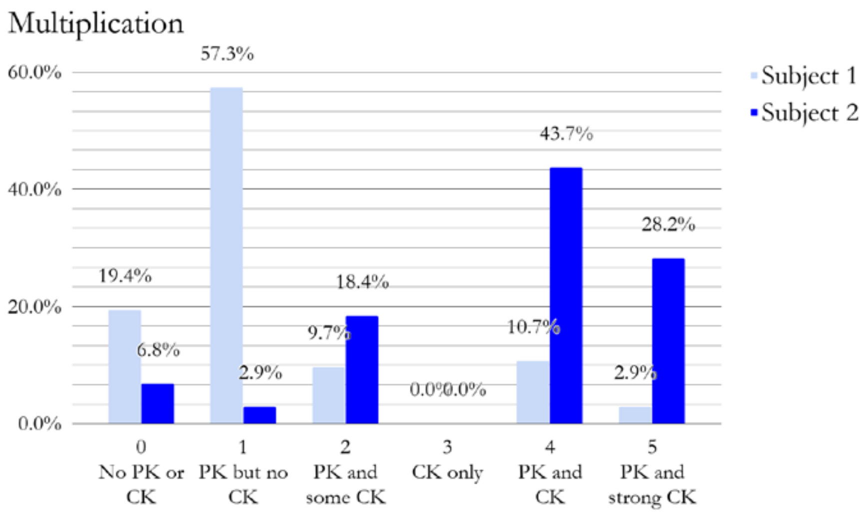


Figure 2: Comparison between multiplication scores in the PSTs' first- and third-year examinations

Comparatively, the responses to the multiplication tasks in their first (*Median* = 1, *Range* = 5) and third year (*Median* = 4, *Range*= 5) demonstrated increased conceptual knowledge. This shows that the PSTs' knowledge of multiplication of fractions has shifted from being primarily procedural knowledge and little or no conceptual knowledge, to stronger conceptual knowledge, again, suggesting the impact of RRTL.

For the multiplication tasks, the mean of the ranks of their third-year responses was 44.73, while the mean of the ranks of their first-year responses was 23.60. A Wilcoxon Signed Ranks Test showed there had been a statistically significant improvement from their first- to their third-year responses (z = -7.65, p <0.001, r = 0.75).

Changes across subtraction and multiplication responses

Between the PSTs' first and third years, the data showed there had been changes in the two dimensions of fraction knowledge as demonstrated through their responses to the subtraction and multiplication fraction problems. Some participants (n=7, 6.80%) were able to show strong growth in conceptual and procedural knowledge in both the multiplication and subtraction tasks in their third year, but were not able to do so in their first year. None of the PSTs lacked both conceptual and procedural knowledge in both tasks in their third year, compared with eight participants (7.77%) in their first year. Overall, it would seem that the PSTs' procedural and conceptual knowledge has improved considerably between their first and third years in both multiplication and subtraction of fractions. Again, although this may partly be attributed to external factors, the scale of the improvement suggests it is a result of the RRTL approach.

Qualitative Data Analysis

The qualitative data were collected through semi-structured interviews where the PSTs were given their first- (Subject 1) and third-year (Subject 2) examination responses in the interview and asked to comment on their thinking. Responses were analysed to gain richer insights into these PSTs' knowledge and their experiences of the RRTL approach to teaching fraction operations and concepts.

Interviewees' pre- and post-test responses

Below are the four interviewees' first- and third-year examination responses in fraction multiplication and subtraction tasks (Figures 3-18). Some of the PSTs did not identify the errors in their exam responses and thus did not comment on these. Summaries are provided of each interviewee's procedural and conceptual knowledge as evidenced in their examination responses and/or interviews.

Matt's responses

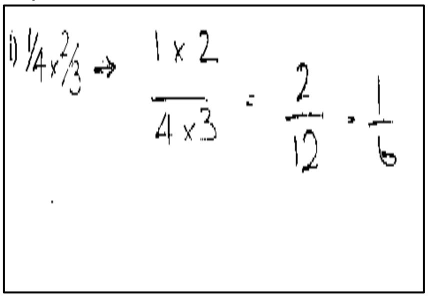


Figure 3: Matt's 1st-year multiplication response. Code 1 (PK).

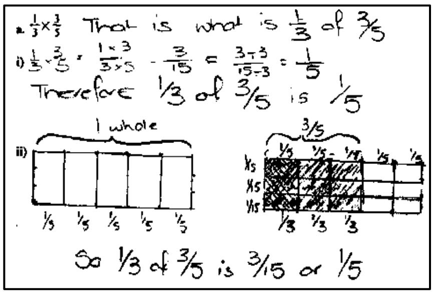


Figure 4: Matt's 3rd-year multiplication response. Code 5 (PK and strong CK).

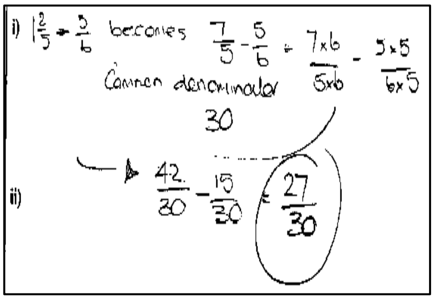


Figure 5: Matt's 1st-year subtraction response. Code 1 (PK).

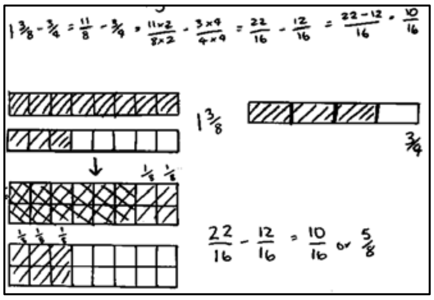


Figure 6: Matt's 3rd-year subtraction response. Code 2 (PK, some CK).

Matt demonstrated procedural understanding of multiplication and subtraction in both Subject 1 and 2 despite making a basic calculating error in his first-year subtraction response (Figure 5). Matt's demonstration of conceptual knowledge increased considerably from his first- to third-year studies, from making no attempt to provide drawn representations for either operation in Subject 1 (Figures 3 and 5) to showing considerable conceptual understanding of the multiplication and subtraction operations in Subject 2 (Figures 4 and 6). It is worth noting that while Matt demonstrated strong conceptual understanding of equivalence in his multiplication response (Figure 4), the third-year subtraction task does not demonstrate this understanding. The joining of the two wholes in his subtraction drawings demonstrates the misconception that the whole changes size when finding an equivalent fraction (transformation of 1 3/8 to double the size, Figure 6). He did not comment on this when reflecting on his examination responses.

Ashley's responses

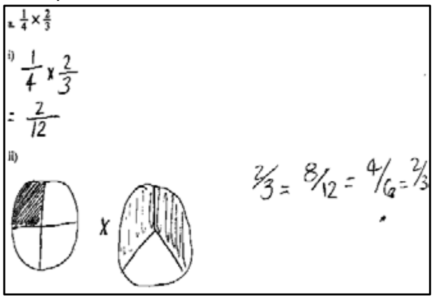


Figure 7: Ashley's 1st-year multiplication response. Code 1 (PK).

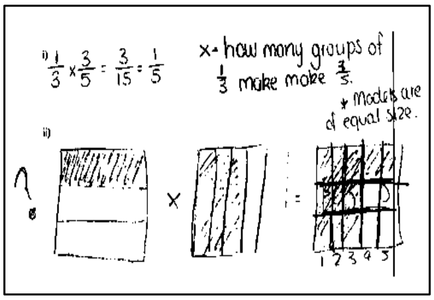


Figure 8: Ashley's 3rd-year multiplication response. Code 1 (PK).

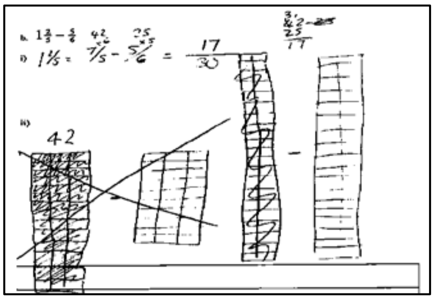


Figure 9: Ashley’s 1st-year subtraction response. Code 1 (PK).

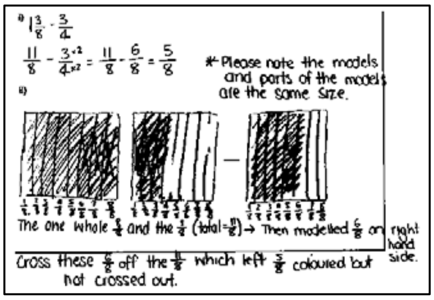


Figure 10: Ashley’s 3rd-year subtraction response. Code 4 (PK and CK).

Ashley could use the common algorithms to achieve correct answers in both the multiplication and subtraction tasks in her first year. In the multiplication task she did not simplify her answer (Figure 7, part i). There is no clear reason for providing equivalent fractions for $\frac{2}{3}$ (Figure 7, part ii) and, in interview, Ashley did not know why she had written them. Ashley’s demonstration of conceptual knowledge did not increase for multiplication but improved considerably for subtraction.

In both multiplication tasks (Figures 7 and 8) Ashley represented the individual fractions from the operation correctly. In Subject 1, she did not attempt to draw a representation for her answer or the operation, whereas in Subject 2, she attempted both. However, in the Subject 2 multiplication response (Figure 8) Ashley appears to have confused the language she used to describe the operation of multiplication with language used to describe division, yet her drawing of an array model for $\frac{1}{3} \times \frac{3}{5}$ seems to model the algorithm appropriately. In interview it became evident that she had no clear understanding of the concepts represented in her drawing and she had, in fact, focused on the uncoloured area of the rectangle rather than the overlap of the multiplicand and multiplier. Ashley stated that she had not been able to reconcile the answer she had achieved through her use of the algorithm and the answer she produced when counting up the uncoloured squares in her drawing.

In Subject 1 subtraction response (Figure 9), Ashley attempted, but had difficulty, representing the minuend and subtrahend, unsure of how to represent $\frac{42}{30}$ and $\frac{17}{30}$. In interview she reflected on her confusion, saying she realised at the time that the wholes needed to be the same size but she was not sure how to illustrate that. She did not draw a representation of the subtraction operation or her answer. In her Subject 2 response (Figure 10) Ashley was able to demonstrate an understanding of how to relate fractions as numbers to fractions as parts of things, where the wholes used to demonstrate the part/whole relationships in the minuend, subtrahend and difference need to be the same size. While she did not draw a representation of the equivalence of $\frac{3}{4}$ and $\frac{6}{8}$, she successfully demonstrated the subtraction process.

Tamara's responses

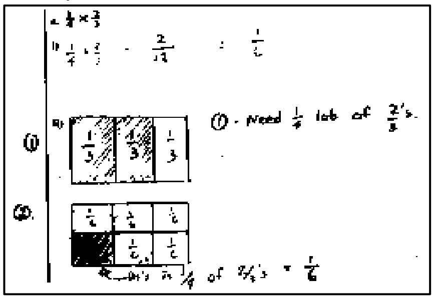


Figure 11: Tamara's 1st-year multiplication response. Code 5 (PK, strong CK).

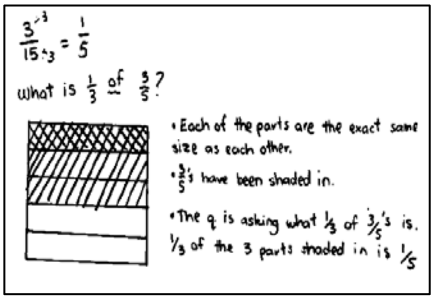


Figure 12: Tamara's 3rd-year multiplication response. Code 5 (PK, strong CK).

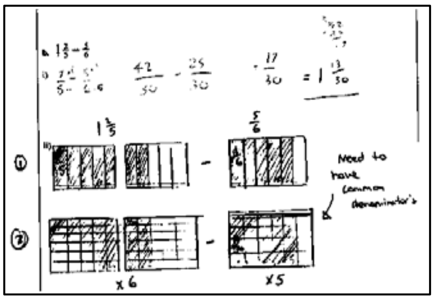


Figure 13: Tamara's 1st-year subtraction response. Code 2 (PK and some CK)

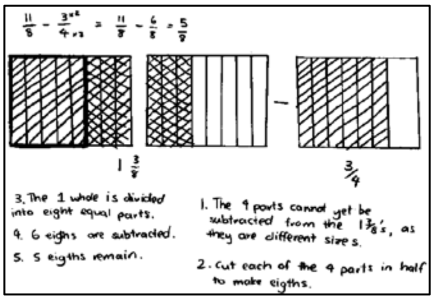


Figure 14: Tamara's 3rd-year subtraction response. Code 5 (PK, strong CK).

Tamara demonstrated procedural understanding in utilising multiplication and subtraction algorithms in Subject 1 and Subject 2. However, she made an error in her first-year subtraction response by incorrectly converting $\frac{17}{30}$ to $1\frac{13}{30}$ (Figure 13; 17/30 cannot be simplified to 1 13/30 as this changes its value). Tamara was not asked directly about her error, and interestingly, she did not realise her procedural error when reflecting on her examination responses in interview. She stated "I did the subtraction and then turned it back to the proper form." She was also unaware of the conceptual inadequacy of her drawn representation, in that it did not represent the operation appropriately and so did not highlight her error in converting a proper fraction to a mixed numeral. While Tamara demonstrated a solid understanding of equivalence in both subtraction responses, her drawn representation for Subject 1 (Figure 13) does not include a drawn representation of the process of subtraction or her answer. Her conceptual understanding of the multiplication operation is clear in both Subject 1 and 2 responses.

Jenny's responses

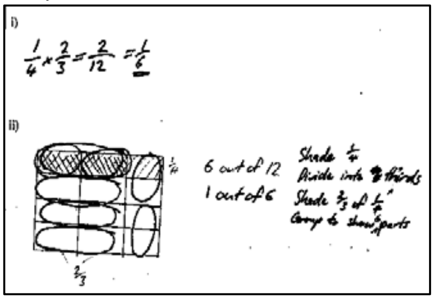


Figure 15: Jenny's 1st-year multiplication response. Code 5 (PK, strong CK)

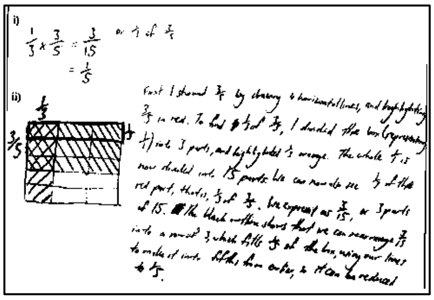


Figure 16: Jenny's 3rd-year multiplication response. Code 5 (PK, strong CK)

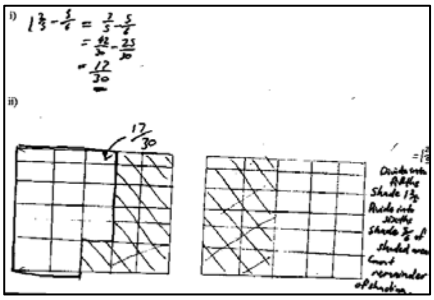


Figure 17: Jenny's 1st year subtraction response. Code 4 (PK and CK)

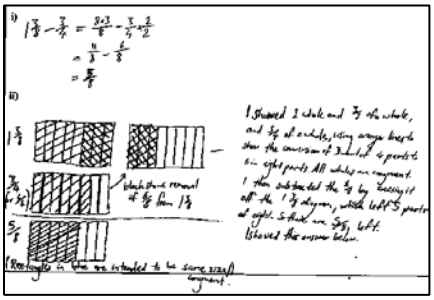


Figure 18: Jenny's 3rd year subtraction response. Code 5 (PK, strong CK)

Jenny demonstrated procedural and conceptual understanding of both operations in Subject 1 and 2. She remembered the fraction and operation concepts being demonstrated in lectures and tutorials in Subject 1, feeling that it was "one of the big things on fractions that I took out of Subject 1 … knowing how to visually show [the concepts]". Her drawing and explanation of the Subject 2 subtraction task (Figure 18) demonstrated the concept of equivalence, which was not clearly demonstrated in her Subject 1 response.

Interviewees' development of procedural knowledge

During the interviews, all four participants felt confident in their procedural knowledge for subtracting and multiplying fractions. Three of the four students (Ashley, Tamara, and Jenny) were confident in these when they commenced Subject 1. Alternatively, Matt felt "it was almost like relearning primary school maths," although he quickly felt confident with the algorithms too. All four interviewees demonstrated procedural knowledge of how to use the common multiplication and subtraction algorithms in both their first- and third-year examination responses, although Matt and Tamara produced incorrect answers in Subject 1 subtraction.

Interestingly, Matt and Tamara did not recognise that they had made errors in their examination responses when viewing them in the interview. Matt had incorrectly calculated 5 ×

5 as 15, although this may have resulted from a misreading of his own handwriting mistaking the first '5' for a '3' (Figure 5). More concerning was the error made by Tamara who incorrectly converted $\frac{17}{30}$ to $1\frac{13}{30}$ in her 1st year subtraction response (Figure 13). This does not seem to be a thoughtless error, as Tamara has incorrectly divided the denominator by the numerator, an inappropriate process with no connection to the concepts involved in converting between improper fractions and mixed numerals. That neither of these PSTs identified the errors suggests they did not look closely enough at their examination responses, or that their number sense was insufficient to distinguish the errors.

Interviewees' development of conceptual knowledge

In comparing examination responses from their first- and third-year subjects, all interviewees demonstrated an improvement in their representations of fractions and fraction operation concepts in one or both operations. While Matt and Ashley could only demonstrate procedural knowledge in their first-year responses, all interviewees demonstrated at least some conceptual understanding of important concepts in their third-year responses.

RRTL as a scaffold of conceptual understanding

In analysing the interviewees' examination and interview responses three features of the RRTL approach emerged as important in scaffolding PSTs' conceptual understanding of fraction concepts and operations: the development of appropriate language, the process of producing a visual representation and connections made between their representations.

Language as a scaffold for conceptual understanding

Three of the four interviewees (Matt, Jenny, and Ashley) referred to the use of language in scaffolding their understanding of fractions concepts, particularly in relation to the multiplication and division operations.

During the interview, Matt focused on the importance of language in supporting his understanding of the multiplication concept, saying "language played a really big part … in understanding what it meant." He described the introduction of language to scaffold his conceptual understanding of multiplication by a fraction ($\frac{1}{3} \times \frac{3}{5}$ means $\frac{1}{3}$ of $\frac{3}{5}$), as "that Eureka moment in the lecture, going 'that's what it means!' and being able to go 'I understand in my head,' not just how to do something, but I understand in my head what it actually looks like." He maintained that if he had the opportunity to redo his examination responses again he would focus more on the use of scaffolding language in his explanations.

Jenny described the lecturer's approach to teaching fractions; "she taught us to use words to describe what's happening." Jenny had forgotten how to model the multiplication task in Subject 2, so left it and came back to it. In rewriting the question $\frac{1}{3} \times \frac{3}{5}$ as $\frac{1}{3}$ of $\frac{3}{5}$ she comments "I knew linguistically what was going on, and that helped me picture it, physically, what was going on."

Ashley also referred to the use of scaffolding language, in describing how she would develop her drawn representations, but confused language for multiplying by a fraction with that used when dividing by a fraction (Figure 8). This confusion appears to be related to dependence on memorising the "rules" for drawing representations and led to her incorrect interpretation of her correct drawn representation, a problem that is discussed below.

Visual representations as a scaffold for conceptual understanding

All interviewees found the process of constructing visual representations of fraction and fraction operations a scaffold for their conceptual understanding of key concepts and engaged them in thinking about representations that would be appropriate to support their own students' conceptual understandings.

Of the interviewees, none had experiences with drawing representations of fraction operations prior to Subject 1, and while they were used explicitly in Subject 1 to scaffold understanding, both Matt and Ashley found it very difficult to comprehend. Matt's comment typifies the interviewees' responses: "I've never ever been exposed to, been taught, remember learning this … visualisation stuff. It's always been about the procedure."

Ashley reflected:

> I didn't understand the models in the first year, … obviously I can colour in a quarter and a half, and a third, and all those things, but to model them as an actual operation was difficult. It was more [that] I could say, you know, there's a half, there's a quarter but to model it in a picture together was just not as easy.

After Subject 2, Ashley was still not "one hundred per cent confident" with representing fraction operations with regional models, although stated that she felt "heaps better if you were to ask me to teach in a classroom." She attributes this to the RRTL method, stating "I felt that it helped a lot and by the end I had some idea of what was happening."

While Matt had found the development of appropriate language an important scaffold for multiplication and division of fraction, it was not as helpful as producing the representation for the subtraction operation. He found the "visualisation stuff" more helpful and appreciated understanding, for the first time, how to represent the operations visually rather than just focusing on the procedure. He felt more prepared to support his own students to develop basic fraction concepts as he could now represent fractions and fraction operation concepts visually.

Tamara identified the visualisation of concepts through drawing representations as the aspect of the RRTL most helpful in developing her conceptual understanding. It was the first time she had been required to draw a representation of fractions concepts and operations and stated "it definitely puts more meaning into it. … [It] makes it more concrete … than just working with numbers. When you're doing just the operations, you're not really thinking what's happening." She liked the "hands-on" nature of the activity where she was "actually cutting it up and thinking about why and how [the algorithm] works." In reflecting on her first-year subtraction response (Figure 13), she considered the pedagogical implications for the representations, realising the direction she cut the pieces when making equivalent fractions produced pieces that were the same area but not the same shape and concluded that this could be confusing for children.

Jenny found the explicit focus on drawing representations in Subject 2 scaffolded her "thinking about why and how [each fraction operation] works." She appreciated moving away from "manipulating numbers" to working with concrete representations. Jenny felt she had problems with remembering procedures without understanding the reasons behind them. The process of drawing representations, where she asked herself questions such as "what am I supposed to do here?" and "what does it look like on the page?" was helpful in making sense of operations and checking her answers.

Making connections between representations to scaffold conceptual understanding
Two of the four interviewees (Jenny and Ashley) identified the connections they made between their calculations, drawn representations and explanations useful for consolidating their conceptual understanding of fraction concepts and/or operations.

Jenny felt she had a strong procedural knowledge of fraction operations before Subject 1 and had some grasp conceptually, although quite abstractly, about the concepts involved in fraction operations. She felt that she was more likely to remember a procedure if she knew what she was doing conceptually. She demonstrated this in interview when her model prompted her to realise she had incorrectly used division language to describe her multiplication response in Subject 1.

Jenny originally made a simple calculation error in her subtraction response in Subject 2 (Figure 18), and found in producing her drawn representation that " … trying to explain it helped me figure out I had a mistake." Jenny's confidence in the correctness of her examination responses developed as she "had the words to connect to the model … and compare against the operation." While Jenny was confident in her conceptual knowledge in Subject 1, she felt that the RRTL approach "consolidated and solidified" her conceptual understanding because of the need to explain it, stating that she hoped it would assist in her teaching of fractions as well.

Ashley recalled her examination experience in Subject 2 where differences in her calculation and drawn representation highlighted a possible error in her answer (Figure 8), and her frustration with her inability to identify where she had gone wrong. She did not develop robust conceptual understandings of fraction operations in either Subject 1 or Subject 2, although the process of drawing representations appears to have assisted the development of some conceptual understanding of equivalent fractions. In discussing her examination response to the subtraction task in Subject 2, Ashley made connections between her "rule" for making equivalent fractions with numbers and her drawn representation (Figure 10). She states her rule as: "what you do to the top you do to the bottom" to justify turning $\frac{6}{8}$ into $\frac{3}{4}$ by halving the numerator and denominator. She goes on to connect this with the drawn representation by explaining "if you halve them again to make quarters (pointing to her representation of eighths in her drawing of $\frac{6}{8}$), you've got two in each section, so I knew that there was three-quarters that was shaded." Despite this developing understanding, Ashley could not clearly explain why making equivalent fractions was necessary when adding or subtracting and believed she could not give a reason to children for this, other than "that's the only way addition and subtraction work."

Difficulties in moving from procedural to conceptual knowledge

While all interviewees had demonstrated stronger conceptual understandings of fraction concepts and operations after the introduction of RRTL, interviews indicated Ashley still struggled with conceptual understanding of fraction operations. It became apparent that her previous experiences of mathematics, which focused on procedures and rules, constrained her approach to undertaking mathematical tasks and consequently the development of conceptual knowledge.

Ashley improved in her ability to draw representations of the operations after the RRTL approach undertaken in Subject 2 (Figures 8 and 10). However, in interview it was evident that her approach had been to memorise several "rules" for drawing representations, such as "make the wholes the same size," "don't use circles," and "make the parts the same size to add or subtract." She had developed some conceptual understanding of these rules. For example, she explained that in the subtraction problem, $1\frac{3}{8} - \frac{3}{4}$, if she had drawn $\frac{3}{4}$ and not made the equivalent fraction $\frac{6}{8}$ she would have had difficulty subtracting ("crossing them off") and explained that circles are not the most suitable shape to use when representing fractions and fraction operations "because there is no way … that you can easily [partition] two circles of equal size like you can with a square or a rectangle."

However, the reasons for all of the rules Ashley had tried to memorise were not clear to her and when she described multiplication $\frac{1}{3} \times \frac{3}{5}$ as one third "groups of" three-fifths, she could not make sense of the need to divide $\frac{3}{5}$ into three parts (Figure 8). In the examination, she could not remember the "rule" for overlapping the two fractions or the division language to assist her to draw the model, and she focused on an incorrect part of the model (the uncoloured section $\frac{4}{15}$) for her answer. When the answer from her representation did not match the answer from her calculation ($\frac{3}{15}$), she had no way to reason through her solution and identify the problem. Ashley approached understanding and drawing representations in the same way as she approached using algorithms, trying to remember the rules and procedures, without understanding the concepts underpinning the rules, what Skemp (1978) described as "rules without reasons" (p. 9).

Discussion and Implications

Our research set out to assess the effects that RRTL approach had on PSTs' growth of conceptual and procedural knowledge of fractions. Comparison of participants' responses to a set of fraction tasks produced in their first and third year showed significant improvement in their conceptual knowledge of fraction concepts and operations. Although procedural knowledge was not the focus of our teaching, more PSTs were also able to produce a correct answer using algorithms in their third year compared to their solutions generated in the first year.

The introduction of RRTL as a teaching approach was prompted by our analysis of PSTs' knowledge that was activated during the solution of context-free fraction problems indicating that prospective teachers had difficulty producing visual representations of fractions concepts and operations. According to Barmby et al.'s (2009) Representational Reasoning model of mathematical understanding, teachers must be able to construct and reason with visual models that are interconnected with their verbal, written, and symbolic representations of core concepts in mathematics. Interconnections were argued to help PSTs develop both conceptual and procedural knowledge. We suggest that conceptual knowledge is evidenced by multiple representations and the construction of connections between such representations. In the present study, the solution of multiplication and subtraction problems involved processing that demanded the use of conceptual knowledge to generate linguistic and visual representations. Equally important is the establishment of appropriate links among these translations. Thus, the building and strengthening of these connections can be expected to enhance the growth of teachers' conceptual knowledge. Findings reported in the present study suggest that the RRTL approach could be employed to assist the development of PSTs' conceptual knowledge of fractions.

We suggest that both procedural and conceptual knowledge are intertwined and build on each other, as highlighted by Kieran's (2013) notion of "conceptual nature of the learning of procedures" (p. 169). Both strands of knowledge are essential for developing knowledge for teaching fractions, and both work in tandem to support the PSTs' knowledge for teaching mathematics. From a theoretical and practical viewpoint, our position is that PSTs need to develop deep procedural *and* conceptual knowledge of fractions. Both forms of knowledge are structurally complex on their own and future teachers need an understanding of their nature and relations as argued by Rittle-Johnson et al. (2015).

Qualitative analysis of data revealed that memorising a step-by-step procedure for drawing and explaining representations may mask what is fundamentally procedural knowledge as conceptual knowledge. This is a limitation of data collection and analysis involving examination responses without being able to clarify PSTs' thinking. While this may be problematic in analysing the data, it is equally important to address the issue it raises in the utilisation of RRTL in future subjects. It is envisioned that greater focus will be given to PSTs' construction of multiple visual representations (e.g., number line and discrete models as well as regional models), and exploring the connections between multiple representations more deeply. Opportunities will be given for PSTs to analyse a range of good and poor quality representations and to offer assistance to hypothetical students in overcoming their misconceptions. The challenge for future research in this space is to ensure that problems are sufficiently complex such that their representations require the accessing and use of robust conceptual understandings.

In making judgements about PSTs' growth of procedural and conceptual understanding we have drawn on non-contextualised multiplication and subtraction problems. While the data provide a degree of insight into the changing character of participants' knowledge, we are guarded in making claims about their conceptual knowledge of fractions that involve multiplication and subtraction *in general*. Exposure to a wider range of non-contextualised and contextualised multiplication and subtraction problems could generate more reliable data to support our claims about fostering PSTs' conceptual/procedural knowledge than those used in our pre- and post-tests. We also noted that, for a number of participants (e.g., Ashley), RRTL worked well in fostering conceptual knowledge in subtraction problems but not in multiplication problems. It would seem that there is room to fine-tune RRTL to suit the structure of problems.

The coding scheme that was used in the present study may have inadvertently led readers to conclude that conceptual knowledge of fractions is more complex and, indeed, desirable than procedural knowledge. It is important to iterate that the coding here does not reflect such a view as the values for the coding were used on a nominal scale. Whilst there is empirical support for the RRTL implemented within an existing teacher education course, the present study did not employ an experimental design with a treatment-control design which would have permitted us to control competing potential threats to validity and reliability through extraneous factors influencing the effect of RRTL. We suggest that future studies use a quasi-experimental randomised design to implement RRTL with more sensitive measures of procedural and conceptual knowledge in pre- and post-tests.

The present study showed that PSTs progressed through their first content and pedagogy subject with limited conceptual knowledge of fractions. The tasks that were used here were designed to build on participants' prior knowledge and track changes in their knowledge that were brought about by RRTL. It is possible that the growth of conceptually rich knowledge may not be sustained over a longer period. We suggest that future studies need to employ a range of complex tasks that could motivate PSTs to reflect on their knowledge development and applications to classroom practice.

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