



Developing a model for problem-solving in a Grade 4 mathematics classroom

Author:Susan Nieuwoudt¹**Affiliation:**

¹Faculty of Educational Sciences, North-West University, Potchefstroom Campus, South Africa

Correspondence to:

Susan Nieuwoudt

Email:

susan.nieuwoudt@nwu.ac.za

Postal address:

PO Box 20189, Noordbrug
2522, South Africa

Dates:

Received: 21 Aug. 2014

Accepted: 29 Sept. 2015

Published: 18 Dec. 2015

How to cite this article:

Nieuwoudt, S. (2015). Developing a model for problem-solving in a Grade 4 mathematics classroom. *Pythagoras*, 36(2), Art. #275, 7 pages. <http://dx.doi.org/10.4102/pythagoras.v36i2.275>

Copyright:

© 2015. The Authors.
Licensee: AOSIS
OpenJournals. This work is licensed under the Creative Commons Attribution License.

The teaching of problem-solving through the development of a problem-solving model was investigated in a Grade 4 mathematics classroom. Learners completed a questionnaire regarding their knowledge of mathematical problem-solving, their attitudes towards problem-solving, as well as their experiences in solving problems. Learners' responses revealed overall negative beliefs towards problem-solving as well as a lack of knowledge about what problem-solving in mathematics entails. The teacher then involved the learners in a structured learning programme where they worked in cooperative groups of six on different kinds of mathematical problems to solve. The groups regularly engaged in discussions about the different strategies they were using to solve a specific problem and eventually succeeded in formulating a generic problem-solving model they could call their own. The model was effectively used by the learners to solve various mathematical problems, reflecting their levels of cognitive development to a certain extent.

Introduction and orientation

Problem-solving has to be the primary goal of the teaching and learning of mathematics, giving each learner the opportunity to engage in problem-solving activities (NCTM, 2000). Learners not only learn mathematics while solving problems, but also develop problem-solving skills and strategies while doing mathematics (Lesh & Zawojewski, 2007; Schoenfeld, 1992, 2013).

The identification and solving of problems using critical and creative thinking, working in groups and recognising that problem-solving contexts do not exist in isolation are some of the general aims set for education and training in South Africa (Department of Basic Education, 2011). Moreover, problem-solving is also part of every content area in the South African Intermediate and Senior Phases Mathematics curricula.

South African Grade 8 learners performed poorly in the Trends in International Mathematics and Science Study (TIMSS) (Howie, 2004; Reddy, 2006). The TIMSS evaluated, among others, acquired mathematics knowledge as well as the use of logical thinking while solving problems (Heideman, 1999). Brenner, Herman, Ho and Zimmer (1999) attribute the outstanding performance of learners from Singapore, Korea and Japan to effective teaching and learning of problem-solving skills at school. South African mathematics learners' poor results, in contrast, seemed to relate to learners' inadequate mathematics knowledge and skills, especially problem-solving skills, and to poor mathematics teaching and learning (Howie, 2004; Reddy, 2006). This is a reflection of the situation not only in Grade 8 mathematics classrooms but also in Grade 4, where learners encounter problem-solving for the first time in a more formal (structured) way than before. Although issues related to problem-solving in mathematics have been widely researched at secondary school level, little is known about problem-solving strategies at Grade 4 level.

The aim of this article is to report on the process by which Grade 4 mathematics learners develop a problem-solving model while solving problems.

More specifically, the following research question was addressed: How can problem-solving be taught in a Grade 4 mathematics classroom?

Conceptual and theoretical framework

The research reported in this article was executed from a social-constructivist perspective regarding the learning of mathematics. Learners construct their own mathematical knowledge by connecting mathematical facts, procedures and ideas (Hiebert & Grouws, 2007). Understanding or meaningful learning involves not only internal or mental representations of individual learners, but also social and cultural aspects. The development of mathematical concepts and mathematics learners' problem-solving abilities is highly interdependent and socially constructed (Lesh &

Read online:

Scan this QR code with your smart phone or mobile device to read online.



Zawojewski, 2007). Therefore, the teaching of mathematics through problem-solving provides opportunities for learners to gain understanding and attain higher levels of achievement (Rigelman, 2007).

Problem-solving refers to a mathematical situation that poses a mathematical question to which the solution is not immediately accessible to the solver, because they do not have a way to relate the data to a solution (Callejo & Vila, 2009). For the purpose of this article, mathematical problem-solving refers to a person's efforts to solve a problem that they have not encountered before. Through solving a given problem, a person should learn some mathematics.

Perspectives on problem-solving vary from a more traditional approach to a models and modelling approach. Traditionally, problem-solving involved the following steps: mastering the prerequisite mathematical ideas and skills, practising the newly mastered ideas and skills in solving word problems, learning general problem-solving processes and, finally, applying the learned ideas and skills to solve real-life problems. Lesh and Zawojewski (2007) view problem-solving as modelling: in response to a real-life problem situation, the problem solver will engage in mathematical thinking as they produce or develop a sensible solution for the problem. This suggests that people learn mathematics through problem-solving and that they learn problem-solving through doing mathematics. For Schoenfeld (2013), solving problems is part of the doing and sense-making of mathematics. In doing mathematics, learners investigate, make conjectures and use problem-solving strategies to verify those conjectures.

Most current problem-solving models have adapted Polya's four-phase model of understanding the problem, devising a plan, carrying out the plan and looking back (Polya, 1945/1973). Lester (1985) added meta-cognitive behaviour to Polya's model. Schoenfeld (1992) included managerial processes in the teaching of problem-solving, to be discussed with learners while they are solving problems. Fernandez, Hadaway and Wilson (1994) introduced a dynamic and cyclic interpretation of the model, including meta-cognitive processes (self-monitoring, self-regulating and self-assessment). According to this model, a learner starts solving a problem by engaging in thought to understand a given problem, then moves into the planning stage. After some time spent on making a plan, the learner's self-monitoring of understanding creates the need to understand the problem better and the learner returns to the understanding-the-problem stage.

In their research on the nature of problem-solving behaviour, Lester and Kehle (2003) come to the conclusion that the knowledge of good problem-solvers not only exceeds the knowledge of poor problem-solvers, but also is more connected. Good problem-solvers pay more attention to the structural features of problems, while poor problem-solvers' attention is focused on surface features. Furthermore, good problem-solvers are better users of meta-cognition during problem-solving.

Schoenfeld (1992) refers to meta-cognition as the ability that enables problem-solvers to break down a problem into sub-problems, solving the sub-problems and eventually solving the original problem. Wilson and Clark (2004) report on students' use of meta-cognitive language to describe how they go through a meta-cognitive cycle (awareness, evaluation, regulation, evaluation) during problem-solving activities.

Primary school mathematics learners are not mathematical problem-solvers by nature; therefore, they have to be taught problem-solving skills and strategies (McCormick, Miller & Pressley, 1989; Lesh & Zawojewski, 2007). This can be done by using problem-based mathematics lessons (Van de Walle, Karp & Bay-Williams, 2013). These lessons consist of three parts, namely a 'before' part when pre-knowledge is assessed and the problem is presented to the learners, the 'during' part when the learners attempt to solve the problem and the 'after' part when the learners discuss and reflect on their solutions.

Problem-solving can be successfully executed in small groups (McLeod, 1993). The interaction between the teacher and learners as well as among the learners working together in small groups can improve the quality of the teaching and learning of mathematics (Berry & Nyman, 2002). Rather than working on their own, learners in groups have more opportunities to participate in problem-solving activities, discover problem-solving strategies for themselves and report back to other groups than when working on their own (Cangelosi, 2003).

The understanding of the role of beliefs and dispositions in problem-solving has not changed much since Schoenfeld's work in 1992 (Callejo & Vila, 2009; Lesh & Zawojewski, 2007). Certain beliefs with respect to mathematical problem-solving sometimes have negative influences on learners' mathematical thinking, such as: mathematics problems have only one correct answer; there is only one correct way to solve a mathematics problem; only a few learners understand mathematics – other learners are supposed to memorise and apply what they have learnt without understanding; learners who have understood the mathematics they learnt will be able to solve any problem in five minutes or less (Schoenfeld, 1992).

A supportive problem-solving environment can change learners' dispositions towards problem-solving (Yudariah, Yusof & Tall, 1999) from negative to positive. Middleton, Lesh and Heger (2003) conducted problem-solving sessions among learners where they had to solve mathematical problems in small groups. During the sessions learners not only shared their mathematical thinking processes while collaborating in groups, but also revealed their beliefs and dispositions with respect to the mathematics dealt with in a specific problem.

Although there is a strong relationship between learners' approaches to problem-solving and their belief systems, it is difficult to determine a causal relationship between specific beliefs and problem-solving activities (Callejo & Vila, 2009). Learners' beliefs regarding the level of effort required to



solve a mathematics problem, as well as their self-confidence in mathematics problem-solving, influence the learners' involvement in solving a given problem.

From the above arguments it should be clear that school mathematics can be taught through problem-solving and that learners learn mathematics while solving problems. Therefore, the development and use of a problem-solving model has the potential to assist learners in the learning of mathematics.

Empirical investigation

Aim of the investigation

This article reports on one aspect of a broader study (Graaff, 2005), namely an empirical investigation into teaching problem-solving through developing a problem-solving model in a Grade 4 mathematics classroom.

Research method

A qualitative research method by means of a case study was employed in a Grade 4 mathematics classroom in an urban school in Gauteng, South Africa. A purposive sample of one Grade 4 mathematics class was chosen from the three

Grade 4 classes (the population) taught by the participating teacher.

The Gauteng Department of Education, as well as the school, granted written permission for the research. Parents of all the participating learners granted informed consent. We guaranteed that learners' identities would not be revealed.

The class was divided into six small groups of six learners each. These learners' performance in mathematical problem-solving was studied for a period of eight months. For the duration of the investigation, all mathematics topics were taught through problem-solving.

Initially (before they had to solve any mathematical problems), a questionnaire regarding their experiences in problem-solving, their attitudes towards problem-solving, their efforts at problem-solving and their knowledge of solving problems was completed by the class group. Learners from one small group (the investigative group, from now on referred to as IG or Group A) were also interviewed with respect to their beliefs about mathematics, problem-solving and group work.

The small groups, labelled A–F, were each given different kinds of problems to solve. The teacher used problem-based

TABLE 1: Summary of different groups' efforts towards a problem-solving model.

Problems	Suggested problem-solving 'model'	Comments and discussion
Problem 1 The chairs in the school hall of Polokola Primary School are arranged in 35 rows of 48 chairs each. How many chairs are there in the school hall?	<ol style="list-style-type: none"> 1. Draw the problem. 2. Find the answer by counting. 	One of the members of Group A (IG group) suggested that they represent the chairs by drawing lines. Each row of lines was counted and the total (48) written down at the end of each row. When the group reported back to the class, they could neither show any calculation nor provide an answer to the problem, they showed only their drawing. Another group (Group B) drew a rectangle, representing the hall, as well as drawing the chairs in the hall, and then counted the chairs. They came up with the correct answer.
Problem 2 Nine children attended a birthday party. On one of the plates on the table were 24 cocktail sausages, among other food. The birthday girl wanted everyone to have the same number of sausages. How many sausages did each child get to eat?	<ol style="list-style-type: none"> 1. Make a plan. 2. Draw a picture to illustrate the problem. 3. Write down the calculations (in this case division) 4. Find the answer. 	Members of Group A recognised this problem as division and wrote: $\frac{24}{9} = 2$ remainder 6. When the teacher asked them to explain what happened to the 6 sausages, they did not know. Members of Group B drew a picture to solve the problem, indicating the division of sausages among the children. According to their drawing only 5 sausages, and not 6, were left over. One of the group members came up with an answer of $2\frac{1}{5}$, obviously the wrong answer.
Problem 3 A wall is built by laying 17 rows of bricks, using 69 bricks for each row. How many bricks are used?	<ol style="list-style-type: none"> 1. Read the problem. 2. Decide which operation to use for the calculation. 3. You need to know your multiplication tables. 4. Do the calculation. 5. Write down the answer to the problem. 	Some learners of Groups C and D realised that they had to multiply 69 by 17, but did not know how to do the calculation. They clearly knew that multiplication involves repeated addition, but could not find the correct answer. A learner from Group C found the wrong answer by adding the product of 17 and 9 to the product of 17 and 6 (instead of 60).
Problem 4 A string of beads is made by using 3 red beads for every 5 blue beads. How many red beads are there in a string containing 60 blue beads?	<ol style="list-style-type: none"> 1. Read the problem 2. Understand the problem 3. Try to solve the problem. 4. Count to find the answer. 	Although one of the group members (Group D) read the problem to the others, there was no reaction – apparently they did not understand the problem. The teacher realised that it could be that the group members did not understand the phrase '3 red beads for every 5 blue beads'. One of the group members explained it to the other group members. From the observations it became clear that she had a good vocabulary, as well as a sound number sense. The teacher encouraged the other groups to try solving the problem by drawing a circle, and then adding 3 red beads, 5 blue beads, and so forth.
Problem 5 A frog fell into a well 10 m deep. For every 3 m the frog jumped to get out of the well, it slid back 2 m. How many jumps does the frog need to get out?	<ol style="list-style-type: none"> 1. Draw the problem. 2. Count the jumps to get the answer. 	One of the members of Group A tried to draw the efforts of the frog, indicating that the frog needs 9 jumps to get out of the well. The learners did not realise that the problem involved number patterns.
Problem 6 Six people arriving at a party, started greeting one another by handshakes. How many handshakes did they share?	<ol style="list-style-type: none"> 1. Understand the problem. 2. Act it out. 3. Find the answer. 	One of the learners of Group A responded immediately by saying that the answer is $6 \times 6 = 36$ handshakes. The learners of Group F did not understand the problem and decided to start greeting one another, counting the handshakes, and came up with 30 handshakes.
Problem 7 A girl has 2 skirts and 3 blouses that can be mixed and matched. How many outfits can she put together?	<ol style="list-style-type: none"> 1. Understand the problem. 2. Draw a picture. 3. Find the answer. 4. Make sure that the answer is correct. 	One of the learners of Group E (a boy) did not know the word 'blouse'. Learners found it difficult to solve a problem when they don't understand some of the words in a word problem.



mathematics lessons (Van de Walle et al., 2013) to teach mathematics to the Grade 4 learners. While solving a problem, group members had to design a problem-solving 'model' (see Table 1), indicating step by step how a learner should go about solving the specific problem. After a problem had been solved, different groups from the class had the opportunity to illustrate and explain their problem-solving models.

After the Grade 4 class, with the assistance of their teacher, had developed their problem-solving model (see Figure 4), the model was used to solve more mathematics problems.

Analysis and discussion of the findings

Learners' responses to the questionnaire revealed negative attitudes and beliefs towards problem-solving that could be attributed to a lack of exposure to problem-solving in previous grades. When confronted with questions about solving problems, learners showed little confidence in answering these questions:

1. Can you solve a mathematics problem (like the example)?
2. Do you understand what is asked in the problem?
3. Where would you start solving the problem?
4. Do you have a plan in mind (to help you to solve the problem)?

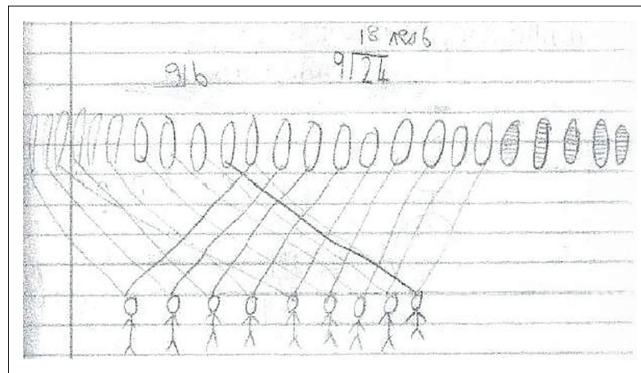
This is consistent with the findings of Callejo and Vila (2009:123) that even high school mathematics learners are reluctant to solve unknown problems because of negative beliefs towards problem-solving.

From the interviews with the IG learners (Group A) the following became evident: although the group members did not regard mathematical problems as difficult as such, and were not afraid to attempt solving a problem, three learners admitted that they did not have an idea where to start to solve a given problem. Whereas only one group member read the problem more than once, trying to understand the problem, the other two learners used trial and error to solve the problem, but did not show any indication that they had tested their solution. One must conclude that they had a lack of knowledge about what problem-solving in mathematics entails.

As has been indicated, the initial data gathering about learners' attitudes towards and knowledge about problem-solving was followed by the implementation of a problem-solving approach in the classroom. I now provide a few brief illustrating examples of group discussions during the problem-solving sessions as a background to the summary of the findings in Table 1.

Problem 2

Nine children attended a birthday party. On one of the plates on the table were 24 cocktail sausages, among other food. The birthday girl wanted everyone to have the same number of sausages. How many sausages did each child get to eat?



Source: Graaff, M. (2005). *Probleemoplossing en die onderrig en leer van wiskunde in graad 4* [Problem solving and the teaching and learning of mathematics in Grade 4]. Unpublished master's thesis. School for Natural Sciences and Technology for Education, Faculty of Education Sciences, North-West University, Potchefstroom, South Africa (p. 71). Available from <http://dspace.nwu.ac.za/handle/10394/785>

FIGURE 1: Group B's drawing for Problem 2.

Two of the members of Group B read the problem to the other group members. Although Luke read fluently, Carin struggled to pronounce some of the words. James drew 24 cocktail sausages and nine learners beneath the sausages (see Figure 1).

Ockert: Let's distribute the sausages among the learners.

[James drew lines between the sausages and the nine learners. Adeli distributed the next nine sausages, while the others kept on counting.]

Group members: 1, 2, 3, 4, 5, 6, 7, 8, 9.

Ockert: We divide each sausage into nine parts [referring to the remaining five sausages.]

[Adeli wrote down the answer as $2\frac{1}{5}$, clearly not the correct answer.]

(Graaff, 2005, p. 70)

Problem 3

A wall is built by laying 17 rows of bricks, using 69 bricks for each row. How many bricks are used?

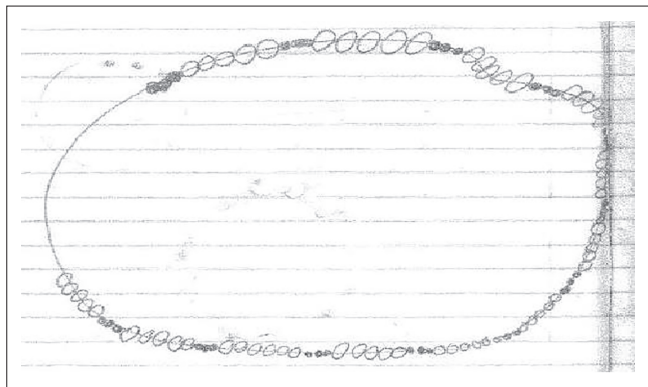
While one learner from Group A was reading the problem to the other members of the group, another learner started writing down '69' 17 times, indicating the rows of bricks in the wall. Another learner knew she had to multiply 69 by 17, but could not find the answer. A learner from Group C thought that adding the sum of 17×9 and 17×6 would give the correct answer.

Problem 4

A string of beads is made by using three red beads for every five blue beads. How many red beads are there in a string containing 60 blue beads?

One of the members of Group A, Jaco, read the problem to the others. Two of the girls in the group did not understand what 'three red beads for every five blue beads' meant. Another girl, Ronel (with encouragement from the teacher) tried to explain it to the others:

Ronel (to Jaco): Draw a big circle, put three red beads on the circle. [Jaco drew the circle with the 3 beads on the circle (see Figure 2).]



Source: Graaff, M. (2005). *Probleemoplossing en die onderrig en leer van wiskunde in graad 4* [Problem solving and the teaching and learning of mathematics in Grade 4]. Unpublished master's thesis. School for Natural Sciences and Technology for Education, Faculty of Education Sciences, North-West University, Potchefstroom, South Africa (p. 69). Available from <http://dspace.nwu.ac.za/handle/10394/785>

FIGURE 2: Group A's drawing for Problem 4.

Ronel: Jaco, you have to draw five blue beads on the circle. We have to complete the circle using 50 blue beads.

Jane: No, 60 blue beads.

Ronel: Oh yes, 60.

Jaco: What now?

Ronel: Well, just continue the same way. Now you draw five blue beads and three red beads. [Jaco continued like this until there were a total of 60 blue beads. The group members then counted the red beads.]

Group members: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36.

Ronel: There are 36 red beads on the string.

(Graaff, 2005, p. 68)

Problem 7

A girl has two skirts and three blouses that can be mixed and matched. How many outfits can she put together?

Amanda (a member of group E) took a pencil and drew the three blouses and two skirts, showing the different combinations in Figure 3a. Ben did not know the word 'blouse' and Amanda explained (to the other boys in the group) that a blouse is a girl's shirt. Jake showed Amanda that there are more ways to combine the skirts and blouses (see Figure 3b). Although the other group members agreed with Jake's solution, Carol drew her own picture, in order to assure herself that Jake had drawn all the possible combinations.

From the solutions to problems 1 to 7 (see Table 1), as well as other problems solved during the investigation, the following became clear:

1. Some learners experienced difficulties in understanding a given problem.
2. Most learners realised they had to do something (make a plan) to solve a given problem.
3. Learners used different problem-solving strategies (draw a picture, do a calculation, act it out, etc.) to solve a problem.

4. Learners were not able to solve some of the problems because their calculations were wrong.
5. Learners did not always check solutions to given problems.

Not all the stages of the problem-solving models used by Polya (1945/1973), Schoenfeld (1992, 2013), Fernandez et al. (1994), and others referred to earlier in the article, are reflected in Table 1. This result is in line with the views of McCormick et al. (1989) and Lesh and Zawojewski (2007) that learners are not problem-solvers by nature, and that to become successful problem-solvers, learners have to be supported in discovering problem-solving strategies. In addition, the social interaction between learners during the solving of the respective problems assisted them in exploring and constructing 'new' mathematical knowledge.

Each group had the opportunity to display and explain their model to the other groups. During the class discussions each group tried to convince the other groups that their model could be used to solve any problem in Grade 4 successfully. The teacher asked the learners the following questions:

Teacher: Can your group's model for problem-solving be used to solve a mathematics problem?

Learner 1: Yes.

Teacher: Will you always get the right answer when you use this model?

Learner 2: Sometimes, but not always.

Teacher: How would we know that the answer to the problem is wrong?

Learner 1: When you mark it wrong.

Learner 2: When the different groups' answers are not the same.

Learner 3: When we haven't answered the question.

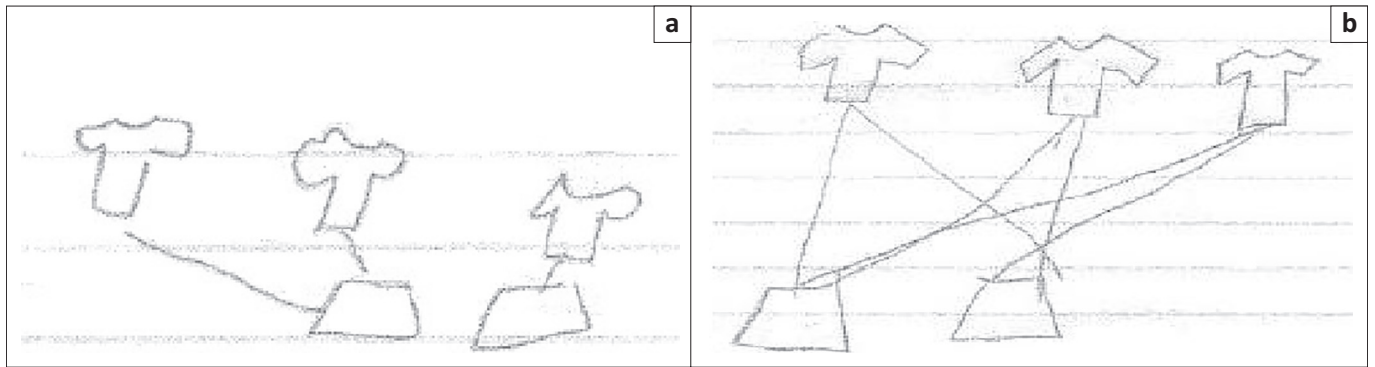
Teacher: Yes! How do we know that we haven't answered the question?

Learner 4: At the end, after we have done everything.

(Graaff, 2005, p. 79)

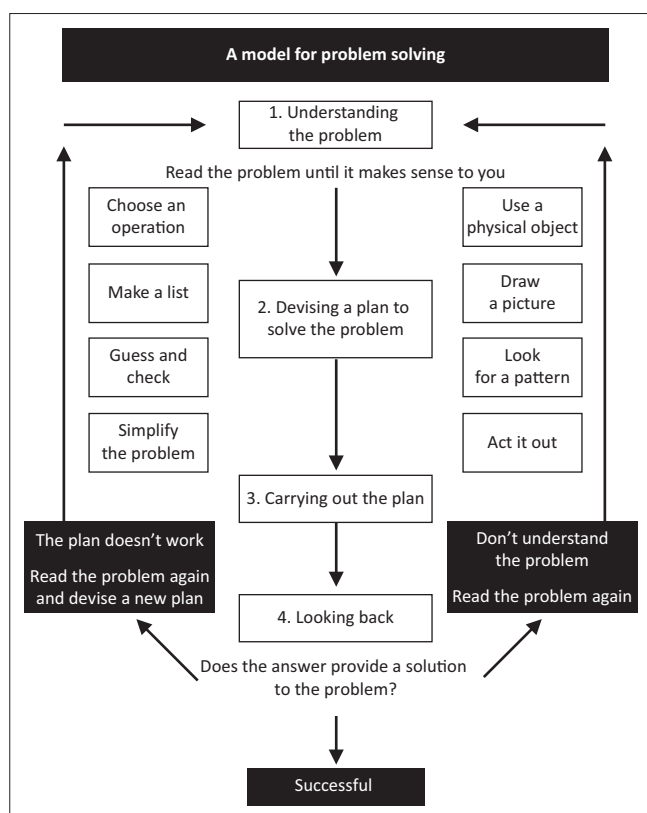
The teacher then asked the Grade 4 learners how they would guide other learners when (1) they didn't understand a problem, (2) they had to make another plan when their plans to solve a problem did not work out and (3) how would they (the learners) know that their solution to a specific problem was correct. Learners reworked their models, resulting in the problem-solving model illustrated in Figure 4. This cyclical model closely resembles the mathematical problem-solving 'method' originally initiated by Polya (1945/1973), and adapted by Fernandez et al. (1994) and others.

When the groups applied the developed model to solve some other mathematical problems, the initial observations of the teacher were confirmed, namely that learners with a well-developed number sense solved problems with more ease than those with a weak number sense, that learners' ability to perform the basic operations correctly enabled them to



Source: Graaff, M. (2005). *Probleemoplossing en die onderrig en leer van wiskunde in graad 4* [Problem solving and the teaching and learning of mathematics in Grade 4]. Unpublished master's thesis. School for Natural Sciences and Technology for Education, Faculty of Education Sciences, North-West University, Potchefstroom, South Africa (p. 72). Available from <http://dspace.nwu.ac.za/handle/10394/785>

FIGURE 3: Group E's drawings for Problem 7.



Source: Adapted from Graaff, M. (2005). *Probleemoplossing en die onderrig en leer van wiskunde in graad 4* [Problem solving and the teaching and learning of mathematics in Grade 4]. Unpublished master's thesis. School for Natural Sciences and Technology for Education, Faculty of Education Sciences, North-West University, Potchefstroom, South Africa (p. 81). Available from <http://dspace.nwu.ac.za/handle/10394/785>

FIGURE 4: The problem solving model developed by the teacher and Grade 4 learners.

solve problems and that learners needed basic mathematical knowledge and skills to solve problems.

Conclusions

Grade 4 learners, assisted by their mathematics teacher, were able to develop a problem-solving model while trying to solve novel mathematics problems. Although Grade 4 learners were able to use the developed model to solve other mathematical problems, the success of the teaching of mathematics through problem-solving depends on more than one factor.

For problem-solving to be effective in a primary school mathematics classroom, the mathematics teacher has to plan thoroughly for teaching, involve the learners actively in the learning-teaching activities and play a crucial role as facilitator by teaching problem-solving with the aid of a guideline such as a problem-solving model.

Mathematics teachers need to understand the role of problem-solving in learners' everyday lives, as well as the importance of problem-solving in the mathematics classroom. By incorporating problem-solving in their classrooms, teachers will enable learners not only to attain one of the general aims of the South African curriculum, namely to identify and solve problems and make decisions using critical and creative thinking, but also to attain a specific aim for school mathematics, namely to apply mathematics to solve problems, using acquired knowledge and skills.

Acknowledgements

This article is partially based on the research by Magda Graaff (Graaff, 2005) in one of her Grade 4 mathematics classrooms in a Primary School in Randfontein, Gauteng, South Africa.

Competing interests

The author declare that she has no financial or personal relationship(s) that may have inappropriately influenced her in writing this article.

Author's contribution

S.N. (North-West University) wrote the manuscript based on the original research by Magda Graaff (Randfontein Primary School) under the supervision of S.N.

References

Berry, J., & Nyman, M. (2002). *Co-operative learning and small group learning and mathematics*. Available from http://www.tech.plym.ac.uk/maths/CTMHOME/research_teaching_&_learning_Coop

Brenner, M.E., Herman, S., Ho, H., & Zimmer, J.M. (1999). Cross-national comparison of representational competence. *Journal for Research in Mathematics Education*, 30(5), 541–557. Available from <http://www.jstor.org/stable/749773>



- Callejo, M.L., & Vila, A. (2009). Approach to mathematical problem solving and students' belief systems: Two case studies. *Educational Studies in Mathematics*, 72, 111–126. <http://dx.doi.org/10.1007/s10649-009-9195-z>
- Cangelosi, J.S. (2003). *Teaching mathematics in secondary and middle school: An interactive approach*. Upper Saddle River, NJ: Prentice Hall.
- Department of Basic Education. (2011). *National curriculum statement Grades 10–12 Mathematics*. Pretoria: DBE.
- Fernandez, M.L., Hadaway, N., & Wilson, J.W. (1994). Problem solving: Managing it all. *The Mathematics Teacher*, 87, 195–199.
- Graaff, M. (2005). *Probleemoplossing en die onderrig en leer van wiskunde in graad 4* [Problem solving and the teaching and learning of mathematics in Grade 4]. Unpublished master's thesis. School for Natural Sciences and Technology for Education, Faculty of Education Sciences, North-West University, Potchefstroom, South Africa. Available from <http://dspace.nwu.ac.za/handle/10394/785>
- Heideman, N. (1999). Probleemoplossing, AMESA en wiskundekompetisies [Problem solving, AMESA and mathematics competitions]. *Suid-Afrikaanse Tydskrif vir Natuurwetenskap en Tegnologie*, 18(3), 74–75. <http://dx.doi.org/10.4102/satnt.v18i3.723>
- Hiebert, J., & Grouws, D.A. (2007). The effects of classroom mathematics teaching on students' learning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte, NC: Information Age Publishing.
- Howie, S.J. (2004). A national assessment in mathematics within an international comparative assessment. *Perspectives in Education*, 22(2), 149–162. Available from http://repository.up.ac.za/bitstream/handle/2263/4809/Howie_National%282004%29.pdf
- Lesh, R., & Zawojewski, J. (2007). Problem solving and modelling. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–803). Charlotte, NC: Information Age Publishing.
- Lester, F.K. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41–69). Hillsdale, NY: Lawrence Erlbaum Associates.
- Lester, F.K., & Kehle, P.E. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh, & H. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 501–518). Mahwah, NJ: Erlbaum.
- McCormick, B.C., Miller, G., & Pressley, M. (1989). *Cognitive strategy research: From basic research to educational applications*. New York, NY: Springer.
- McLeod, D. (1993). Affective responses to problem solving. *Mathematics Teacher*, 86(9), 761–763.
- Middleton, J.A., Lesh, R., & Heger, M. (2003). Interest, identify, and social functioning: Central features of modeling activity. In R. Lesh, & H. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 405–433). Mahwah, NJ: Erlbaum.
- National Council of Teachers of Mathematics (NCTM). 2000. *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Polya, G. (1973). *How to solve it: A new aspect of mathematical method*. (2nd edn.). Princeton, NJ: Princeton University Press. (Original work published 1945)
- Reddy, V. (2006). *Mathematics and science achievement at South African schools in TIMSS 2003*. Cape Town: HSRC Press. Available from <http://www.hsrcpress.ac.za/product.php?productid=2148>
- Rigelman, N. (2007). Fostering mathematical thinking and problem solving: The teacher's role. *Teaching Children Mathematics*, 13(6), 308–319.
- Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan. Available from http://gse.berkeley.edu/sites/default/files/users/alan-h-schoenfeld/Schoenfeld_1992%20Learning%20to%20Think%20Mathematically.pdf
- Schoenfeld, A. (2013). Reflections of problem solving theory and practice. *The Mathematics Enthusiast*, 10, 9–34. Available from http://www.math.umd.edu/tmme/vol10no1and2/1-Schoenfeld_pp9_34.pdf
- Van de Walle, J.A., Karp, K.S., & Bay-Williams, J.M. (2013). *Elementary and middle school mathematics. Teaching developmentally*. (8th edn.). Boston, MA: Pearson.
- Wilson, J., & Clark, D. (2004). Towards the modeling of mathematical metacognition. *Mathematics Education Research Journal*, 16(2), 25–48. <http://dx.doi.org/10.1007/BF03217394>
- Yudariah, B.T., Yusof, M., & Tall, D. (1999). Changing attitudes to university mathematics through problem solving. *Educational Studies in Mathematics*, 37, 67–82. <http://dx.doi.org/10.1023/A:1003456104875>