MISCONCEPTIONS AND ERRORS IN LEARNING INTEGRAL CALCULUS

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ABSTRACT

The paper presents the results of a case study examining students' difficulties in the learning of integral calculus. It sought to address the misconceptions and errors that were encountered in the students' work solution. In quantitative study, the marks obtained by 147 students of Diploma in Computer Science in advanced calculus examinations were used as a measurement to evaluate the percentages of errors. Further, qualitative study examined the types of errors performed by 70 diploma students of the advanced calculus courses in their on-going assessments. The students encountered more difficulties in solving questions related to improper integrals for standard functions (63.1 percentages of errors). The three techniques of integration, namely by parts, trigonometric substitution and partial fraction with combined percentage errors of 42.8 also contributed to this. The types of conceptual errors discovered are symbolic, standard functions recognition, property of integral and technique determination. The procedural errors are due to the confusion between differentiation and integration process while the technical errors have foreseen the students struggling with poor mathematical skills and carelessness. The results will thus be useful to Mathematics educators who are keen in designing functional teaching and learning instruments to rectify the difficulties and misconceptions problems experienced by calculus students.

Keywords: *misconceptions; errors; integral calculus; integration techniques; learning difficulties*

INTRODUCTION

Integral calculus has been considered as a challenging topic by many students. Each level of difficulties in acquiring a good working knowledge of integral calculus varies across curriculums, institutions' educational practices, the students' accumulative mathematical skills and norm cultures of its countries. According to Tall (2012), it is impossible for university to deliver its programmes without calculus. Differentiation and integration are essential topics for many science and technology courses where solid knowledge on derivatives and integrals as well as its applications are foremost (Tall, 2011; Metaxas, 2007; Pepper, Stephanie, Steven & Katherine, 2012).

As mathematics learning contributed higher rates of school failure as compared to other discipline of learning at international and transcultural level (Coronado-Hijón, 2017), addressing the errors and misconceptions in mathematics learning is important for university students. There seems to be some consistency in the pattern of common mistakes found in every round of semester classes. These repetitive mistakes compounded by years of erroneous concepts on certain important basic mathematical skills can seemly be undaunting. When the students produce numerous similar mistakes again and again, this learning difficulty can cause them to give up on learning Mathematics. Poor understanding on the concepts of functions, limits and derivatives leads to difficulties in learning integral calculus (Dane, Cetin, Bas & Sagirli, 2016; Hashemi, Abu, Kashefi, Mokhtar & Rahimi, 2015; Tall, 2009; Orton, 1983).

Misconceptions and errors are inter-related, but they are also distinct. The Oxford dictionary defined a misconception as *a view or belief that is incorrect because of faulty thinking and understanding*. An error is *a mistake, slip, blunder or inaccuracy and a deviation from accuracy*. The misconception indicates a misunderstanding of an idea or concept whereas the error indicates incorrect applications or executions of the concepts, theories or formulas. The evidence of misconception is based on how many errors produced. According to Green, Piel and Flowers (2008) and Li (2006), the students' misconceptions produced systematic errors. Specifically, any misunderstandings occurred on either the students' procedural knowledge or conceptual knowledge, or both. Since errors produced were comparatively consistent, obvious and known, as it occurred throughout the many years of students' mathematics learning. The corrections using assisting expert knowledge and tools were often helpful (Li, 2006; Smith, DiSessa & Roschelle, 1993). When the errors were noticeable, the misconceptions were usually undetectable without detailed observation. Occasionally, the misconceptions could even be shrouded in accidentally correct answers (Smith et al., 1993). Riccomini (2005) theorised that unsystematic errors as unexpected, non-repeating wrong answers which could easily be corrected by the students themselves, with minimal instruction from facilitators.

Donaldson (1963) classified the students' mathematics errors into three types; namely structural, arbitrary and executive errors. In Donaldson's (1963) work, high school and college students managed to utilise basic integration techniques to solve mathematics problems, but unfortunately they misunderstood the principal concepts (Orton, 1983). Avital and Libeskind (1978) categorised three types of difficulties that the students faced in mathematical induction; namely conceptual, mathematical and technical difficulties. Seah (2005) classified the students' potential errors and misconceptions while solving integration problems into three categories: namely conceptual, procedural and technical errors. Seah (2005) described the conceptual errors as an inability to comprehend concepts and relationships in problems; the procedural errors as having conceptual understanding but failing to perform manipulations or algorithms; and the technical errors as Mathematics knowledge inadequacy and carelessness. At times, the multiple errors were expected and even seen in a single work solution.

A Mathematics error that is due to carelessness is less serious, but an error that results from misconception must be addressed and replaced. Some students might imagine, assume and conceive ideas incorrectly, which was beyond the expectation of a teacher, and it usually remained hidden. A good teaching by an experienced instructor must reveal this misconception or else, it will become a hindrance for the students to learn advance materials (Smith et al., 1993). Correcting the students' misconception improved achievement and ensured strong mathematical skills foundation. Askew and Wiliam (1995) postulated that effective learning took place when the students made mistakes first without realization of any possible misconceptions, but later they learnt the trick through open discussions. Even though the misconception could not simply be avoided, strategies for reducing the misconceptions were important and they must also be implemented (Swan, 2001).

Sofronas (2011) had found that the mistakes are often made by the first-year calculus students. Students were either weak in the mastery of calculus concepts or calculus fundamental skills and they were not able to establish the links between concepts and skills. Therefore, these make students difficult to understand the topics of advanced calculus. Muzangwa and Chifamba (2012) reported that majority of the errors and misconceptions on the learning of calculus were due to knowledge gaps in basic algebra. Poor understanding on basic concepts affected students' choice of strategy in tackling mathematics problems (Shamsuddin, Mahlan, Umar & Alias, 2015). At times, teaching approach that over emphasises procedural aspects and neglects the solid theoretical side of calculus also lead to difficulties and misconceptions in calculus, as stated by Bezuidenhout (2001). Thus, the errors and misconceptions committed by students should be identified and rectified in order to enhance the students learning in higher education. With regard to this, documenting the students' misconceptions and errors in the learning of integration techniques is crucial for the understanding of students' cognitive in view of effective calculus learning.

OBJECTIVE

The objective of the study was to determine the students' learning difficulties with regards to integral calculus. Essentially, it sought to address the misconceptions and errors that were encountered in the students' work solution.

METHODOLOGY

To answer the objective of the study, the research design was divided into two parts, namely quantitative and qualitative designs. The first part was a quantitative design which sought to study on the students' difficulties in solving integral calculus problems. It involved 147 students of Calculus II for six consecutive semesters and all students were taught by the same lecturer. The Calculus II was an advanced calculus course offered in the third semester of Diploma in Computer Science in a public university in Sarawak, Malaysia. In the consecutive six examinations, the five main important types of integral questions, namely improper standard integral, integration using completion of the square, integration by u-substitution, integration by parts and integration using partial fractions were selected for the study. These five main important types of integral questions contributed an average of 47 per cents in the final examination. All the selected questions in the six examinations were comparatively similar in function types and instructions. The marks obtained by the students in those questions were used as a measurement to evaluate the percentages of errors.

The second part was a qualitative design, which examined the types of errors performed by the students of advanced calculus course in their on-going assessments for the Semester November 2014 to April 2015. The Calculus II was undertaken by 12 students of Diploma in Computer Science. On the other hand, the Calculus II for Engineering students was undertaken by 12 students of Diploma in Electrical Engineering and 46 students of Diploma in Chemical Engineering. For Engineering students, Calculus II was undertaken in their third semester of study. The common errors performed in the solution steps of integral calculus questions were qualitatively analysed and categorised. A framework developed by Seah (2005) was used as a basis to classify and extend the different possible errors and the misconceptions that the students encountered in solving integration problems (refer Table 1). Tactic *noting patterns and themes* was used to determine what type of error goes with what type of question.

Table 1: Classification of Errors (Seah, 2005)

Types of Errors	Characteristics
Conceptual	Misunderstanding of concept. For example, failure to evaluate the total area of bounded region which is both above and below the x-axis.
Procedural	Improper conduct of algorithm. For example, failure to perform trigonometric rules for integration process.
Technical	Insufficient basic knowledge. For example, error in manipulating binomial expansion.

The first type of errors was *conceptual* errors. Due to failures to comprehend the concepts in problems or errors that arose from failures to appreciate the relationships involved in the problems. The second type of errors was *procedural*. The procedural errors were those which arose from failures to carry out manipulations or algorithms despite having understood the concepts behind the problems. The third type of errors was *technical* errors which were errors due to lack of mathematical knowledge and carelessness.

FINDINGS AND DISCUSSIONS

The findings of the data analysis was carried out to determine the students' difficulties in learning integral calculus and some common errors were made by the diploma students in advanced calculus courses from a public university in Sarawak, Malaysia.

STUDENTS' DIFFICULTIES ON INTEGRAL CALCULUS

The Calculus II course has a significant portion of integration questions, which ranges between 45-49 per cents. The students' performance on the questions related to standard functions, u-substitution and techniques such as by parts, trigonometric substitution, partial fractions and completion of the square in the examinations was recorded. The data analysis was conducted for the six consecutive semesters (June-September 2013, November 2013-March 2014, June-September 2014, November 2014-March 2015, June-September 2015 and November 2015-March 2016). The selected exam questions were of similar types and instruction throughout the six semesters. A total of 147 diploma students of Diploma in Computer Science were involved in the study. Firstly, the original marks obtained by the students for the selected type of questions were recorded. Secondly, the average marks ("0" = zero mark … "5" = full marks) for each type of topical questions in every semester, were calculated (refer Table 2). Thirdly, both average errors ("0" = zero error … "5" = full errors) and the percentage errors for the corresponding topical questions were computed (refer Table 3).

Table 2: Comparison of Average Marks for Six Consecutive Semesters

Semester	Sep-13	Mar-14	Sep-14	Mar-15	Sep-15	Mar-16
Number of Students	49	38	41	12	3	4
Technique						
Completing the square	4.09694	3.72368	3.73171	2.95833	5.00000	4.87500
u-substitution (with hint)		3.20395	3.60366			4.37500
u-substitution (without hint)	2.87500	2.84868	2.36585	3.00000	4.45833	3.50000
By part, trigonometric substitution, partial fraction	2.32568	2.21749	2.96494	2.13542	3.776042	3.73047
Standard function of Improper integral			2.40854	2.54167	0.25000	2.18750

For algebraic integrals which required the elementary process of completing the square, the percentage errors were 18.7. For proper integrals related to the u-substitution where a hint was given, the percentage errors were 25.4, and when there was no hint given, the percentage errors increased to 36.5. Integrals which apply u-substitution comprised algebraic, exponential, logarithmic and trigonometric functions. Integrals of the type by parts, trigonometric substitution, and partial fractions accounted for 42.8 percentage errors. Improper integrals involved standard functions, i.e. exponential and algebraic functions contributed about 63.1 per cent errors.

Table 3: Comparison of Average Marks and Error Scores among the Techniques of Integration

Technique	Average mark	Average error	% error
Completing the square	4.06428	0.93572	18.71447
u-substitution (with hint)	3.72754	1.27246	25.44927
u-substitution (without hint)	3.17464	1.82536	36.50712
By part, trigonometric substitution, partial fraction	2.85834	2.14166	42.83320
Standard function of Improper integral	1.84693	3.15307	63.06145

The relative errors in the three categories of integral types are shown in *Figure 1*. Integrals for basic functions, whether it was proper or improper integral, contributed errors of 40.9 per cent. The first integration technique, i.e. u-substitution accounted for 31.0 percentage errors. Subsequently, there were three techniques of integration, i.e. by parts, trigonometric substitution and partial fractions, had a combined percentage errors of 42.8, which was actually above average.

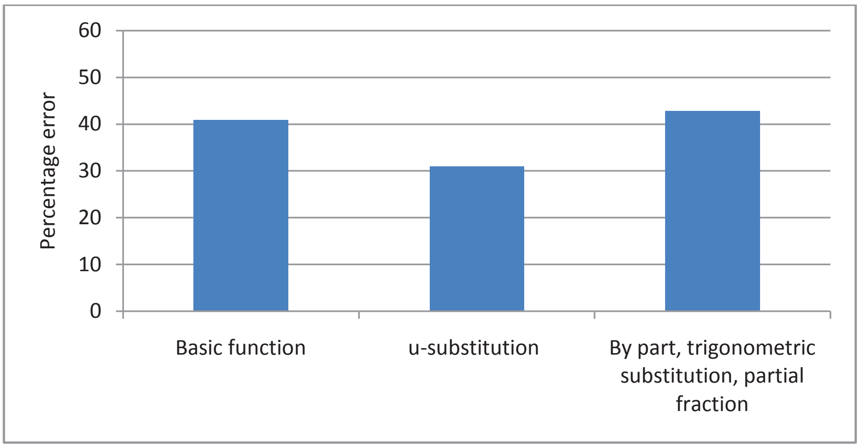

Figure 1: Comparison of Percentage Errors on Integral Types

A BROAD VARIETY OF ERRORS

The various errors produced by the students were similar. The commonality of these mistakes could be because of several reasons such as misinterpretation of questions, misconceptions, wrong assumptions or carelessness. The errors were categorized using the framework errors developed by Seah (2005), i.e. conceptual, procedural and technical. Specifically, the conceptual errors were sub-categorised into four types: symbolic errors, standard functions recognition errors, property of integral errors and techniques determination errors. The mistakes occurred would be sub-categorised because of the confusion between differentiation and integration process as they belonged to the procedural errors. The mistakes occurred due to poor basic mathematical skills and carelessness thus contributed to the technical errors (Seah, 2005).

Conceptual Errors: Symbolic

For the equation formulae, $\int f(x)dx = F(x) + \text{c}$; where $f(x)$ acts as integrand function, and c acts as constant of integration, the symbol '*dx*' is equally important. In Sample 1, the variable '*x*' was taken lightly, and the symbol '*dθ*' was completely ignored in the 1^{st}, 6^{th}, and 7^{th} solution steps. In the 8^{th} step, '*dx*' was used by default without considering the actual variables of the integrand function. The errors related to symbols and notations pertaining to integral calculus might seem trivial, but needless to say, they were very inaudible. These might arise due to lack of emphasis or understanding that every symbol or notation represents a specific, definite meaning of its own. The students did not realise that the structure of mathematical expression became void or invalid when they used the wrong symbols. The students' difficulties with symbols, notations and variables were identified, as one of the problems in calculus (Tall, 1985; White and Mitchelmore, 1996).

Missing and incorrect usage of 'with respect to variables' symbol

Written Sample 1:

$$\int \frac{\sqrt{16-9x^2}}{x} dx$$

$$\int \frac{\sqrt{16-16\sin^2\theta}}{\frac{4\sin\theta}{3}} \frac{4\cos\theta}{3}$$

In first step:

$$4\int \frac{1}{\sin\theta}\left(1-\sin^2\theta\right)$$

In sixth step:

In seven step: Let $u = \sin\theta \quad du = \cos\theta$

In eight step:

$$4\int \frac{1}{u}\left(1-u^2\right) dx$$

Conceptual Error: Standard Functions Recognition

Integral comprises standard functions could be evaluated by applying the standard formulae of integration. It is a very straightforward process, and also generally introduced as fundamentals to basic calculus syllabus. In Samples 2a and 2b, the errors were caused by inability of students to produce the right kind of inverse functions for specific standard functions. In the 2nd steps, both students failed to use "$\cosh^{-1}$" and "$\tan^{-1}$" respectively. The students were unable to distinguish the patterns of several similar standard functions, and hence they failed to memorise and produce the correct results.

Errors due to failures to identify the correct standard formulae

Written Sample 2a:

$$\int \frac{du}{\sqrt{4u^2 - 25}}$$

In first step: $a = 5,\ x = 2u,\ x' = 2$

In second step: $\frac{1}{2}\mathbf{sin}^{-1}\frac{2u}{5} + c$

Written Sample 2b:

$$\int \frac{1}{(x-2)^2 + 3}dx$$

In first step: $x' = \mathbf{(}x - 2\mathbf{)}' = 1\mathbf{,}\ a = \sqrt{3}$

In second step: $\frac{1}{\sqrt{3}}\mathbf{tanh}^{-1}\frac{x-2}{\sqrt{3}} + c\mathbf{;}f\ |x-2| < \sqrt{3}$

The Samples of 3a and 3b were fragments of solutions for the problems that belonged to integration by partial fractions. The process of splitting the rational functions into sums of partial fraction was done correctly in Sample 3b, but not in Sample 3a. However, both students failed to write the correct standard function integrals for the distinctive rational functions. The power rule integration should be used instead of logarithmic rule integration (in bold). In Samples 3c and 3d, students encountered difficulties in rewriting improper integral into proper integral by applying the one-sided limit notation. It is noted that certain students had insufficient fundamental knowledge and understanding on the concepts of limit to tackle questions on improper integrals. The elementary topics of limit and continuity should be mastered by the students as they advanced to calculus of integration (Orton, 1983; Bezuidenhout, 2001). The 'division by zero error' produced in Sample 3d has showed a serious misconception problem.

Errors due to failures to recognise standard functions

Written Sample 3a:

$$\int \frac{4}{x} - \frac{6}{x^2} - \frac{4}{x+2}\,dx$$

In final step: $4\ln|x| - 6\mathbf{ln}|x| - 4\mathbf{ln(}x+2\mathbf{)}^2 + c$

Written Sample 3b:

$$\int \frac{-2}{x} + \frac{3}{x+2} + \frac{2}{(x+2)^2}\,dx$$

In final step: $-2\ln|x| + 3\ln(x+2) + \mathbf{2\,ln}\left|\mathbf{(}x+\mathbf{2)^2}\right| + c$

Written Sample 3c:

$$\int_0^2 \frac{1}{2-x}\,dx$$

In first step: $\lim_{a\to \mathbf{2}} \int_0^a \frac{1}{2-x}\,dx$

In third step: $\lim_{\boldsymbol{a}\to 2}\left[\frac{\mathbf{(2-\boldsymbol{x})^{-2}}}{\mathbf{2}}\right]_{\mathbf{0}}^{\boldsymbol{a}}$

Written Sample 3d:

$$\int_0^2 \frac{1}{2-x}\,dx$$

In first step: $\lim_{x\to 2^+} \int_0^2 (2-x)^{-1}\,dx$

In second step: $\lim_{\boldsymbol{x}\to \mathbf{2}^+}\left[\frac{\mathbf{(2-\boldsymbol{x})^{0}}}{\mathbf{0}}\right]_{\mathbf{0}}^{\mathbf{2}}$

In overall, these students had misconceptions about the derivatives of logarithmic functions. Apparently, the inability to perform standard integral problems indicated failures to grasp the relationship between differentiation and integration processes. It also indicates the students' difficulties in recognising standard functions formulae. The students did not seemly know how to identify whether the problem was a standard function, or otherwise; they did not know how to manipulate the problems in order to apply the standard formulae, and they did not even know when to use the standard formulae for standard functions. All these contributed to the students' difficulties in recognising the formulae.

Conceptual Errors: Property of Integral

Another misconception problem is the misunderstanding on the property of integral. In Samples 4a and 4b, the students were required to determine the solutions for both questions, without any hints. The students failed to evaluate the derivatives of exponential (in Sample 4a) and trigonometric (in Sample 4b) functions, due to carelessness or weak memory. In the 2nd step of Sample 4b, students failed to use $(\mathbf{sec}^2\, x - 1)$. By utilising few so-called 'brilliant-creative-logical' twists and manipulation, the solutions in both samples eventually indicate poor mastery on the techniques of u-substitution.

Misconceptions on property of integral

Written Sample 4a:

$$\int \frac{3e^{3x}}{\sqrt{4e^{6x} - 25}}\, dx$$

In first step: $3\boldsymbol{e^{3x}} \int \frac{1}{\sqrt{(2e^{3x})^2 - 5^2}}\, dx$

In second step: $x' = (2e^{3x})' \rightarrow \mathbf{4}\, e^{3x},\ a = 5$

In final step: $12\, \boldsymbol{e^{3x}} \cosh^{-1} \frac{2e^{3x}}{5} + c$

Written Sample 4b:

$\int \tan^3 x \sec x dx$

In second step: $\int \tan x \ \mathbf{(1 - sec^2} \ x\mathbf{)} \ \sec x \ dx$

In third step: Let $u = \sec x$, $\frac{du}{dx} = -\sec x \tan x$

In fifth step: $\frac{1}{\mathbf{-\sec x}} \int (1 - u^2) u \ du$

In final step: $\frac{1}{-\sec x}\left[\frac{\sec^2 x}{2} - \frac{\sec^4 x}{4}\right] + c$

Conceptual Errors: Techniques Determination

In the following two samples, the students were required to evaluate the integrals by using suitable methods. Both samples show that the students decided to use by parts technique to find the integrals. In Sample 5a, since the integral function is a product of algebraic and exponential functions, the reason for choosing by parts technique might be seen as a good choice. However, it was unexplainable why by parts technique was chosen in Sample 5b. There were also numerous basic calculus errors in the giving solution steps. The failures to recognise u-substitution as an appropriate method to solve the integrals, indicates that the students had problems with techniques determination. This is a case of misidentification of methods in solving the integrals.

Misidentifications of integration techniques

Written Sample 5a:

$\int x e^{x^2} dx$

In first step: Let $u = x$, $du = dx$; $dv = e^{x^2} dx$, $v = \boldsymbol{e}^{x^2}$

In second step: $\boldsymbol{uv - \int vdu} = xe^{x^2} - \int e^{x^2}\ dx$

Written Sample 5b:

$\int \tan^3 x \sec x dx$

In first step: Let $u = \tan^3 x$, $du = 3\mathbf{sec}^2\ x\ dx$; $dv = \sec x\ dx$, $v = \mathbf{sec}\, x\, \mathbf{tan}\, x$

In second step: $\boldsymbol{uv - \int vdu} = (\tan^3 x)(\sec x \tan x) - \int (3\sec^2 x)(\sec x \tan x)\ dx$

The techniques of integration are essential and compulsory topics which must be mastered by any students of advanced calculus. Nevertheless, most students are easily overwhelmed by the diversity of techniques, and they are also at lost in identifying suitable methods to solve integral problems. This type of conceptual errors appeared as one of the hardest problems to rectify because it is concerned with the students' cognitive ability to visualise several types of integration methods, as a whole entity. Simultaneously, it is also involved mental aptitudes in recognising functions, choosing suitable methods and deciding the method that works best. The ability to determine and perform integration technique is vital in the understanding of integral calculus (Sofronas et al., 2011).

Procedural Errors: Confusion between Differentiation and Integration

In Samples 6a and 6b, the correct techniques of integration were used to evaluate the integrals. However, both samples exhibited confusions between differentiation and integration process of trigonometric and inverse trigonometric functions in the initial steps. The students did not realise the mistakes occurred since the subsequent steps and answer were properly written. There was a lack of connections between their procedural and conceptual knowledge and failures in retaining what they learnt (Naidoo, 2007). The procedural errors usually deem as less serious than the conceptual errors. Unfortunately, they can also lead to unnecessary marks deduction therefore they should be avoided at all.

Confusions between differentiation and integration process

Written Sample 6a:

$$\int \sec^4 x \tan^3 x \, dx$$

In second step: $u = \sec^2 x$, $\frac{du}{dx} = \mathbf{tan}\, x$

Written Sample 6b:

$$\int x \tan^{-1}(x^2) \, dx$$

In 1st step: $u = x$, $du = dx$; $dv = \tan^{-1} x \, dx$, $\boldsymbol{v = \frac{2x}{1 + x^4}}$

Technical Errors: Basic Mathematical Skills

Some of the basic mathematical skills errors discovered were additive, arithmetic operation and cancelling errors on rational functions; completion of the square error; radical and right angle errors on trigonometric functions; and, trigonometric identity and exponent errors (refer Samples 7-12c). These miscellaneous technical errors were spanned from basic algebraic skills to functions. These errors were contributed by several bad, impractical years of Mathematics learning process initially. Those who did not possess solid Mathematics knowledge saw themselves struggling to learn the integral calculus. The students' massive mistakes were found in algebra and its functions which led to poor performance in calculus exams (Tally, 2009).

Additive and arithmetic operation errors on rational functions

Written Sample 7:

$$\int \frac{dx}{(x+1)^2 + 5}$$

In first step: $\int \frac{\mathbf{1}}{(\boldsymbol{x}+\mathbf{1})^{\mathbf{2}}} + \frac{\mathbf{1}}{\mathbf{5}} \boldsymbol{dx}$

Cancelling errors on rational functions

Written Sample 8a:

$$\int x \tan^{-1}(x^2)\, dx$$

In third step: $uv - \int vdu = \frac{x^2 \tan^{-1}(x^2)}{2} - \int \frac{2x^3}{2+2x^4}\, dx$

In fourth step: $\frac{x^2 \tan^{-1}(x^2)}{2} - \int \frac{\mathbf{1}}{\mathbf{2}+\boldsymbol{x}}\, \boldsymbol{dx}$

Written Sample 8b:

$$4\int \frac{\cos^2 \theta}{\sin \theta}\, d\theta$$

In first step: $4\int \frac{1-\sin^2 \theta}{\sin \theta}\, d\theta$

In second step: $\mathbf{4}\int \mathbf{1} - \mathbf{sin}\,\theta\, \boldsymbol{d}\theta$

Completion of the square errors

Written Sample 9:

$$\int \frac{dx}{x^2 - 2x + 4}$$

In first step: $x^2 - 2x + 4 = \frac{-1}{2}\left(-2x^2 + 4x + 8\right)$

In second step: $\frac{-1}{2}\left(-2x^2 + 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 + 8\right)$

Radical errors on trigonometric functions

Written Sample 10:

$$\int \frac{12\sec\theta}{\sqrt{36\sec^2\theta - 36}} 6\sec\theta\tan\theta\, d\theta$$

In first step: $\int \frac{\mathbf{1}}{\mathbf{6\sec}\theta - \mathbf{6}} \cdot 12\sec\theta \cdot 6\sec\theta\tan\theta$

Right angle trigonometry errors

Written Sample 11a:

$x = 6\sec\theta$

In first step: $\mathbf{sin}\,\theta = \sqrt{x^2 - 6^2}$

Written Sample 11b:

$x = 6\sec\theta$

In first step: $\tan\theta = \frac{\sqrt{\mathbf{36 - x^2}}}{\mathbf{6}}$

Trigonometric identity and exponent errors

Written Sample 12a:

$$\int \sec^4 x \tan^3 x\, dx$$

In second step: $\int \sec^2 x \, \mathbf{(1 - tan^2 \mathit{x})} \, (\tan^2 x)(\tan x) \, dx$

Written Sample 12b:

$\int \sec^4 x \tan^3 x \, dx$

In second step: $\int \sec^2 x \, \mathbf{(1 - tan^2 \mathit{x})} \, (\tan^2 x)(\tan x) \, dx$

Written Sample 12c:

$$\int \sec^2 x(1+u^2)u^2(u) \frac{du}{\sec^2 x}$$

In first step: $\int u^3 + \; u^{\mathbf{6}} \, du$

Technical Errors: Carelessness

The technical errors can also be contributed by the students' carelessness. Some of the carelessness mistakes include substitution, arithmetic and missing brackets, as shown in Samples 13a, 13b and 13c, respectively. Generally, the students had no misconceptions or procedural errors, but the mistakes were made unwillingly, which might be affected by time-constraint factor or other personal reasons. Unless the students were more attentive and alert, these errors could actually be avoided.

Errors due to carelessness

Written Sample 13a:

$$\int \frac{\sqrt{16-9x^2}}{x} dx$$

In second step: $\int \frac{\sqrt{\mathbf{16-(4\sin\theta)^2}}}{\frac{\mathbf{4}}{\mathbf{3}}\mathbf{sin}\,\theta} \cdot \frac{d\theta}{\frac{\mathbf{4}}{\mathbf{3}}\mathbf{cos}\,\theta}$

Written Sample 13b:

$2A = 5$

In first step: $A = \mathbf{\frac{2}{5}}$

Written Sample 13c:

$$\int e^x \sin\left(\frac{x}{2}\right) dx$$

In fourth step: $e^x \sin\left(\frac{x}{2}\right) - \frac{1}{2} e^x \cos\left(\frac{x}{2}\right) - \int -\frac{1}{4} e^x \sin\left(\frac{x}{2}\right) dx$

CONCLUSION

Improving student's performance in the calculus is indeed a daunting task. High failure rates in integral calculus have been around for decades. The key element to any successful integral calculus achievement includes the ability to pinpoint what difficulties and errors the students are experiencing. The documentation of misconceptions and errors in the learning of integration techniques thus provides handy and useful resources to students and lecturers. This significant result is useful to Mathematics educators who are keen in designing functional teaching and learning instruments to rectify the difficulties and misconceptions problems experienced by calculus students. Future studies need to focus on relevant significant analysis, utilising the concept questions of integral calculus in classroom lectures, and include wider varieties of data related to erroneous concepts. Another suggestion is, to construct valid and reliable instruments, such as questionnaire or structured interview questions. The study should also be extended to the students studying integral calculus in local and international universities, and even into classrooms employing a diverse of pedagogical settings.

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