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Investigating Geometric Habits of Mind by Using Paper Folding

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Abstract: Paper folding studies are quite effective in the development of students' visual and spatial skills. The "paper" used in these studies is a genuine tool that can support the development of geometric habits of mind as well as the visual-spatial skills. This is an action research aimed to investigate the potential of paper folding to improve students' geometric thinking skills and to enhance their achievement in national exams. The improvement in their geometric thinking was investigated based on the framework of the Geometric Habits of Mind. This study was carried out with three students studying in the 11th grade. Four geometry questions were asked to students, and they were expected to solve these questions by paper folding. The solution process was video-recorded. Video-recordings were transcribed, and the obtained data were qualitatively analyzed within the framework of the components of the geometric habits of mind. As a result of the study, it was seen that the students were able to reach solutions more easily by concretizing the intangible questions through paper folding. The students were able to comprehend the fact that the main components of triangles don't change; that is, they are preserved (the angles and the sides do not change).

Key words: geometrical thinking, geometric habits of mind, investigating invariants, paper folding

1. Introduction

School geometry includes concepts associated with each other, systems of axiomatic notation, and mathematical reasoning methods of the relations and transformations of spatial objects (Crompton, 2017). NCTM (2000), when explaining the standards of the geometry programs, indicated that all students from kindergarten to twelfth grade should be able to define spatial relations and that students' use of their visual-spatial skills in the process of solving geometry problems would positively affect the perpetuity of instruction. However, in Turkey, it is seen that despite the revision of the curriculum at the quality of a reform, the visual-spatial skills neither in the teaching-learning activities in the classroom nor in the evaluation process received the due gratitude (Sevimli et al., 2008). Clements and Battista (1992) stated that because achievement in geometry and visual-spatial skills had strong connections with each other, it was necessary that they should be integrated into the instructional curriculum and that various activities, which would improve students' such skills, should be brought into the classroom environment. The fact that visual support in problems related to geometric shapes affect success in a positive way brings forth the importance of the process of conversion of concept-related mental representations to visual representations (Dreyfus, 1991). It is also thought that the richness of students' concept image related to geometric shapes and their interactions between shapes and mathematical knowledge can positively affect their spatial skill development (Arcavi, 2003). In recent years in Turkey, the types of questions that require students to use their visual-spatial skills and geometrical thinking have been encountered in national examinations. These questions were used in the Student Selection Examination (SSE) in 2002, SSE in 2007, the Transition to Higher Education Examination (THEE) in 2011, the Undergraduate Placement Exam (UPE) in 2012, THEE in 2015, and THEE in 2017. These questions require imagination in the mind of the form the paper takes when folded, and they are the kind of questions that can be solved by the use of paper as a manipulative artefact. For example; students encountered a question like the following in 2017 THEE.

ABCD rectangle with short edge 12 units and long edge 18 units was folded from the corners B and D along the lines AL and KC so that |KB| = |LD| = 4 units as seen in the figure.

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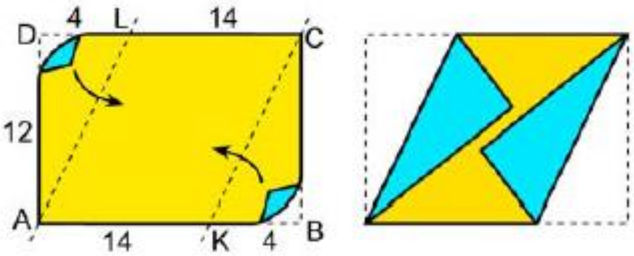
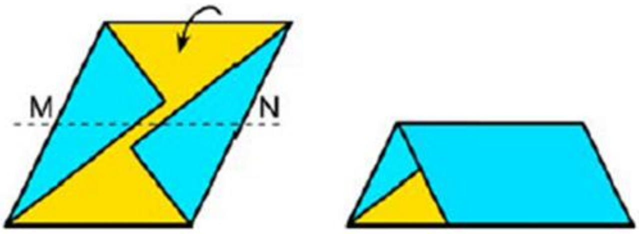


Next, this shape obtained was refolded along the line MN, where M and N were the midpoints of their corresponding sides, as seen below to form a trapezoid.



According to this, how many square units are in the area of this trapezoid?

Figure 1. *Students encountered a question in 2017 THEE*

Students have difficulties in solving the types of questions in Figure 1, and teachers have difficulties in having students to gain the ability of geometric thinking. According to Wiles (2013), students can use paper folding to produce assumptions during reasoning while thinking geometrically, and this process can be examined under the theoretical framework of the geometric habits of mind. This framework, called GHoM, provides teachers with a definition language for geometric thinking and a perspective to analyze students' work (Driscoll, et al., 2008, p.3). Moreover, the correctness of a student's answer can only be seen through clues derived from the thinking process. For this reason, this framework allows teachers to analyze what students do during the stages of thinking and to reflect on whether this thinking is productive for geometric problem solving. From this point of view, the Geometrical Habits of Mind have been accepted as a theoretical framework in this study to improve the geometric thinking ability which is the basis of the aim of the geometry course. "This framework, termed the Geometric Habits of Mind (GHoM), provides teachers with language to describe geometric thinking and a lens through which to view and analyze their work and the work of their colleagues and students." (Driscoll, et. al., 2008).

The aim of this study is to support students' geometrical thinking skills by the help of paper folding. This study bears importance as it is thought that students will be able to improve their geometric thinking skills with paper folding activities and to increase their success in national examinations.

2. Theoretical Framework and Literature

Van de Walle et al. (2004) defined geometric thinking as "the ability to think and draw conclusions about geometric situations." Studies done by Piaget et al. (1956, 1960) and Pierre and Dina van Hiele (1984) are among the first studies on the development of geometric thinking in individuals. These research have received considerable interest but have also been subject to various criticisms. These criticisms have also led many other researchers to work in the field of geometric thinking. In addition to the above-mentioned approaches in the development of geometric thinking and the analysis of this development, in recent years the Geometric Habits of Mind can offer a different perspective.

2.1. Geometric Habits of Mind

The mental habits that Cuoco, Goldenberg, and Mark (1996) introduced are handled in two forms: general mental habits reduced to each discipline and mental habits specific to mathematics. The framework of the geometric habits of mind (GHoM) and the framework of the mathematical habits of mind (MHoM) put forward by Driscoll et al. (2007, 2008), which have an important role in the development of geometric thinking, are concepts developed on the work of Cuoco, Goldenberg and Mark (1996). The mathematical habits of mind have emerged based on the need to help students to think about the "path followed by mathematicians" (Lim & Selden, 2009). On the other hand, these habits that support the application and learning of mathematics as well as advanced mathematical thinking (Leikin, 2007) have been equated with mathematical power (Çimen, 2008). The purpose of these habits have been determined to help learners learn and adopt the ways of thinking (Cuoco, Goldenberg & Mark, 1996; Lim & Selden, 2009). Geometric Habits of Mind (GHoM) contains geometry-specific components compared to the framework of the mathematical habits of mind. Researchers working on this roof (Driscoll et al., 2007; Driscoll et al., 2008), composed of four geometric habits associated with each other, stated that the roof they constructed is a perspective for geometric thinking. The structure of GHoM focuses on identifying evidence for geometric thinking (Driscoll et al., 2008; Koç & Bozkurt, 2012). The main four components of geometric habits of mind are identified as Reasoning with relationships, Generalizing geometric ideas, Investigating invariants, and Balancing exploration and reflection (Driscoll et al., 2008).

Table 1. *GHoM patterns*

Geometric Thinking Habits	Indicators of the GHoM Habits	Student Indicators
Reasoning with relationships	Focus on relationships among separate figures	Determines the relationship between the properties of geometric shapes
	Focus on relationships among the pieces in a single figure	Identifies/classifies the properties of shapes
	Use special reasoning skills to focus on relationships	Associates more geometric shapes with proportional reasoning (congruence-similarity)
Generalizing geometric ideas	Seek solutions from familiar cases or known solutions	Makes generalizations from the special case to explain the problem situation
	Seek a range of solutions using assumed simplifying conditions	Adapts a general situation in a problem for the special case
	Seek complete solution sets or general rules	Can think of all possible situations based on the data in the problem
Investigating invariants	Use dynamic thinking and searching	When a geometric shape is transformed in any way or enlarged/reduced in a specific rate, solves the problem by determining which features of the shape have changed and which ones remained fixed
	Check evidence of effects	Can imagine the geometrical structure as mobile so as not to disturb the conditions of the problem and can explain the emergent effect
Balancing exploration and reflection	Put exploration in the foreground	Can make additional drawings to help solve the problem / Can develop different strategies for solving the problem
	Put end goals in the foreground	Asks questions related to retro-metacognitive capacity in problem solving / Can make explanations through mathematical language for correctness of problem solving

[1]Note: Adapted from Driscoll et. al. (2008)

When the relevant literature is examined, it can be seen that GHoM is a form of generative thinking that supports the learning and application of geometry. This way of thinking means the examination of geometric relations and the reasoning through these relations, the generalization of geometric ideas, the investigation of the changing and unchanging properties of geometric structures, and with all these components, the evaluation of a geometrical structure (Driscoll, et. al., 2007). Driscoll et al. (2007), as a result of the studies they carried out between 2004 and 2008, expressed how teachers could define ways of generative thinking in order to develop geometric thinking of the 5th to 10th grade students. In the study, ways of thinking for both students and adults to be successful geometric problem solvers were defined, and analyses of evidence for geometric thinking were given. As a result of the study, a framework for mental habits in geometry was defined and this framework was proposed to be used as an instructional tool in the development of geometric thinking.

Koç and Bozkurt (2012) conducted their studies on the performances and geometrical reasoning skills of mathematics teacher candidates. Özen and Köse (2013) carried out a study on the mathematics teachers' geometric problem-solving process. Köse and Tanışlı (2014), in their study on determining the geometric habits of primary school teacher candidates, found that the candidates did not have different ways of thinking in the context of geometric habits. Wiles (2013) stated that all the steps of the geometric habits of mind could be analyzed by detail in a paper folding study.

2.2. Paper folding

Japanese paper folding art is used as an educational tool in many countries as it develops the skills of creativity, analysis, synthesis and evaluation in students (Tuğrul & Kavici, 2002). In addition, it is a commonly used tool in geometry education. It can be used to have students in early grades to enjoy mathematics and to construct basic concepts of geometry (Boaks, 2008; Polat, 2013; Johnson, 1957; Olson, 1975; Prigge, 1978). There are also studies on advanced algebra and geometry with regard to paper folding (Alperin, 2000; Geretschlager, 1995; Auckly & Cleveland, 1995; Krier, 2007) although they do not involve education.

As with the Euclidean geometry, there are also postulates and axioms in the paper folding. These postulates and axioms explain how the concepts such as line segment, angle, perpendicularity and congruence are defined in the paper folding world and are expressed differently by different researchers (Alperin, 2000; Geretschlager, 1995; Auckly & Cleveland, 1995; Olson, 1975). Nevertheless, the shortest and understandable postulates that are capable of serving in this work are the postulates known as the Huzita axioms. According to Huzita's axioms, the process of paper folding can be reduced to seven simple postulates (Krier, 2007).

Postulate 1: Given two points P1 and P2, one can fold a single crease which passes through them.

Postulate 2: Given two points P1 and P2, one can fold a crease placing P1 onto P2.

Postulate 3: Given two lines L1 and L2, one can fold a crease placing L1 onto L2.

Postulate 4: Given a point P1 and a line L1, one can fold a crease which will be $\perp$ to L1 and pass through P1.

Postulate 5: Given two points P1 and P2, and a line L1, one can fold a crease that places P1 onto L1 and passes through P2.

Postulate 6: Given two points P1 and P2 and two lines L1 and L2, one can fold a crease that places P1 onto L1 and P2 onto L2.

Postulate 7: Given a point P and two lines L1 and L2, one can fold a crease placing P onto L1 which is $\perp$ to L2.

We used only the first five of these postulates for this study, because the creases referred in the first five postulates are useful for showing congruent angles and segments. And, we investigated the process in terms of GHoMs, especially for "investigating invariants". The reason for the selection of "investigating invariants" from the GHoM steps is the standpoint regarding this step as the abstraction phase of the geometric concepts (Wiles, 2013) necessary.

3. Method

The following section covers the research model, participants, process and data analysis.

3.1. The Model of the Study

One of the researchers, with the help of teaching geometry lessons for secondary school students for years, saw that students had difficulties in solving some of the questions requiring spatial thinking. He found that students had a very hard time especially in solving the questions such as the folding of triangles and quadrilaterals. It was observed that students could not recreate such folds in their minds. The researchers of this study were convinced that students could overcome these difficulties with paper folding efforts and decided to conduct their research by the action research method.

Mertler (2008) states that action research provides teachers with the opportunity to work in their own classroom, thereby improving their qualifications and effectiveness by better understanding their teaching methods, students and evaluation systems. Action research aims to solve problems in a program, organization or community (Patton, 2015). One of the reasons for the choosing of action research in the study was to estimate that the difference between research and action would decrease at an imperceptible rate during the resolution of the identified problem (Patton, 2015).

3.2. Participants

Our purpose was support students' geometric thinking skills while they try to solve two-dimensional (2D) problems requiring the use of visual skills and to help them solve paper folding problems like the ones presented in Figure 1. For this purpose, the paper focuses on examining the contribution of the paper folding process in the development process of the GHoM patterns. This study was conducted with three students studying in the 11th grade of a high school. These students were studying at a school with a maximum of 10 class members. While the mathematical success of these students was moderate, their interest in mathematics was higher than their success. Despite their interest in mathematics, they had difficulty in solving questions that require visual-spatial skills. They asked for help from their teachers in order to overcome the difficulties they faced. For this reason, these students were preferred to work with in accordance with the purposeful sampling method in order to better analyze their solution processes, visual-spatial skills, additional drawings and the changes in their ideas. Students voluntarily participated in this study.

3.3. Tasks

Task 1

A rectangle sheet of paper was given to students. The corners of the rectangle were marked as point-A, point-B, point-C and point-D. Next, the students were asked to fold a crease placing point-B onto side CD and passing through point-A. The figure depicting this action is as follows:

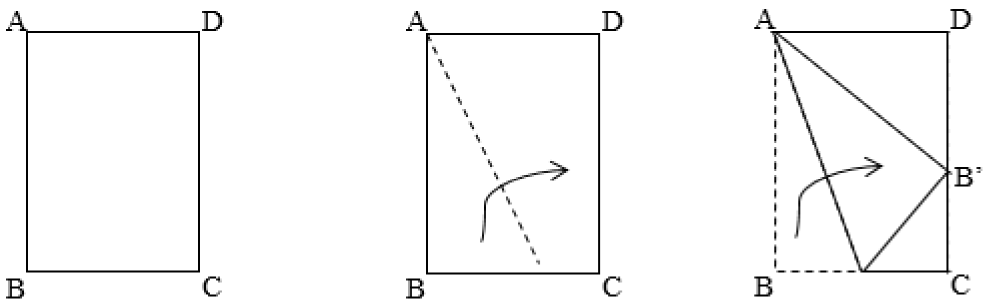


Figure 2. *Task 1*

It was expected that students could determine which angles and sides of the resultant triangles correspond to each other after folding. By the help of this folding, it was possible to say that students could construct in their minds congruent segments and angles of triangles.

Task 2

In this task, the students were given the materials, which had the question presented in Fig. 3 and were appropriately prepared for folding. The problem state for the students to solve in the question was the calculation of the area of triangle AKB' to be formed after folding. |AB| = 20 centimeter, |AD| = 16 centimeter. When the rectangle is fold along AK, point B hits point B' on side |DC|. The following figure was given to students and they were asked to calculate the area of triangle AKB'

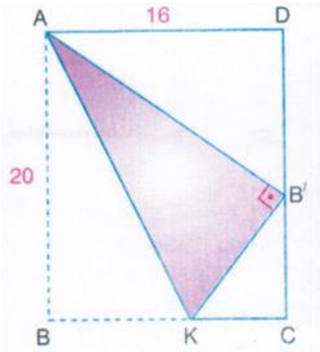


Figure 3. *Task 2*

Students were expected to find the sides of triangle AKB' through paper folding and to calculate its area. In this task, students were able to notice that the angles and sides were preserved by the paper folding material.

Task 3

Students were given the following figure. The instruction was "The area of the first shape is 18 square centimeter larger than the area of the second shape. Calculate AD."

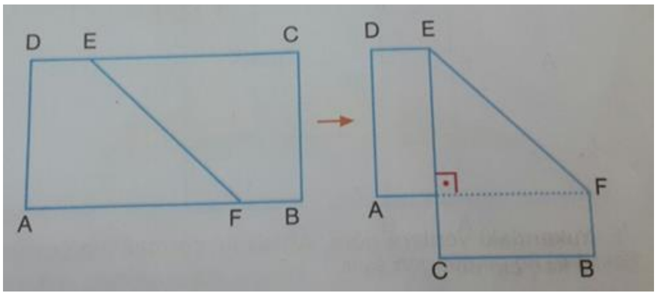


Figure 4. *Task 3*

The paper folding was performed during the task and it was directed to the student as a question. What was expected of the students was to determine that the sides did not change when the folding took place and to notice the invisible (disappearing) area afterwards.

Task 4

When the shape on the left in Fig. 5 is folded along side |AE|, corner B is placed on side |DC|. How many degrees are in angle x?

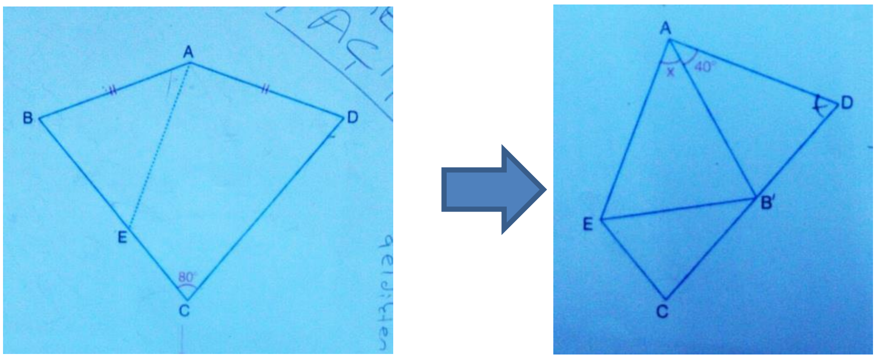


Figure 5. *Task 4*

It was expected of the students to imagine without physically folding the shape that the sides and angles did not change and to complete the solution of the question without folding the paper.

3.4. Process and Data Analysis

In the study, first, the postulates given in Krier (2007) were completed with paper folding by the students. By means of these folds, it was aimed for the students to understand the concepts such as line segment and equality of angles in terms of folding paper. Because, they should know the postulates to accomplish the given tasks. Then, the students were given the four tasks, which were the actual data collection tools in the study, and they were asked to fulfil these tasks. This process was video-recorded and transcribed. The transcripts were independently analyzed with descriptive methods and the results were reported independently by three investigators, taking into account the criteria found in the literature on the GHoM framework and expressed in Table 1. After independent analysis, investigators came together and came to a consensus regarding analysis.

4. Results

Separate analyzes were performed for each task and dialogues and pictures were included to support the analysis results. The results of the study are explained below.

Result of Task 1

Following the understanding of the logic of paper folding and, the accomplishments of the aforementioned postulates through folding, students were asked the first question and were asked to make the desired folds without considering the lengths and to explain the folding that emerged. The students began to do the folding as seen below.

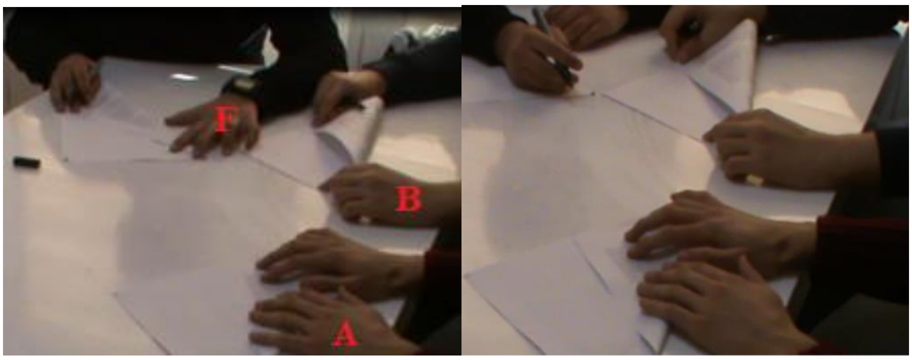


Figure 6. *First folds*

In the behaviors and expressions of students, the indicators of the "investigating invariants" step (13, 17, 19, 26) were found more. Furkan successfully made the fold (Figure 2 folding). Ahmet and Bahadir could not do exactly. In the second picture, Furkan appears to be helping others. Below is the dialogue between the teacher and the students after they did the folds.

12 A (Ahmet): Which sides were equal, guys?

13 F (Furkan): This one and this side of the rectangle were equal (|AB| and |CD|).

14 R (Researcher): Good job! So, Furkan, why were those two sides equal?

15 F: Sir, when we folded, side |AB| here got placed on side |AE| here, so we saw that they were equal.

16 R: Very nice. How did you do it Bahadır?

17 B (Bahadır): Sir, when I folded, corner B of side AB came on this point E, and because angle B was 90°, angle E forming here also became 90°.

18 R: Yes, both that side AB turned out to be equal to side AE and that angle B turned out to be equal to angle E.

19 F: Sir, if we call this F, BF becomes equal to EF.

20 R: How were those sides equal, Furkan?

21 F: Sir, these fold lines completely overlap.

22 A: Are these angles perpendicular?

23 B: No, they do not intersect at right angles. When we fold, because the angles gets placed on top of each other, the fold line becomes bisector, thus the angles becomes equal. (m (∡ EFA) = m (∡ BFA))

24 R: Yes, Furkan, can you tell us what you did?

25 F: When we made a fold line to pass through point A, because angle B completely overlapped with the angle on side CD, called E, it became equal and 90°. Side AB here that we drew along the fold line, as it corresponds to the point at the point we call E on the CD edge. When we draw along the fold line, these sides become equal because side AB overlaps one-to-one with side AE.

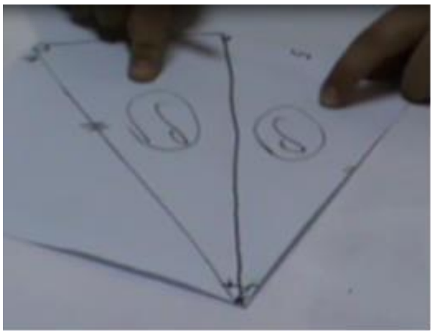


Figure 7. *The paper on which Furkan made the solution*

26 A: Then the angles here becomes equal to each other.

27 F: Yes, when we fold, because the angles overlap they become equal to each other (that is, the fold line becomes bisector)

28 R: How did these sides (|EF| and |BF|) become equal?

29 F: Sir, when we fold, because these sides, too, completely overlap with each other, they become equal

From time to time, it was determined that Furkan exhibited the step of “balancing exploration and reflection” (15, 21, 25, 27, 29), and that Bahadır emphasized the step of “balancing exploration and reflection” (23) even though not as much as Furkan. The following dialogue explains this situation.

However, in Ahmet’s expressions, indicators belonging to the “reasoning with relationships” step were determined rather than the step of “investigating invariants”. This means that Ahmet saw the associations but has not yet begun to examine these associations. In particular, the question Ahmet asks on line 12 can be given as a reference for this situation.

Result of Task 2

In addition to the fold made in Task 1, students were asked to find the area of triangle AKB’ according to the lengths given in Task 2. In this task where the students did not have difficulties with algebraic operations, they made folds as follows:

In students’ expressions, the indicators of the “reasoning with relationships” (38), “balancing exploration and reflection” (59, 61) and “investigating invariants” (40, 43, 45, 57) steps were found from the dialogues given below.

38 B: Let’s show that these angles are equal.

39 R: Yes, very nice. Bahadır said that the angles are equal. Bahadır, which angles are equal?

40 B: The angle is equal to the angle K. (He means angles AKB and AKB’.)

41 R: Very good. Which ones are equal above?

42 B: There is already a 3-4-5 triangle above. (Referring to triangle ADB’. 12, 16, 20)

43 A: It’s equal in places we fold. (He is showing that sides AB and AB’ are equal.)

44 R: Very nice. Good job, Ahmet.

45 B: These angles are equal, too. (Marking angles BAK and B’AK.)

46 A: That place is 90 degrees. (Pointing to angle C)

47 R: Very nice.

48 F: Sir, shall we solve it?

49 R: Solve it, then let’s discuss your solution. (The students continue to solve the question by folding, they also do operations)

50 A: It asks for the area of AKB.

51 R: Yes, the area of the red part.

52 A: It gave us 20 for that place. (Pointing to side AB.)

53 A: Sir, is it 100? Did you find it?

54 F: I found 100.

55 B: I also found 100.

56 R: Okay. Now Furkan, can you tell us your solution.

57 F: Sir, I folded first. I wrote 20 here (side AB’), finding out the places where overlapped and where were equal.

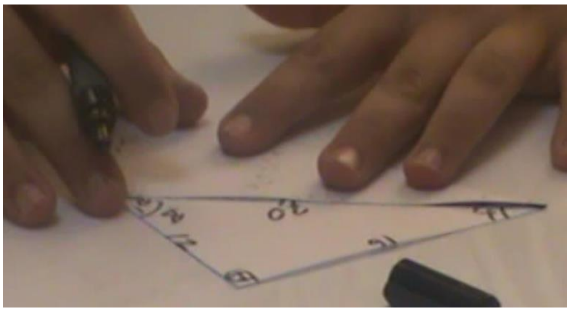


Figure 8. *Furkan’s solution*

58 R: Why did that place become 20, Furkan?

59 F: Sir, because this side completely coincided here. (Coincidence of side AB with AB’)

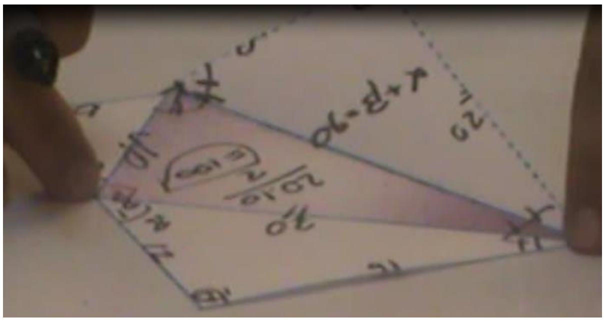


Figure 9. *Folds Furkan did for Task 2*

60 R: Yes, Ahmet. Why?

61 A: Because we folded. The question already told that this place was 20. The same corner coincides here. That’s why this place is 20, too. (Corresponding of corner B to corner B’)

Line 43 is the expression of Ahmet, and it can be said that Ahmet, who had difficulty in passing beyond the “reasoning with relationships” in Task 1, could also use appropriate expressions for the “investigating invariants” step and his doing so by referring to paper folding is an indication that he treated the paper as an artefact.

Result of Task 3

In Task 3, students were given the following figure and the teacher asked “rectangle ABCD is folded along EF. How many centimeters are in AD, if the size of this area is 18 square centimeter less than the area before it is folded?”

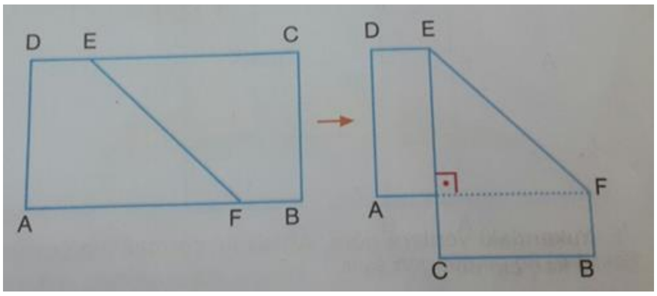


Figure 10. *The picture of Task 3*

The students began to solve the question by folding papers. All three students were able to solve the question easily with the aid of folding. Bahadır solved the question by himself and did not participate in the discussion. Ahmet's progress is clearer in the solution of this problem. Ahmet did the folding like the following and explained to his teacher and friends the solution process as in the dialogue below.

84 A: We are folding like that, and it is forming like this.

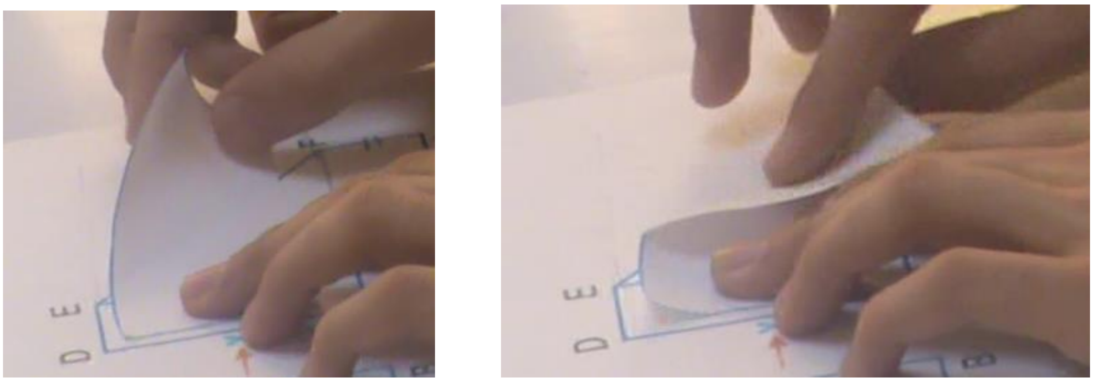


Figure 11. *The folding Ahmet did for Task 3*

85 R: Let's draw from there.

86 F or B: What was the area of the rectangle?

87 R: It wants from us AD anyway.

88 A: That place becomes 18 square centimeter because we folded it here. Is that right? (If we call the corner K, he is referring to triangle EKF.)

89 R: Let's go on, yes, very nice.

90 A: It's 18 square centimeter. This is (KF)...

91 R: You can get help here by folding what you have in your hand.

92 A: Because we folded. Here and here are equal. (KF and CB)

93 F: Well, is our answer 6?

94 R: Yes, we found 6. Ok, other guys. Find, too.

95 A: This place came here. (Observing by folding to where side CB is moved.)

96 F: Ahmet, I can help you if you want to.

97 A: When folded, this place is equal to here. (EK and CB) Because this place is equal to here, that place is equal to here. (KF and CB) Is that true?

98 F: Yes. Right. These places are equal to each other, as well. (AB and EK) You can, therefore, carry x here. (He says to write x on length EK, too.)

99 A: It is here, then. If x multiplied by x and divided by 2 equals to 18, x^2 equals to 36 equals, and x equals to 6. (Calculating the area of triangle EKF)

100 F: What it wants from us is side DA anyway, and we named side DA, x.

101 A: It this place (KE) is 6, then this place (DA) is 6, too.

In the above dialogue, the folding movement of Ahmet in lines 84 and 95 can be referenced to the "reasoning with relationships" step. Lines 88, 92, 97 and 101 are the sentences that can be referenced to the "investigating invariants" step. The phrase in line 98, which was expressed by Furkan, is a reference to the "investigating invariants" step.

After the solution of the problem, the students began to explain the area with justifications when the teacher asked the reasons why they accepted the area of triangle EKF as 18 square centimeter.

113 B: Sir, it was given in the question.

114 R: Okay, it was given in the question, but how did you understand?

115 F: Sir, we are doing it this way. We fold it like this. Here, sir, there are two, which are exactly equal to each other (indicating that they formed a square in the middle and that the diagonal was EF). By folding the square, we are dividing the area into two right along the middle, and we lose the area with 18 square centimeter.

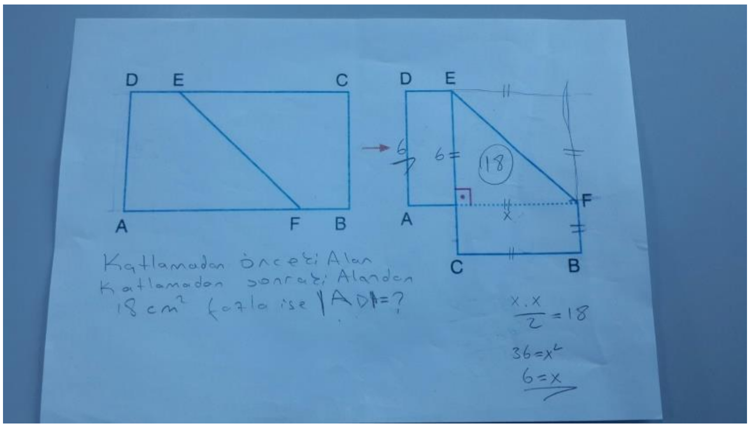


Figure 12. *Furkan's solution for Task 3*

116 R: Yes, very nice.

117 F: Here, a triangle with a square of x disappears. This equals to 18 square centimeter.

118 R: Can we say that the part that disappeared is here, then? (Showing triangle EKF)

119 F: Yes, the part that disappeared is that. That is, it is eliminating two identical pairs by folding on top of each other.

The above dialogue supports that Furkan is at the step of "balancing discovery and reflection." In particular, the explanations he made on lines 115 and 119 are beyond "investigating invariants;" they can be accepted as reference sentences for "balancing discovery and reflection," because they assess the situation and make explanations with justifications.

Result of Task 4

When it comes to Task 4, the last task, students were given a question that they could solve by paper folding, but they were asked to solve the question and figure out angle x and by imagining rather than folding the paper. The question is as follows.

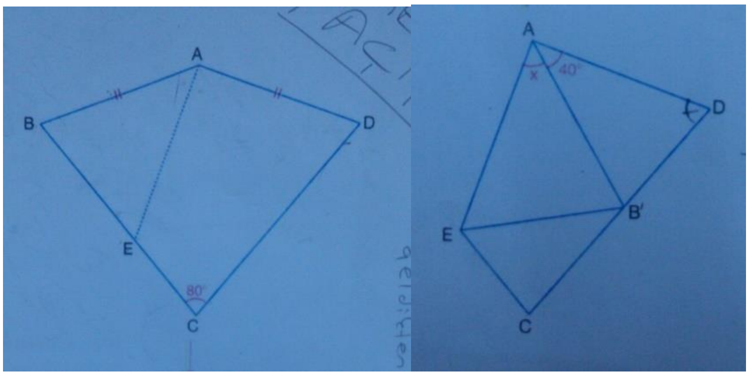


Figure 13. *The picture of Task 4*

All three of the students successfully solved the question, and all three participated in the discussion. In the discussion of the students, references to the steps of "reasoning with relationships" (121, 125), "investigating invariants" (124, 146) and "balancing exploration and reflection" (126, 157), which is the highest level of the geometric habits of mind, can be observed.

121 A: These angles were equal, did not they? (Drawing a line segment connecting corners B', A and E, and referring to angles BEA and AED in the figure that has not been folded.)

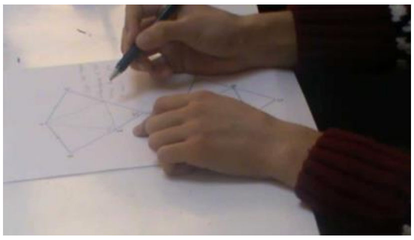


Figure 14. *Paper that Ahmet used to solve Task 4*

122 B: Yes

123 F: The angles in the fold lines are equal, too. (he refers to angles BAE and EAB' in the drawing he made on line 121.)

124 A: Yes, if we call this "x" here, then it is "x" there, too (angles BAE and EAB')

125 F: But, these are not becoming isosceles. (Referring to triangle AB'D)

126. B: When I fold it exactly there, AB does not become equal to here. It becomes equal to AB', which you drew secondly. (he states that triangles EBA and EAB' are congruent.)

...

141 F: Well, because the sum of the inner angles of this quadrilateral (ABCD) is 360, I took the sum of all the angles and equaled to 360. These angles, of which measures are 70, are equal to each other; these become 140 + 80 + 40. There is also 2x. A bisector formed here as a result of the folding.

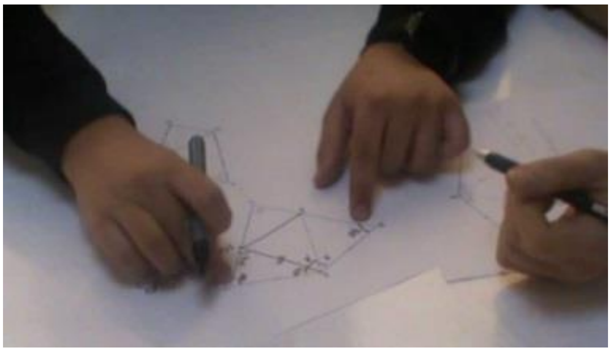


Figure 15. *Ahmet shows the angles that are equal to each other in Task 4*

...

145 R: Let's look at Ahmet's question first. Ahmet, would you like to tell us what you did?

146 A: I showed the congruent angles. Angle B also was divided into two congruent pieces. These were equal, too (angles AEA with BEA). I showed that the sides were equal.

...

156 R: Yes, now, Furkan shall solve the problem by explaining the reasons.

157 F: Well, from here, I pulled angle B onto angle B' on the shape in my mind. In this way, side AB side became equal to side AB'. In question, it was already given that AD was equal to AB. Here I saw an isosceles triangle here (triangle of AB'D). It was 40 degrees. This place became isosceles. Then its other two angles were 70-70. The opposite sides of the deltoid were equal to each other.

Even though the students did not fold the paper, they solved the question by imagining the folds in their minds and by referring to it. Similarly, they described their solutions by linking them with paper folding.

5. Discussion and Conclusion

The mathematical habits of mind are the applications that mathematicians use in their work, such as searching for patterns, experimenting, tinkering, describing, inventing, visualizing, and conjecturing. However, teachers who want to improve these habits in their students are faced with a difficult situation such as finding tasks that can be developed and the increase of such habits even more. Paper folding can be an effective tool in carrying out such a mathematical logic. NCTM (2000), when explaining the standards of the geometry programs, stated that students should be able to define spatial associations and that students' use of their visual-spatial skills in the process of solving geometry problems would positively affect the perpetuity of instruction. Given origami and its relation to mathematics, images of complex (intricate) folded geometric models come to mind. Thanks to this capacity, origami has a great potential to enrich mathematics classrooms (Cipoletti & Wilson 2004; Higginson & Colgan 2001). However, while creating an origami pattern, the person who folds the paper usually follows a series of procedures (see the Postulates). The mathematical thought here is not the folding process but the thought of reasoning about the model. In this reasoning process, Wiles (2013), focusing on paper folding and the geometric habits of mind, state that paper-folding is a support for students in providing (i) opportunities to explore important geometric ideas, (ii) testing ideas and making assumptions, (iii) asking new questions, and (iv) revealing unobserved relationships at first sight. Given these four key elements highlighted by Wiles

(2013), it was found that in paper folding questions, prior to the study, students had difficulty in especially perceiving how sides and angles did not change, how they were equal, and how additional lines were drawn (how to complete the shape). For this reason, through paper folding activities, students were let to realize that they would have easier access to solving abstract questions if the questions were concretized. Following the mastery of paper folding postulates, students were taught through the paper folding activities that the main elements of triangles remain unchanged. It was seen that the students who participated in this activity showed a progress from Task 1 to Task 4 and reached to the solutions of folding questions more easily over time.

It was seen that the students gained the geometric habits of mind through the paper folding activities and were able to reach especially the “investigating invariants” step. The “investigating invariants” step can be a good indicator for students to abstract geometric structures. Increasing the visual-spatial capacities of students is a necessity (NCTM, 2000; CCSSI, 2010) and they are seen as the basis for the students’ mathematical reasoning and proof (Wiles, 2013). “People who think analytically tend to note patterns, structures, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove” (NCTM, 2000, p. 56). In Task 4, even though the students did not fold the paper, they were able to solve the question by imagining the folds in their minds while solving the question and by referring to it. The consistent and reflective explanations of students with regard to geometrical constructions demonstrates how beneficial their gains were.

It is clear that the students’ thinking process has improved in the study. In addition to that, it can be said that they have permanently maintained the indicators of geometric habits of mind they have gained. It was determined that all three of the students participating in the study were easily able to solve the paper folding question (Fig. 1) appearing in the national examination (THEE 2017) conducted three months after the study. It can be said that the problem in question addresses GHoM’s components of the “reasoning with relationships” and “investigating invariants”. It is expected that the use of paper as a teaching tool in geometry teaching can enhance students’ geometrical thinking skills and achievements. Conducting paper folding activities in geometry lessons in schools should be supported. By analyzing the mentioned activity processes, the construction of geometric concepts and the development of geometric thinking of students can be examined. It is thought that paper folding exercises can offer a new breath of geometry teaching because students are both physically and mentally active and feel they become an effective part of the learning process.

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