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The Use of Variables in a Patterning Activity: Counting Dots

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The present paper examines a patterning activity that was organised within a teaching experiment in order to analyse the different uses of variables by secondary school students. The activity presented in the paper can be categorised as a pictorial/geometric linear pattern. We adopted a student-oriented perspective for our analysis, in order to grasp how students perceive their own generalising actions. The analysis of our data led us to two broad categories for variable use, according to whether the variable is viewed as a generalised number or not. Our results also show that students sometimes treat the variable as closely linked to a referred object, as a superfluous entity or as a constant. Finally, the notion of equivalence, which is an important step towards understanding variables, proved difficult for our students to grasp.

Keywords: generalisation, patterning activity, variable

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Uporaba spremenljivk pri zaporedjih: štetje pik

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Prispevek prikazuje, kako dijaki interpretirajo različna zaporedja pik. Zanimalo nas je, kako znajo uporabljati spremenljivke pri zapisovanju splošnega pravila zaporedja. Aktivnost, v katero so bili vključeni dijaki, je imela s pikami predstavljena zaporedja geometrijskih oblik. V raziskavi smo se osredinili na posameznega dijaka z namenom, da bi bolje razumeli, kako dijaki oblikujejo posplošitve. Analiza podatkov nas je pripeljala do dveh kategorij uporabe spremenljivk pri dijakih, in sicer ali so jo uporabljali kot zapis za poljubno/splošno število ali ne. Naši podatki tudi kažejo, da dijaki spremenljivko obravnavajo v tesni povezavi z narisanim členom v zaporedju, ali kot konstanto, ali pa ji pripišejo nepomemben pomen. Pokazalo se je še, da je ideja enakosti, ki je pomembna v procesu razumevanja spremenljivk, dijakom težko razumljiva.

Ključne besede: posploševanje, aktivnosti z zaporedji, spremenljivka

Introduction

The use of variables is a process closely linked to algebraic knowledge, a link that is manifested in many different ways in mathematics teaching and learning. Students encounter variables as early as in their first years of schooling, sometimes in the form of empty boxes signifying the unknowns of an equation. Later, still in the primary school years, students experience the use of letters to signify the elements of a geometrical figure, usually in the formulas that are used to designate the figure's perimeter or area. However, the use of variables becomes really significant in secondary education, when students are expected to be able to create, understand and manipulate symbolic expressions, while at the same time having an ability to "generalize patterns using explicitly defined and recursively defined functions" (NCTM 2000, p. 296). Thus, a "patterning approach", especially in a figural form, has been proposed as a fruitful way to introduce even young students to the notion of the variable: "Figural growing patterns and real-life contexts for developing knowledge of variables seem most suitable to support younger students' conceptual learning and their ability to reason algebraically and express generalizations symbolically." (Wilkie, 2016, pp. 353–354)

What, then, are the actions to be performed in a patterning activity? A patterning activity usually begins with a (free or guided) exploration by the students, followed by discussion and comparisons that are expected to lead them to a general rule (or a set of rules) to describe their pattern. The "linearity" of actions implied in the previous description should not be taken literally; Rivera (2010) eloquently describes the following independent actions, which should be *coordinated* in order to achieve successful pattern generalisation:

> (1) *abductive–inductive action on objects*, which involves employing different ways of counting and structuring discrete objects or parts in a pattern in an algebraically useful manner; and (2) *symbolic action*, which involves translating (1) in the form of an algebraic generalization. (p. 300, italics in the original)

The results of studies on pattern generalisation have revealed students' difficulties in generalising patterns in an algebraic form (e.g., English & Warren, 1998; Orton & Orton, 1999). In particular, there seems to be "a gap between students' ability to express generality verbally and their ability to employ algebraic notation comfortably" (Zazkis & Liljedahl, 2002, p. 400; see also English & Warren, 1998). Other difficulties stem from students' inability to identify and generalise patterns that are useful and valid algebraically (see, e.g., Ellis, 2007a). Acknowledging the results of these studies, we organised a teaching experiment in a Polish secondary school in order to examine how students perceive the notion of the variable in a patterning activity. We were also interested in the effect of the structure of the activity in the whole process. Thus, our main research question was: What are the different uses of variables by secondary school students during their engagement in a patterning task?

Theoretical Framework: Patterning Activities and the Use of Variables

The study of generalisation processes in algebra may be accomplished by the use of different contexts and approaches, but patterning activities seem to be the one of the most prominent. Lee (1996) states that "algebra, and indeed all of mathematics is about generalizing patterns" (p. 103). Patterns provide a rich context for "algorithm seeking" (Mason, 1996) and ample opportunities for students to exercise their creativity and develop their communication and technical skills (Lee, 1996).

Patterns can be categorised into "number patterns, pictorial/geometric patterns, patterns in computational procedures, linear and quadratic patterns, repeating patterns, etc." (Zazkis & Liljedahl, 2002, pp. 379–380). It is obvious that each type of pattern poses different challenges and constraints to students who are asked to generalise. For example, pictorial patterns require "visual perception" – containing sensory perception and cognitive perception – that refers to the identification of facts or properties related to an object (Dretske, 1990, as cited in Rivera, 2010).

At this point, it is important to note that the above categories of patterns should not be perceived as mutually exclusive. Stacey (1989) analysed cases of linear patterns presented pictorially; two such examples are expanding ladders made of matches and Christmas trees. In addition to the (rather expected) result that these problems proved challenging for the whole range of the research group (students aged 8–13 years), a significant finding is "the attractiveness of the simple rule". This means that when the students found a counting method infeasible, they decided to use a simple relationship that applies in direct proportions.

Another alarming result of Stacey's study is that "students grab at relationships and do not subject them to any critical thinking" (Stacey, 1989, p. 163). In other words, the students proposed certain relationships to describe the patterns, without examining their validity. When analysing students' work, we should therefore be attentive to all of the processes that led them to the proposed generalisation. Moreover, we should be cautious regarding the "correct" patterns that we expect the students to reach, in relation to all of the patterns that may be discovered. Ellis's (2007a, p. 195) literature review is revealing concerning the multitude of patterns that we may find in students' work: "Examinations of students' work with pattern activities in algebra show that although students recognize multiple patterns, they may not attend to those that are algebraically useful or generalizable" (see also Blanton & Kaput, 2002; English & Warren, 1995; Lee, 1996; Lee & Wheeler, 1987; Orton & Orton, 1994; Stacey, 1989).

In line with the above considerations, there are also different views on what constitutes a valid generalisation; thus, different interpretative frameworks have been proposed. In her extensive review, Malara (2012) presents various theoretical approaches to generalisation, as well as examining how these approaches inform the teaching of algebra and, in particular, the role of the teacher. The author also presents different approaches to the implementation and analysis of patterning activities and the use of variables. Citing Radford (2006), she offers a comprehensive view of how to identify generalisation:

> The level of the algebraic generalization is reached when pupils detach themselves from the figural context and shift towards the relations between constant and variable elements (numbers and letters). Important elements which intervene in this last process are *iconicity*, i.e. a manner of noticing similar traits in previous procedures, the shifting from a particular unspecified number to the level of variables *summarizing* of all the local mathematical experiences, the *contraction* of expressions which testifies a deeper level of consciousness. (Malara, 2012, p. 71, italics in the original)

Arithmetic and algebraic reasoning are inseparably linked: the generalisation of reasoning conducted on concrete numbers leads to algebraic thinking and, in the final stage, to notation with the use of symbols. Already at the primary school level, such passing from arithmetic to algebra is most often initiated by generalisation through a "variation of parameters" method or by inductive generalisation (Zaręba, 2012).

Among the various approaches to generalisation within algebraic activities, for the purpose of the present paper we decided to focus on Ellis's (2007b) approach, which adopts an "actor-oriented perspective" (Lobato, 2003) in order to grasp how students perceive their own generalising actions. In so doing, we adopt a critical stance towards studies that focus on the observers' perspectives, thus categorising students' actions as correct or not according to predetermined criteria. In Ellis's view, students' activities can be broadly categorised into *generalizing actions* (students' mental acts as inferred through the person's activity and talk: relating, searching and extending) and *reflection generalizations* (students' final statements of generalisation: identification or statement, definition and influence of a previously developed generalisation). As mentioned above, an important characteristic of this taxonomy is that it moves away from the dichotomy between correct-incorrect generalisations and thus helps teachers to "view incomplete or incorrect generalizations as necessary steps in the larger process of developing a habit of generalizing" (Ellis, 2007b, p. 258).

Concerning the second element of our framework, i.e., the use of variables, it is noteworthy that within the patterning approach we may encounter different views on the role of algebraic notation. Kieran (1989) believes that

> generalization is neither equivalent to algebraic thinking, nor does it even require algebra. For algebraic thinking to be different from generalization, [. . .] a necessary component is the use of algebraic symbolism to reason about and to express that generalization. (p. 165)

Along the same lines, according to NCTM's (2000) algebra standard, all students in grades 9–12 should "use symbolic algebra to represent and explain mathematical relationships" (p. 296). Krygowska (1980) differentiates four meanings of a letter in algebraic expressions: as a general name, as a variable, as an unknown and as a constant.

On the other hand, Radford (2011) argues that the use of algebraic notations is neither a necessary nor a sufficient condition for algebraic thinking. Our approach is closer to that of Dörfler (2008), who notes that:

> The knowledge and mastery of algebraic notations will not develop simply from generalizing patterns of various kinds though those provide a suitable context and motivation. Of great importance further would be the negotiation of the intended meaning of the algebraic terms, especially of their ascribed generality (which is not inherent in them). (p. 146)

In line with the above, our aim, from a teacher's point of view, was to establish a learning environment that would allow for fruitful and meaningful discussion in the classroom. From a teacher-researcher's point of view, we aimed to examine whether our approach leads to the intended negotiation, and what kind of shared meanings arise regarding the use of variables.

Context of the Study and Methodology

Context of the study – students' background knowledge

Our research took place in the 2^{nd} grade of a Polish "Gymnasium" (students aged 13–14 years) over a period of two weeks. The class consisted of nine girls and seven boys, and was chosen as a convenient sample. The mathematics teacher of the class was present during the three one-hour sessions, together with the researcher (the first author of the present paper). The students in the class had already been introduced to algebraic processes in previous lessons. Specifically, according to their teacher, they had experience in: describing different relationships between quantities using algebraic expressions, transforming expressions, and using different solving methods for equations and inequalities. According to the textbook, the concept of the variable is a letter that represents a number. According to the teacher, however, the students had a rather intuitive view of the concept of the unknown: the concept of the variable had not been defined in the class, although it had been mentioned during discussions. The students had not encountered the concept of function and did not have much experience with generalising processes.

Data collection

For the purpose of this study, we decided to partially adopt the teaching experiment methodology. Specifically, we designed our study to focus on "the processes of a dynamic passage from one state of knowledge to another" (Cobb & Steffe, 1983, p. 87). Thus, our data are rather qualitative, as we were interested in how the students used variables.

Bearing in mind the importance of design and feedback in the teaching-research process, we prepared three worksheets (Reznic & Tabach, 2002) that included some linear geometric patterns and a series of questions. For the purpose of the present paper, we will only refer to the first instructional unit, based on the worksheet "Counting Dots", as shown in Figure 1 below.

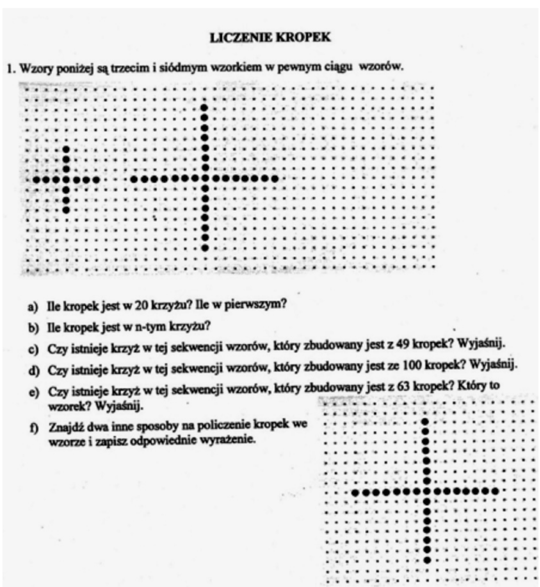

Figure 1. The worksheet given at the first instructional unit.

In the worksheet shown in Figure 1, we read the following:

The following crosses are the third and seventh in a sequence of crosses.

a) How many dots are in the 20th cross? In the first cross?

b) How many dots are in the nth cross?

c) Is there a cross in this sequence with (exactly) 49 dots? In what place? Explain.

d) Is there a cross in this sequence with (exactly) 100 dots? In what place? Explain.

e) Is there a cross in this sequence with (exactly) 63 dots? In what place? Explain.

f) Find two other ways to count the number of dots in a cross and write a corresponding expression.

Students' and observers' roles

The students worked in four groups: three groups had four members and one group had three members (one student was absent). Each group was sitting around a table and had the worksheet and an empty poster at their disposal. The groups were expected to make a short presentation about their findings in front of the class. The teacher and researcher interacted with the students during group work, and then with the whole class during the presentation. Apart from asking questions to prompt the students to give explanations, they supported the students' investigations, eventually by asking "give an example" questions (Zaskis & Hazzan, 1999). In general, we followed Ellis's (2011) view that when the teacher asks for generalisations without providing ready answers or strategies, the students can be led to productive generalising. This is in line with Legutko and Stańdo's (2008) recommendations about teaching in Polish schools in such a way as to develop students' habits of observation, experimentation, self-searching and processing information. This in turn requires the mathematics teacher to engage students in noticing and using analogies, making empirical conclusions, and engaging in recursive reasoning and inductive generalisations. The particular discursive actions that we considered may potentially prove productive for fostering generalisation, were: "[...] highlighting the role of conjecture and justification in classroom discussion, providing access to physical or visual representations of mathematical relationships, revoicing to elaborate or refine student contributions, and encouraging reflection on students' activity." (Ellis, 2011, p. 309)

Method

All of the sessions were video-recorded, transcribed by the first author of the paper and then translated into English. Our data consisted of students' utterances (while interacting within their group, or with the teacher or the researcher, or during their presentation) and their written products, as they appeared in their posters. Since the central phenomenon to be examined was the use of variables, we first located all of the instances in the interactions where there was explicit reference to a variable. We then analysed the utterances in order to identify the meanings assigned to the variables; for this purpose, we did not use any predetermined categories, but rather established categories led by our data (Strauss & Corbin, 1990), as will be shown in the Results section. Finally, we analysed the progress of each group by examining and comparing the utterances used throughout the instructional unit; this was done in order to observe their dynamic passage from the various states of shared knowledge on patterns and the use of variables.

Sample Analysis

As mentioned above, in the last part of the instructional unit, the student groups were asked to present their findings on the blackboard in front of the class. During these presentations, the students were encouraged to exchange their views. In the transcripts that follow, the letter T signifies the teacher and the letter B the researcher. The first transcript comes from Group 2, which consisted of two girls and a boy. The presentation was made by Aneta (A) and Joanna (J).[3] They have already presented their answer to question a) and they proceed to question b).

11 A: It was easy. Now point b. So n is that unknown one…?
12 J: It is that unknown one… that is… well… in the next one, one dot is added on every side, that is times 4 plus the dot in the middle.
13 T: And what can you calculate in this way?
14 J: All of the dots.
15 T: In which figure?
16 A: n times four plus one.
17 T: So in which (figure) can you calculate in this way?
18 J: In every one.

3 All of the names that appear in the excerpts are pseudonyms.

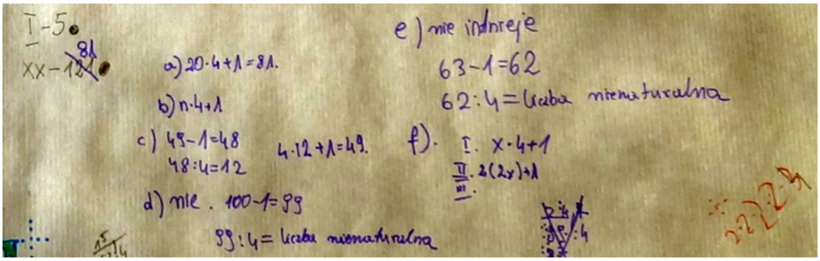

Figure 2. Poster of Group 2.

The first observation is the students' use of the adjective "unknown" to signify the variable n. This use is not in line with the variable n signifying a general case (the n^{th} figure), as is evident in the interaction that follows (13–18), when the teacher is asking for clarifications. The teacher does not receive a correct answer to her question at 15, but when she repeats it, Joanna replies "In every one". We believe that this utterance does not fully reflect the meaning of the variable n in the particular context. This becomes more obvious in the transcript that follows, when the same group is discussing a possible answer to question f). They have come up with the formula 2×(2n)+1 and the discussion is on its correctness and the possible modifications needed. In this discussion, three more students Monika (M), Gosia (G) and Sara (S) from Groups 3 and 4 participate.

52 T: So if 2n means one arm according to you [she means the whole vertical line of the cross, which contains two arms and the central dot], what do you have to change in this formula, if anything, in order for it to be a correct one?

53 S: Move the parentheses.

54 M: Or to put in the parentheses 2 times 2n.

55 J: Maybe minus one in the brackets?

56 M: What? Maybe we can change n into r, in the sense that it is an arm, then it would be correct. It would be two times two arms. Then it would be correct.

57 B: So what does n mean here? In that formula?

58 All: n is also an arm.

59 G: Without the dot in the middle.

60 A: So it is two times two arms, then it is ok.

61 T: Then everything is correct?

62 M: Then it is the same.

63 G: Exactly, n and r, it is the same, because n is an arm, right?

64 S: It is a letter marked.

65 T: And Marta, can you write what you just said? That with the r?

66 M: But it is the same.
67 A: This is the same, just a different letter.

In the above transcript, we first note a correspondence that was proposed in the previous turns between 2n and an arm of the cross. This is an initial manifestation of a category that emerged; in this category, the students treat the variable as closely linked to the referred object (or in this case to a part of it). This is evident throughout the excerpt: in 56, 58, 60 and 63. The letter r, which is suggested by Monika (M), comes from the Polish word "ramie" which means "arm". Monika believes that by changing the letter the formula would become correct; in this way, she expresses her view on the equivalence of formulas (in relation to the notion of the variable).

Results

Our data led us to two basic categories. In the first category, the variable was treated as a generalised number (English and Warren, 1998), while the second category contained the cases in which the variable was not treated as a generalised number; in the latter category, we distinguished three subcategories: (a) the variable being closely linked to the referred object (or to a part of it), (b) the variable being used in a superfluous manner, and (c) the variable being treated as a constant. It is important to note that in most cases the student groups showed a switch between these categories, especially from the second category to the first one.

The variable as a generalised number

This category contains the cases in which the students' acts demonstrate an explicit understanding of the variable n as signifying the general case: the n^{th} cross with 4n+1 dots. Another variable included in this category was k, signifying the number of all of the dots in a cross. It appeared in the formula (k-1):4, which was deployed by two groups for answering questions c), d) and e) of the worksheet.

The variable closely linked to the referred object

The second fragment of the dialogue in our sample analysis illustrates how this category emerged. Throughout the discussions, we found many cases of this category with different letters being used. The most frequent was the one associating n (or r, 2x, 2n) with an arm of the cross (a 'short' or a 'long' arm).

The variable being used superfluously

This category contains the cases in which the use of the variable seemed to somehow exceed that of a generalised number and signified an entity that not only did not play a part in the generalising process, but eventually hindered it. In the following, Joanna from Group 2 provides her answer to question a): "In the first there are five. Then in the second, one dot is added to every side. So if four dots are put to every x, in the 20^{th} we have 81 dots". Here x is used to name a previous figure, but the relation under discussion is not recursive. Joanna does not use the "previous" cross in order to calculate the 20^{th} one, nor does she mention the next cross. Thus, the variable does not assist the group to generalise, but rather creates obstacles in the process of generalisation.

The variable as a constant

An occurrence of this category was observed in the presentation of Group 1 in answering question f). The students proposed the formula (4n+1)+4+4+4+.... The relationship was recursive and they tried to convince their classmates that by using this formula you can calculate the number of dots in the n^{th} cross. What is interesting is that, for them, the expression (4n+1) was constant and represented the dots of the first cross. They even stated that "for n there is always 1, let's assume".

The shift towards the variable as a generalised number

The students who perceived the variable as closely linked to an object (e.g. Monika, who is mentioned in the Sample Analysis section) were able to shift to a generalising view. Another decisive factor for the shift towards the first category of variable use was the interventions of the teacher and the researcher:

P: It will be n·4+1. This is the formula.
T: Ok, where (there is) n what does it mean for you?
G: One arm.
P: That short arm. One. [showing the drawing]
G: One arm – the short one – times 4 plus 1 in the middle.
T: And which drawing does it give us? Which cross?
G [reading question b] ....hm.... the n^{th} cross... [Silence]
P: That is the n^{th} cross, [very unsure] I don't know... [Silence]
T: Can it be, for example the 21^{st} cross?
P: [thinking for a while and then with enthusiasm] It can be! Because for n we can substitute any number. This is for all (showing the figure), right?

In contrast, the other two sub-categories seemed to be a result of the students' need to fulfil the expectations of the teacher (and the task); since they were expected to find a formula, they tried to name some quantities using letters.

The notion of equivalence

English and Warren (1998) state that the notion of equivalence can be explored as soon as the concept of the variable has been established. In the present study, we observed our students' difficulties with this notion: the formulas (k-1):4 and r=(n-1)/4 (and 4n+1, n=4r+1) were characterised as different by most students. The same can be noted in the case of a variable treated as a constant; in the example presented above, the students first discovered the general formula 4n+1 and then used the same expression (as a constant) for the number of dots in the first figure.

Discussion

The main purpose of our teaching experiment was to analyse the use of variables by secondary school students. Our analysis, which was student-oriented, led us to different categories that reflect different students' views. Of greater importance, however, was to examine the possibilities for a shift from a non-generalising to a generalising view of the variable. In this aspect, we observed that perceiving the variable as closely linked to the referred object (or to a part of it) can be seen as a step forward to the variable as a generalised number. Generally, we can conclude that, although the majority of our students managed to overcome their difficulties with the notion of the variable, they still have problems with the notion of equivalence, which we believe is the next step in fully understanding the concept.

The structure of the teaching experiment, the questions posed in the task, and the interventions of the teacher and the researcher proved helpful in the negotiation of meanings in the class. Moreover, we concur with Ellis (2007b) that incomplete generalisations can be viewed as part of the process of generalising and, particularly in our case, of the process of using variables. We thus believe that our study contributes to the existing research on variables, as well as to the specific topic of equivalence. This is especially because the categorisation we propose allows for relating students' activities to their progress in the use of variables, while at the same time being based on data from a teaching experiment and not from laboratory research. Thus, we believe that our findings can be useful to the mathematics teacher-researcher not only in preparing certain activities, but in providing him/her with the means to monitor and evaluate the students' actions, as how students execute algebraic activities is just as important as what they do during such activities.

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