Developing a Framework of Outcomes for Mathematics Teacher Learning

Three Mathematics Educators Engage in Collaborative Self-Study

Damon L. Bahr, Eula Ewing Monroe, & Jodi Mantilla

Abstract

This article synthesizes the literature on what it means to teach mathematics and science to ELLs and abstract from it a set of knowledge and skills teachers might need to teach ELLs effectively. To this end, the article brings together the socio-cultural and linguistic perspectives identifying three areas of effective teaching practice. One argument is that collaborative learning conditions are beneficial in teaching mathematics and science to ELLs. A second contention is that teachers should be able to engage ELLs in mathematics 'talk' and the discourse of scientific concepts by bridging the divide between students' background experiences and the content of mathematics and science lessons. The third area of effective teaching practice forwards the claim that teachers should engage ELLs in talking and writing the language of mathematics and science. To support this point, the linguistic perspective identifies the shared and distinctive features of the academic languages of mathematics and science. Into this discussion, we integrate the insights of the mathematics and science specialists that participated in our panel.

Damon L. Bahr is an associate professor, Eula Ewing Monroe is a professor, and Jodi Mantilla is a visiting professor, all in the Department of Teacher Education of the David O. McKay School of Education at Brigham Young University, Provo, Utah. Email addresses: damon_bahr@byu.edu, eula_monroe@byu.edu, & jodi_mantilla@byu.edu

Introduction

Numerous reform documents in mathematics education call for significant changes in perspectives about the nature of teaching and learning mathematics. "More than simply minor adjustments in current ways of teaching" (Wood & Turner-Vorbeck, 2001, p. 185) are proposed, and by extension, a rethinking of the outcomes[1] for teacher learning (Ball & Cohen, 1999). Such outcomes could create a common conceptual framework of sufficient detail and clarity to benefit mathematics teacher educators not only in their professional practice but also in their study of that practice. As Simon (1995) wrote, "A well-developed conception of mathematics teaching is as vital to mathematics teacher educators as well-developed conceptions of mathematics are to mathematics teachers" (p. 142). Desimone (2009) concurred, suggesting two interrelated benefits: "The use of a common conceptual framework would elevate the quality of professional development studies and subsequently the general understanding of how best to shape and implement teacher learning opportunities for the maximum benefit of both teachers and students" (p. 181). Similarly, Doerr, Lewis, & Goldsmith (2009) wrote, "A shared framework on which to hang existing findings may also help us develop, as a field, a shared theoretical model of teachers' on-the-job learning so that we can build on one another's work more productively" (p. 12). (See also Simon, McClain, Van Zoest, & Stockero, 2009.)

A major step toward creating a framework of learning outcomes for teachers was taken by the National Council of Teachers of Mathematics (NCTM) with its *Professional Standards for Teaching Mathematics* (1991). The stated purpose was "to provide guidance to those involved in changing mathematics teaching" (p. 2). Its successor, *Mathematics Teaching Today* (Martin, 2007), underscored the central message of the first document: "More than curriculum standards documents are needed to improve student learning and achievement. Teaching matters" (p. 3). Nevertheless, as Linda Gojak, recent president of NCTM, expressed, even teaching standards have been insufficient for promoting the teacher learning that results in the "specific actions that teachers . . . need to take to realize [the] goal of ensuring mathematics success for all. . . . " (NCTM, 2014, p. vii). Gojak continued, "We have learned that standards alone . . . will not realize the goal of high levels of mathematical understanding by all students. More is needed than standards" (NCTM, 2014, p. vii).

During our work over the years, we have relied on these standards as well as other related documents (e.g., *Principles and Standards for School Mathematics,* NCTM*,* 2000; *Adding It Up*, NRC, 2002) to create several successive renderings of outcomes to guide our work in teacher preparation and professional development. Although useful, these renderings felt incomplete and incohesive. We needed a comprehensive, structured set of outcomes to frame the teacher learning that would align with these documents and was reflective of current research in our field. Our thinking was corroborated by the authors of *The NCTM Research Agenda*

Conference Report (Arbaugh, Herbel-Eisenmann, Ramirez, Knuth, Kranendonk, & Quander, 2010), a portion of which calls for the identification of "competencies that teachers need to have and prioritizing these competencies as desired outcomes of professional learning opportunities" (p. 18).

The purpose of this article is to describe our *Framework of Outcomes for Mathematics Teacher Learning* (*FOMTL*) and the collaborative self-study (LaBoskey, 2007) we used to construct it. The work portrayed in this study reflects our own long-term journey to refine our vision of teacher learning in mathematics teacher education. In creating our framework of teacher learning outcomes, we synthesized the wisdom inherent in the afore-mentioned documents with a combined teaching experience of more than a century. As described below, our work was a collaborative self-study (LaBoskey, 2007) that bridged to our practice and helped us refine our own teaching and learning as mathematics teacher educators. It does not represent a recommendation or set of recommendations but rather a framework we have found useful and therefore desire to share with our field.

Positions and Domains of Teacher Learning

As previously discussed, we engaged in numerous endeavors to define outcomes for our work in mathematics teacher education. Early in our framework development, we had encountered Wood, Nelson, and Warfield's (2001) work regarding the positions from which teacher learning is studied and developed—*Mathematical Content Knowledge, Student Thinking, Child Development,* and *Social Interaction.* Two positions have developed from a psychological perspective; the third from a sociological, or socio-psychological, perspective; and the fourth from a disciplinary one.

1. Ball, McDiarmid, Wilson, and Shulman (Ball, 1988; McDiarmid & Wilson, 1991; Shulman, 1986, 1987), conjectured that teachers change as the nature of their *mathematical content knowledge* changes.

2. Carpenter and others (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Peterson, Carpenter, & Fennema, 1989) suggested that teacher learning is brought about by changes in perspectives concerning *student thinking.*

3. Schifter, Simon, and Fosnot (Schifter, 1996a, 1996b; Schifter & Fosnot, 1993; Schifter & Simon, 1992) suggested that teachers learn as their perspectives regarding the nature of learning and mathematics change, a *child development* position.

4. Cobb, Wood, and Yackel (Cobb, Wood, & Yackel, 1990; Wood, Cobb, & Yackel, 1991), viewed teacher change as resulting from renegotiation of the *social interaction* that characterizes classroom pedagogy.

We contend that these positions are not necessarily oppositional but, rather, complementary; moreover, we view the research conducted from each of these positions as making valid contributions to the knowledge base on teacher learning.

Hence we argue that a comprehensive set of outcomes for teacher learning would reflect all four positions.

In addition to the four positions, Wood et al. (2001) also affirmed that teacher learning occurs within three domains—beliefs, knowledge, and practice. In our work, we used Rokeach's (1968) definition of *beliefs*, cited by Leatham (2006), as "predispositions to action" (p. 92), and contrasted them with knowledge because beliefs possess a relatively stronger, affective component (Abelson, 1979; Speer, 2005). We defined *knowledge* as truths, facts, principles, or information acquired or constructed via experience, cognition, or association (Merriam-Webster Dictionary, 2012), and *practice* as the work that characterizes a profession, such as the teaching of mathematics. The singular form of the noun *practice* is used in this paper primarily to refer to a teacher's customary way of going about teaching. When the intended meaning of practice refers to ways of dealing with specific teaching situations, the form is usually the plural noun, *practices*.

Research findings from all four positions contributed to our understanding of the beliefs, knowledge, and practices teachers need to construct as they learn to teach, thus giving us a more holistic, comprehensive view of teacher learning. To give structure to our understanding, we created a two-dimensional framework of outcomes organized by these four positions and the domains of *Beliefs, Knowledge,* and *Practice*.

However, despite the high value we placed on the positions and domains as important anchors of our framework, we became less than satisfied with its content and structure. It seemed cumbersome and incohesive, so we felt compelled to start afresh. Almost immediately, we encountered two documents that were pivotal in reshaping our direction. First, a curriculum-oriented collaborative self-study of science and literacy integration by Hall-Kenyon and Smith (2013) convinced us that engaging in the deep discourse that characterizes self-study research could help us frame and develop a satisfying set of outcomes for our students. Second, NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (2014) confirmed and extended our thinking regarding both the necessity for and the content of outcomes for mathematics teacher education. The first document inspired a different approach to developing our outcomes—collaborative self-study, and the second informed our efforts to develop outcomes to guide our own practice. Although our goal was to improve our practice, we hoped to contribute to the national conversation regarding prioritizing " . . . competencies as desired outcomes of professional learning opportunities" (Arbaugh et al., 2010, p. 18) as well.

Creating a Landscape for Teacher Learning

Our self-study is based on a view of learning that is highly influenced by the landscape metaphor. Traditional characterizations of learning, including professional learning, tend to be unidimensional in their orientation. Such characterizations represent learning as moving along a linear path, and by implication, that all learners are assumed to move along the same pathway, the same linear trajectory. Modern characterizations of learning tend to be more multi-dimensional and respectful of individual differences, sometimes employing a landscape metaphor as part of the characterization. "In this . . . metaphor, learning is analogous to learning to live in an environment. . . . Knowing where one is in a *landscape* requires a network of connections that link one's present location to the larger space" (Bransford, Brown, & Cocking, 2000, p. 139).

Fosnot and Dolk (2001) adopted the landscape metaphor in their descriptions of students' mathematical learning. According to these authors, a mathematical landscape includes "landmarks" (p. 18) of big ideas, strategies, and representations leading to various points on the mathematical "horizon" (p. 18) for a given topic, e.g., place value or addition and subtraction. Children may follow different trajectories on the learning landscape for a mathematical topic, yet arrive at the same mathematical horizon. We have found this same metaphor useful in describing teacher learning. As Lesh, Doerr, Guadalupe, & Hjalmarson (2003) suggested,

> The essence of the development of teachers' knowledge . . . is in the creation and continued refinement of sophisticated . . .ways of interpreting the situations of teaching, learning and problem solving. We believe that the theoretical constructs that govern the development of useful models by students are the same theoretical constructs that govern the development of useful models by teachers (p. 227).

Thus, we sought to construct a comprehensive, cohesive, clear, and concise framework of teacher outcomes that would serve as landmarks within the teacher learning landscape and allow for individual trajectories within that landscape. We therefore had two specific purposes for engaging in this study: (a) to uncover and analyze our own outcome lists as experienced mathematics educators, and (b) to investigate how those outcomes evolved as we moved toward a shared vision of the outcomes of our work, a vision we eventually named a *Framework of Outcomes for Mathematics Teacher Learning* (*FOMTL*). In this paper we present the *FOMTL* in the context of its creation. Following a discussion of self-study methodology, we describe the framework's initial conceptualization as a lengthy list of unstructured outcomes, then the series of qualitative analyses resulting in the framework in its current form.

Self-Study Methodology

Self-study, a form of practitioner research that typically employs qualitative methodology, has emerged as a viable means through which teacher educators learn from their own professional practice. Alluding to the potential of self-study as a process for enacting change, Berry and Hamilton (2012) wrote,

> In self-study, researchers focus on the nature and development of personal, practical knowledge through examining, in situ, their own learning, beliefs, practices, processes, contexts, and relationships. Outcomes of self-study research focus

> both on the personal, in terms of improved self-understanding and enhanced understanding of teaching and learning processes, and the public, in terms of the production and advancement of formal, collective knowledge about teaching and teacher education practices, programs, and contexts that form an important part of the research literature on teacher education. (Para. 1)

Recognizing the promise that self-study holds in promoting mathematics education reform (Goodell, 2011), we engaged in collaborative self-study (LaBoskey, 2007) as we sought to reconstruct our earlier framework of outcomes. We examined our own and each other's perspectives about learning outcomes, and the records of the interactions associated with these examinations then served as data for our study.

Data Sources and Analysis

As a starting point, Eula and Damon each spent time brainstorming lengthy lists of potential teacher learning outcomes— Eula's list totaled 53 and Damon's 33. These lists, which served as our initial data sources, were influenced by knowledge gained from our individual and collective research agendas, our own wisdom of practice, and the outcomes constructed during our previous work. We then engaged in a series of dialogues about the lists that served as our first analyses, not unlike constant comparative methodology (Straus & Corbin, 1990). These dialogues, more fully explained below, occurred in the process of six analyses—"passes"—as we coded and recoded our initial outcomes lists using a mixture of *a priori* and emergent coding. Some of these passes were planned ahead of time, and the need for others arose based on what we were learning from the data. This coding enabled us to simplify, refine, and structure our combined lists into a tentative framework as shown in Table 1.The information in the table is more fully described in the Findings section.

This tentative framework went through several more revisions as we used it to guide our methods instruction over two semesters. Our primary data sources then consisted of written and video records of numerous additional dialogues about our changing conceptions of the framework, supplemented by our written plans for course sessions, various in-class and homework assignments and assessments, and written post-instruction notes and reflections. These data served as a history of our journey, helping us to make sense of current dialogues by remembering the lessons of the past. The dialogues produced a steady stream of revised outcomes, some of which resulted in modifications to our practice during each semester. Thus we engaged in a series of miniature, design-based inquiries (Design Based Research Collective, 2003) that involved frequent application of our thinking to our classroom practice for testing and refinement (LaBoskey, 2007).

Our dialogues, therefore, served as both an ongoing source of data and a major component in the data analysis process. Interspersed among these dialogues were applications of our current thinking to our work with students. Dialogue became a

Table 1
A Framework of Outcomes for Mathematics Teacher Learning–Version 1[1]

	Teacher Learning Positions			
	Mathematical Content Knowledge	**Student Thinking**	**Social Interaction**	**Child Development**
Teacher Beliefs	As students cycle through *The Continuum of Mathematical Understanding*, they make connections within and across three domains—Conceptual, Procedural, & Representational—and mathematical understanding is constructed.	"Children can solve problems in novel ways before being taught how to solve such problems. The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. The teacher should allow the children to do as much of the thinking as possible" (Philipp et al., 2007, p. 475), thus honoring the ways students think.	"…Thinking with others promotes children' s cognitive progress because it encourages . . . individuals to bypass their own cognitive limitations and enables children to develop virtual dialogue, the capacity to think alone" (Rochat, 2001, p. 139). This " . . . discourse embeds fundamental values about knowledge and authority. . . ." (NCTM, 1991, 21, 34).	Recognizing that in an equitable learning environment all students can learn, " . . . teaching should be grounded in how students learn. . ." (Schifter & Fosnot, 1993, p. 193) and " . . . acknowledges learners as active knowers" (Noddings, 1990, p. 10) who engage in "active construction, not merely passive absorption . . . (and in) invention, not imitation" (Baroody & Ginsburg, 1990, p.52). This construction often begins with concrete thinking and moves toward the abstract.
Teacher Knowledge	Engaging in the "Mathematical Practices" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) while studying elementary mathematics produces a "Profound Understanding of Fundamental Mathematics" that is intricately inter-connected, multi-faceted, longitudinally coherent, and structured according to "big ideas" (Ma, 1999, p. 120).	Because children's initial mathematical thinking is best supported by real-life contexts (Philipp et al., 2007), their initial problem solving strategies and representations model the actions and contexts inherent in the problems (Carpenter et al., 1999; Fosnot & Dolk, 2001). Teaching involves not only knowing "the variety of ways children think about and make sense of mathematical ideas but also how children's thinking about these ideas develops over time" (Wood, 2001, p. 9).	Interaction patterns eliciting greater involvement foster more complex levels of thinking, thinking that can be hierarchally organized and categorized according to *The Framework of Cognitive Complexity* (Bahr & Bahr, 2017). Those categories relate to teacher purpose—comprehension, connection, and consensus—and connect closely with the *Learning Cycle*.	Student learning progresses through a *Learning Cycle* consisting of three phases—Develop, Solidify, and Practice—that can be distinguished on the basis of the differing purposes, tasks, and student thinking that characterize them (Hendrickson, Hilton, & Bahr, 2008).

		Domains of Practice		
		Lesson and Unit Design	Questioning and Assessment	Discussion Orchestration
Teacher Practice	Practices	Teachers conduct 3-stage inquiry via the *Teaching Cycle* (Hendrickson, S., Hilton, S. C., & Bahr, 2009, renamed Launch-Explore-Discuss by these authors), which enables teachers to think through an inquiry-based lesson, accommodates a number of instructional models, and varies depending upon where students are in the Learning Cycle.	The purposes and processes of questioning and assessment vary according to the stages of the Teaching Cycle. During the Launch, assessments are made regarding task comprehension. During Explore, observed thinking is compared to anticipated thinking in order to prepare for the Discuss. During the Discuss, thinking is compared to lesson purpose in order to inform the next teacher move.	Teachers conduct 3-stage mathematical inquiry, "Launch, Explore, and Summarize" (Schroyer & Fitzgerald, 1986) by establishing the "socio-mathematical norms" (Yackel & Cobb, 1996, p. 458) that make " . . . possible all students' active participation in . . . [the] discourse and makes demands on students to engage in increasingly more complex thinking" (Wood & Turner-Vorbeck, 2001, p.192).
	Practices	The Learning Cycle (Develop-Solidify-Practice) enables teachers to think through a unit " . . . to determine what mathematics will be taught when by designing tasks that are challenging yet within reach of students" (Reys & Long, 1995, p. 296).	Formative and summative assessments, using both traditional and alternative processes, are important vehicles for determining where students are in the landscape, in the Learning Cycle, and where they need to go, as well as for establishing accountability for learning.	The moves a teacher makes during all three stages of the Teaching Cycle, but particularly during the discussion, determine whether or not the cognitive demand of the task is maintained throughout the cycle.
	Models	"Big Develop" cycles surface thinking about a whole unit and "Small Develop" cycles surface thinking about specific objectives within units. They are often launched with word problems.	During Development, the teacher can expect student thinking that is varied, possibly misconceived, focused on *how* or *what* rather than *why*, and is context dependent.	The fundamental purpose in orchestrating discussions during Develop cycles is to promote overall comprehension.
	Models	Solidify cycles invite students to examine and extend thinking surfaced in Develop cycles. They focus students on looking for patterns; using symbols, structure, and precise language; constructing arguments; critiquing others' reasoning; and making connections.	Student thinking during Solidify is less varied, well conceived, more focused on *why* (justification), less dependent on context, more generalized, and reflective of connections.	Discussions during the Solidify phase focus on connection among students' thinking and making thinking explicit
	Models	Practice cycles encourage students to refine and make solid thinking fluent, i.e., accurate, efficient, and flexible.	Student thinking during Practice becomes refined and fluent, i.e., accurate, flexible, and efficient. Justifications may become generalized proofs.	When moving to Practice, the focus of the discussion is upon consensus and generalization. In Practice, the Explore and Discuss stages often occur simultaneously.
	Strategies	Develop cycles follow a guided inquiry design, with tasks that have a low threshold but high ceiling (easy entry but challenging for all), are contextualized, allow multiple ways to solve that are not necessarily predictable, require relevant background knowledge, create disequilibrium, and do not commonly produce mastery.	During Develop, student ideas reflect that they recognize there is something to wonder about (implicit pattern or structure), are varied, and possibly misconceived. Their strategies are purposeful, context-dependent, and the steps are implicit (not explicit). Their representations are used for exploring, sense making, describing, recording, and reflecting the mathematics inherent in the problem.	During Develop cycles, the most common levels of thinking within the Discussion include short answers, brief statements, descriptions, elaborations, clarifications, translations, and representations.
	Strategies	Solidify cycles maintain some elements of inquiry and follow these designs: Concept Attainment, Resolution of Conflict, Comparison and Relationships, Concept Development, Socratic Method, and Synectics.	During Solidify, ideas become concepts when students can articulate structure or pattern and its components and give names to roles within those structures or patterns. Their strategies become algorithms when the steps of the algorithm are identifiable and repeatable, and not so context dependent. Their representations become tools when they are used for reasoning and explaining.	Discussions during Solidify cycles are characterized by comparison, relation, discerning patterns and/or structure, reasoning, and transfer, thinking that supports the examining and extending of thinking surfaced during Develop cycles.
	Strategies	Practice cycles resemble Direct Instruction at times but do not introduce anything a student has not already constructed. They include games, routines, programmed reviews, and textbook exercises.	During Practice, concepts become definitions or properties—precise, concise statements that capture the essential nature of the structure or pattern of the definition or the characteristic of the property. Algorithms become procedures when there is a reduction in steps through combining or eliminating. Tools become models when they are used across multiple contexts and embody a definition, property, or procedure.	Students are prepared to move into Practice when, during the discussion, they engage in challenging or supporting, justifying, refining, generalizing, and proving. During Practice their thinking becomes accurate, efficient, flexible, and refined.
	Techniques	Flesh out the math in a Common Core-based textbook unit using topic landscapes; then create or utilize, operationalize, and sequence objectives.	Observe students and ask good, initial, probing questions; then really listen to understand student thinking. Then ask additional questions that move a student's thinking forward or that help a student adjust a misconception.as they represent their thinking.	Monitor thinking during the Explore, looking for anticipated thinking and other worthwhile ideas. Use anticipated thinking to select the student thinking to share and use development sequences to order that sharing--as part of lesson planning.
	Techniques	Find/create a Big Develop task to surface thinking about the whole unit and anticipate the thinking the task will elicit based on research on thinking within mathematical topics.	Interpret the underlying mathematical issues with which a student is grappling. Distinguish between errors and misconceptions and listen for the sense even when misconceptions appear.	Assign roles to listening students and hold them accountable for the listening until socio-mathematical norms are firmly established.
	Techniques	As guided by the Learning Cycle and the anticipated thinking, develop purpose statements that align with the objectives.	Compare observed student thinking to the research on thinking within mathematical topics, thus assessing where a child is on the landscape and in the Learning Cycle.	At the "Magical Moments" when a student makes a seemingly profound statement, decide whether pursuing it will move the class toward the purpose of the lesson. If a student communication is worthy of pursuing at this time, decide on who should pursue it—the sharer, other students, or the teacher.
	Techniques	Find or create accompanying Small Develop, Solidify and Practice tasks of varying cognitive demand to promote connections across the domains and phases of the *Continuum of Mathematical Understanding* (Hendrickson et al., 2009) according to unit objectives.	Re-launch a task when assessments during Explore or Discuss warrant a re-launch. Adjust the sequence of tasks within a lesson or unit as informed by assessing student thinking.	Determine at what cognitive level to guide the pursuit according to where students are in the Learning Cycle.

1 *The Continuum of Mathematical Understanding* and the *Learning Cycle* are two components of the *Comprehensive Mathematics Instructional Framework* (Hendrickson, Hilton, & Bahr, 2009.) that is designed to provide a "framework for the kinds of specific, constructive pedagogical moves that teachers might make" (Chazan & Ball, 1999, p. 2). The *Framework of Cognitive Complexity* provides guidance for teachers as they seek to enhance the engagement of their students in mathematical discussions.

means for generating new hypotheses about the nature of our framework and then reflecting on the data obtained from testing hypotheses about the framework in multiple cycles throughout the study, enabling us to investigate the credibility of our data while simultaneously furthering our ideas (Pinnegar & Hamilton, 2009).

In recurring cycles we investigated our own changing perspectives by engaging in increasingly deeper examinations of the framework, a process that resulted in a string of consensually agreed-upon revisions. During each of the first six passes through the data, we discussed the meanings of the outcomes among ourselves. During an additional 12 analyses, as we were using the outcomes to guide our methods coursework and supporting our students in constructing their own meanings for them, what we were learning informed our revisions. These latter cycles consisted of trying to help our students construct their own meanings for the outcomes and develop the learning they defined, then reflecting upon what we learned as we re-examined and revised the framework. As analysis continued, the structure of the framework reflected the naturally occurring themes that typically characterize qualitative research (Guba, 1978).

Findings

In this section we describe the evolution of our framework from two unstructured lists totaling 86 outcomes to its current form. We describe the specific analyses associated with the first six passes through the lists, during which the outcomes themselves served as data. We also describe the revisions resulting from these analyses–as the outcomes were transformed into a tentative framework, which is shown in Table 1. We then describe the subsequent 12 analytical passes, which occurred while we were implementing the outcomes with our methods students. The revisions resulting from these analyses transformed the tentative framework into the current version of the framework as it appears in Table 2.

First Pass—Domains

Because we sought to create a framework that was both comprehensive and cohesive, we revisited the three domains that commonly comprise studies of teacher learning—*Beliefs, Knowledge,* and *Practice* (Wood, Nelson, & Warfield, 2001)—to code our lists. These domains, according to Arbaugh et al. (2010), may well function as "placeholders for an array of . . . outcomes" (p. 18). Thus, beliefs, knowledge, and practice not only served as *a priori* codes but also became foundational in structuring our framework.

We worked together to categorize each of our competency lists, being sure to reach consensus through negotiation rather than capitulation. Both lists contained a reasonable distribution of outcomes across the three domains. Categorizations of 's list resulted in 17 belief statements, 17 knowledge statements, and 19 statements

Table 2

A Framework of Outcomes for Mathematics Teacher Learning—Final Version[2]

	Teacher Learning Positions			
	Mathematical Content Knowledge	Student Thinking	Child Development	Social Interaction
Teacher Beliefs	• Mathematical work is a sense-making endeavor based upon numerical, spatial, and logical thinking (NCTM, 2000). • Mathematics understanding consists of knowing how and why it works (Philipp et al., 2007). • Mathematical understanding is constructed in an instructional environment characterized by problem solving, reasoning, gentle argumentation, modeling and real-life application, strategic tool use, attention to precision, and looking for and using structure (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).	• Students acquire informal mathematical knowledge by interacting with their environment and thus "can solve problems in novel ways before being taught how to solve such problems" (Philipp et al., 2007, p. 475). • "Each time one prematurely teaches a child something he (sic) could have discovered himself (sic), that child is kept from inventing it and consequently from understanding it completely" (Piaget, 1970, p. 715). • Children's thinking is "generally different from the ways adults would expect them to think about mathematics" (Philipp et al., 2007, p. 475).	• In an equitable learning environment ALL students, including students of varied socio-economic status, cultural backgrounds, ethnicities, languages, and with exceptionalities can learn (NCTM, 2000). All means all. • "Teaching should be grounded in how students learn" (Schifter & Fosnot, 1993, p. 193), a process of "active construction, not merely passive absorption . . . (and in) invention, not imitation" (Baroody & Ginsburg, 1990, p. 52). • Mathematical thinking moves generally from the concrete to the abstract (National Gov., 2010; Piaget & Cook, 1952) or from less formal to more formal thinking (Bonotto, 2005).	• Instructional tasks should be designed at a level that all children can enter with some degree of success but that will stretch their thinking via interaction with their peers (Vygotsky, 1978). • Because "thinking with others . . . enables children to develop . . . the capacity to think alone" (Rochat, 2001, p. 139), teaching should be based upon seeking to "understand rather than to be understood" (Covey, 1989). • "Mathematical discourse embeds fundamental values about knowledge and authority . . ." relating to mathematical correctness (NCTM, 1991, 21).
Teacher Knowledge	• A "Profound Understanding of Fundamental Mathematics" (Ma, 1999, p. 120) is constructed in three *cognitive* domains—Conceptual, Procedural & Representational (Fosnot & Dolk, 2001; Hendrickson, Hilton, & Bahr, 2008) and five *mathematical* domains—Number & Operations, Algebraic Reasoning, Geometry, Measurement, & Data Analysis/Probability (NCTM, 2000). • Legitimate mathematical work is characterized by varying levels of thinking (e.g., the *Framework of Cognitive Complexity*, Bahr & Bahr, 2017).	• Rather than transmitting information . . ." (Richardson, 2001, p. 279), teachers use the *LES Instructional Model* (Launch-Explore-Summarize (Discuss); Schroyer & Fitzgerald, 1986) to engage students in legitimate mathematical inquiry. • The degree of teacher guidance associated with the use of the *LES Instructional Model* is dependent upon the teacher's assessment of the location of the students' thinking in the *Learning Cycle*.	• Students' thinking models the actions and contexts inherent in the problems they solve (Carpenter et al., 1999; Fosnot & Dolk, 2001), and serves as landmarks on topical "landscapes" that define individual learning trajectories. • As students make connections within and across "landmarks," their understanding moves through a *Continuum of Mathematical Understanding* according to the phases of the *Learning Cycle* (Hendrickson, Hilton, & Bahr, 2008).	• Participants in mathematical discussions assume one or more roles—the students who "share" their thinking, the students who listen and interact with the thinking of the sharing students, and the teacher, who orchestrates the discussion and also participates as a listener (Wood & Turner-Vorbeck, 2001). • Teachers establish "socio-mathematical norms" (Yackel & Cobb, 1996, p. 458) that make "possible all students' active participation in . . . [the] discourse" (Wood & Turner-Vorbeck, 2001, p.192).

<table>
<tr><th colspan="6">Categories of Inquiry-Based Teacher Practices (reflecting connections among beliefs and knowledge within and across Teacher Learning Positions)</th></tr>
<tr><th></th><th colspan="2">Unit and Lesson Planning Before Instruction</th><th colspan="3">Questioning, Assessment, and Decision-Making During Instruction</th></tr>
<tr><td rowspan="2">Teacher Practices, Nested Strategies, and Techniques</td>
<td rowspan="2">In unit design, teachers use their knowledge of the mathematics and the developmental landscapes that are associated with a particular topic, along with an understanding of the Learning Cycle, to determine what mathematics will be taught when.

• Create a list of informal goals based on . . .
Mathematics study
Landscapes – problem types, landmarks, numerical or spatial levels
Textbook—lists, lesson titles, lesson tasks and exercises, and assessments
• Create a Big Develop task to surface thinking about entire unit
Anticipate thinking from the task
Compare thinking to the informal goals list
• Design an end-of-unit performance assessment
Select or create a task similar to the Big Develop
Decide upon criteria and create a rubric
Add prompts to ensure validity
• Plan for spaced practice via routines and games</td>
<td rowspan="2">Teachers use the LES Instructional Model and the Learning Cycle to design lessons that promote high levels of interactive discourse and result in deep mathematical understanding in harmony with Common Core goals.

• Continue planning the Big Develop lesson
<u>Launch</u>—materials, grouping, assessing task comprehension, making task comprehensible for early language learners
<u>Explore</u>—anticipated thinking, assessing task-related thinking, language tools for early language learners
<u>Summarize (Discuss)</u>—selected and sequenced sharing based on anticipated thinking, listening student roles and responding procedures using the Framework of Cognitive Complexity, scaffolding early language learner participation
• Design subsequent lessons
Find or create Small Develop, Solidify, End-of-solidify assessment, and single-domain practice tasks using Instructional Models
Anticipate thinking from tasks where necessary
Create purpose statements and outcomes for each Teaching Cycle within the lesson
Plan lesson stages</td>
<td colspan="3">Using the Framework of Cognitive Complexity, teachers consistently observe and ask questions in order to "advance, . . . assess" (NCTM, 2014) and interpret student thinking, informing " . . . the decisions about the next steps in instruction" (Williams, 2013, p. 43) as the lesson proceeds through the planned LES Instructional Model.</td></tr>
<tr><td>Launch - Assess general class comprehension of the task giving attention to students with special needs
• <u>Ask</u> questions about task components
• <u>Interpret</u> responses to ascertain task understanding
• <u>Decide</u> whether to proceed to Explore, or clarify the task first</td>
<td>Explore - Assess and advance task-related thinking while interacting with students
• <u>Ask</u> an open-ended question
• <u>Interpret</u> responses
What is the student thinking?
What is the math inherent in the thinking?
Where is the thinking in the Learning Cycle and on the landscape?
What is the relationship of this thinking to anticipated thinking and purpose?
• <u>Decide</u> the next move with this student(s)
What is the next question to be asked?
Continue with this student or another?
Should the task be re-launched for this student(s)?
Should this student share in the Summarize?
• <u>Decide</u> the next move with the class
Should the task be re-launched?
Should the class move to Summarize?
Should Summarize plan be altered? How?
Who will actually share and in what order?</td>
<td>Summarize (Discuss) – Assess and advance the thinking of sharing and listening students
• <u>Ask</u> for contributions to the discussion
• <u>Interpret</u> contributions as in Explore
What is the student thinking?
What is the math inherent in the thinking?
Where is the thinking in the Learning Cycle and on the landscape?
What is the relationship of this thinking to anticipated thinking and purpose?
• <u>Decide</u> when and how to pursue
Should the student's comment be pursued?
If so, who should pursue it?
Pursue at what thinking level of the Framework?
If not pursuing student comment, who should speak next?
Should Summarize plan be altered, and if so, how?
• <u>Decide</u> which task to launch next</td></tr>
</table>

2 *The Continuum of Mathematical Understanding* and the *Learning Cycle* are two components of the *Comprehensive Mathematics Instructional Framework* (Hendrickson, Hilton, & Bahr, 2008.) that is designed to provide a "framework for the kinds of specific, constructive pedagogical moves that teachers might make" (Chazan & Ball, 1999, p. 2). The *Framework of Cognitive Complexity* provides guidance for teachers as they seek to enhance the engagement of their students in mathematical discussions.

related to practice, and Eula and Damon's list consisted of 13 belief statements, 6 knowledge statements, and 14 statements related to practice. This classification process was the first of several analyses that resulted in adding structure to our outcomes lists.

Second Pass—Positions

As introduced in the Positions and Domains of Teacher Learning section, Wood, Nelson, and Warfield (2001) suggested that research investigating the processes by which teacher learning occurs has been conducted from four positions, each with its own theoretical roots—*Mathematical Content Knowledge, Student Thinking, Child Development, and Social Interaction*. In our second analytical pass, therefore, we recoded the outcomes according to these four positions, using the same process of negotiation and consensus as was used in the first analysis. This classification process added another structure to our outcomes lists. Combining our first and second classifications produced a two-dimensional structure, a grid with 12 cells—beliefs, knowledge, and practices related to each of the positions: student thinking, child development, social interaction, and mathematical content knowledge.

Third Pass—Combining and Eliminating Redundancy

In our third analytical pass, we combined both lists into one by eliminating redundant outcomes. This process required us to be very explicit in our use of vocabulary and in explaining what each of our outcomes meant or did not mean. Seventeen outcomes on Eula's list did not appear on Damon's, eight of Damon's did not appear on Eula's list, and some of Eula's were restatements of previous outcomes. Our negotiations yielded a net list of 44 with an average of three or four outcomes per the 12 above-mentioned domain/position classifications. The grid as it relates to beliefs and knowledge outcomes appears as Table 1. Subsequent paragraphs describe a different structure relating to practice outcomes.

Fourth Pass—Beliefs and Knowledge Connections

The nature of the fourth analysis was unanticipated. This analysis arose as a continuation of the previous positions-based categorization. As we examined the outcomes labeled either as beliefs or as knowledge that were categorized within the same position, we noticed that they tended to relate conceptually in a deeper way than by mere position categorization. It became clear that if a teacher possessed a specific belief, that teacher would likely be disposed to act in acquiring related knowledge. Thus the results of this analysis revealed an internal structure that ensured cohesion between two of its three domains, beliefs and knowledge, as shown in Table 1.

Fifth Pass—Higher Order Position Connections and Resulting Practices

The need for a cross-categories analysis of outcomes in the practice domain arose after we had noticed the connections between beliefs and knowledge during the fourth pass. We observed that each practice outcome seemed to require the application or use of knowledge relating to all four positions, as in the case of the practice of orchestrating discussions. First, a teacher should know the kinds of questions to use to probe student thinking while a discussion is taking place, and second, be able to identify the mathematics inherent within that thinking. Third, the teacher should know where that thinking fits within the developmental landscape of that particular mathematical domain, and fourth, know the kinds of moves to make in order to take full advantage of that thinking when it is shared. Therefore, as Nelson (2001) wrote, making connections among the work represented by the four positions "create[d] a more complete description" (p. 251) of teaching, and a more cohesive one.

Because a position level categorization was insufficient for grouping outcomes we labeled as practices, we examined the relevant data for emergent themes we could use, expecting the outcomes to cluster naturally into a small number of groups. This analysis resulted in three categories—curriculum planning; questioning and assessment; and orchestrating discourse during exploration and discussion—that we labeled *Domains of Practice* as shown in Table 1. Interestingly, these categories are similar to the domains of practice Boerst, Sleep, Ball, & Bass (2011) used to guide their preservice methods course at the University of Michigan, a connection we discovered after creating our own domains.

Sixth Pass—Grain Sizes

Boerst et al. (2011) added their insight into the specification of practice-oriented practices by discussing the decomposition of the work of teaching.

> To make the complex work of teaching mathematics more learnable by beginners, we aim to decompose practice into smaller tasks or routines that can be articulated, unpacked, studied, and rehearsed (Grossman et al., 2009; Grossman & Shahan, 2005; Lampert, 2001, 2005). Such decomposition temporarily reduces complexity by holding some aspects of teaching "still" or by routinizing some components of the work so that beginning teachers can attend to and practice particular skills or focus on specific problems. (p. 2849)

They continued by describing varying sizes of decomposition from large domains to "intermediate- and technique-level practices (p. 2871)." These hierarchical levels of decomposition indicate successively smaller "grain sizes" (p. 2850), with smaller grain-sized practices nested within larger ones.

> The other product of this approach to decomposition is an articulation of *connections* among nested teaching practices. Moving along a strand from practices of

> larger grain size to those of smaller grain size details with increasing specificity how a practice can be implemented. In turn, starting with a technique and moving up a strand to practices of increasingly larger grain size makes visible the purpose that those techniques are serving. (p. 2854)

We therefore analyzed the practice domain outcomes, looking for an appropriate number of grain sizes while categorizing the outcomes according to size. That is, we examined which outcomes conceptually nested within other outcomes to determine the appropriate number of grain sizes. Using our own wisdom of practice, we also examined each category and added outcomes of various grain sizes when a category appeared wanting.

This planned analysis revealed four levels of generality/specificity, or grain size, as shown in Table 1, which we labeled as *Practices, Models, Strategies*, and *Techniques*, terms commonly used by curriculum and instructional designers (e.g., Gunter, Estes, & Schwab, 1999; Burden & Byrd, 2013). The term *practice* now refers both to a domain within our framework, along with beliefs and knowledge, and to a grain size of outcomes within the practice domain. Thus, decomposed practices were labeled as nested models*,* decomposed models were labeled as strategies*,* and decomposed strategies were labeled as techniques. We also sequenced the outcomes within and across grains sizes if the outcomes suggested that teacher practices should be performed in some sort of order while designing or implementing instruction.

Four Coordinated Analyses

Because the set of outcomes was now organized by a structure as described above, we referred to it as a framework, albeit far from complete. We continued to revise and improve it while using it to guide our teaching of preservice methods students. In this process, the framework of outcomes no longer served simply as data we were analyzing, but also as a series of results stemming from our analysis of data obtained from other sources as mentioned previously. We performed various combinations of the following four types of analyses as we passed through the data 12 more times. By this point, Jodi had joined our faculty and had become an integral partner in our study.

1. *Distinguishing between outcomes and course content.* In some of the passes, our analyses included determining if any of our outcomes were actually descriptions of course content.

2. *Reviewing the literature.* We examined the literature acquired from our early framework development along with additional sources we located.

3. *Refining the language of the framework.* We continued the refinement process by considering how best to state our outcomes and to structure the language of the outcomes. The former process consisted of ongoing wordsmithing—looking for the right word here, deleting a word there, etc. The latter process considered the

sentence or phrase structures that would more clearly present outcomes within each teacher learning domain and deliberating about the language needed for structures across domains and grain sizes and within the practice domain.

4. *Reorganizing within and across domains and positions.* We addressed several issues during this stage of framework development that went well beyond the work of wordsmithing as described in the previous section. We divided some outcomes and combined others. We moved some outcomes from one domain or position to another. Some outcomes were removed from the framework altogether because they were redundant.

Twelve Passes in the Context of Framework Implementation

We used the four types of analyses in varied combinations during 12 analytic passes associated with using the framework to guide our methods instruction. As we discuss the results, we share the data and sources from which they were obtained, the processes of analysis we used and how we used them, and the resulting changes in our framework. We do not relate all of the changes but, rather, a few of the most prominent ones, thus providing the reader an overall sense of our analysis and revision efforts.

Our early attempts at sharing the framework with our students revealed that they were generally overwhelmed with the sheer amount of information (Session Notes beginning 9/4/14). This experience influenced all of our analyses, but its first effect occurred in the fourth pass as a reduction in the number of grain sizes in the practice domain from 4 to 3.

After engaging in a series of activities designed to help our students construct the outcomes for themselves, we shared the framework with them in its then current form. We referred to specific outcomes when engaging in activities related to those outcomes and even asked students to self-assess their growth using the framework. It became apparent (Session Notes, 9/4/14) that the outcomes in the beliefs and knowledge domains were difficult to interpret and were therefore not as meaningful for student use as we had hoped. We needed to clarify our thinking both for ourselves and for our students. In multiple iterations of wordsmithing and reconceptualizing, we added to, reworded, deleted, split, and combined outcomes. For example, we noticed the need to emphasize the conceptual and procedural aspects of mathematical work (Philipp, et al., 2007; NCTM, 2014) and added this idea as an outcome in the 11th pass. We added outcomes about sense making, the student mathematical practices from the *Common Core State Standards—Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and the authority structures associated with discourse patterns.

We also changed the positions and/or domains of some outcomes. We moved, in the sixth pass, the outcome about making connections across domains of understanding to child development, thinking it more precisely defined a belief related to how students learn. We also moved the outcome about authority structures to the social interaction position, recognizing that it is more aligned with social interaction patterns than with the nature of mathematics content.

We consistently asked students for formative feedback at key points throughout the semester. One piece of that feedback (Formative Feedback Summary, 2/12/15) suggested that our students needed more help with unit and lesson design, including how to utilize a traditional textbook in designing inquiry-based lessons. Thinking that improving the way our unit design outcomes in the practice domain were written would help us improve our teaching in this area, we split the curriculum-planning category into two categories, *unit planning* and *lesson planning*, in the sixth pass. We added more outcomes of smaller grain size to the unit-planning category in the tenth pass, then discovered we had added too much detail, so we combined, simplified, and reorganized those outcomes.

While considering discussion orchestration in the context of the Launch-Explore-Summarize Instructional Model (Schroyer & Fitzgerald, 1986) in class one day, we were greeted by several puzzled facial expressions and one brave student who queried, “What is Launch-Explore-Summarize?” This time we were the ones mystified because we had discussed this model on more than one occasion (Session Notes 10/2/14). To help ourselves correct this problem, we first added a knowledge outcome to the social interaction position in pass seven, and then revised it and moved it to the student thinking position in the 11th pass. We made the Launch-Explore-Summarize model the organizational structure of the lesson planning category in the sixth pass and the questioning and assessment and discussion orchestration categories in the ninth pass, all within the practice domain. We also made one more structural change related to the model in conjunction with the next major revision as explained in the following paragraph.

In a feedback session following a methods course field practicum (Session Notes 12/2/14), several students expressed interest in learning more about assessment. We had attempted to strengthen the questioning and assessment category in the practice domain in the fourth pass by including outcomes related to fundamental notions of assessment, such as *validity*, *reliability*, *formative*, *summative*, etc., prior to receiving this feedback. At this time we sought to create outcomes that more clearly reflected the integral role assessment played in the in-the-moment decision-making that occurs while a lesson is implemented, including questioning and interpreting student responses (see Williams, 2013). As we made the 11th pass, we combined two subcategories, questioning and assessment and discussion orchestration, into one category, which we titled questioning, assessment, and decision-making. Then, continuing with the use of the Launch-Explore-Summarize Instructional Model (Schroyer & Fitzgerald, 1986) as an organizing structure, we used the model to structure the entire category, then used the frame *ask, interpret, decide* to specify the fine grained outcomes related to Launch-Explore-Summarize, all in the 12th pass.

During the first semester of implementation and 2 months after our students had studied the learning landscapes described in the *Progressions for the Common Core State Standards in Mathematics* (Core Standards Writing Team, 2013), we were mystified by a student's remark that he would like to know where he could find research available on how students' thinking changes over time (Session Notes 12/2/14). We modified our outcomes to show the pivotal role this knowledge should play in the practices of task selection, anticipating student thinking, interpreting student thinking, and in the decisions involved with selecting and sharing student thinking during a discussion. In our earlier versions, the notion of landscapes was part of the unit planning outcomes, but in the 12th pass we made it an integral part of several outcomes within the questioning, assessment, and decision-making category of the practice domain.

When watching our students teach each other and teach small groups of children, we discovered that assigning roles to listening students during the sharing of student thinking in a discussion (see Wood and Turner-Vorbeck, 2001) was often forgotten or not done very well (Session Notes 10/3/14; 2/6/15). After adjusting our instruction as a reflection of changes in the questioning, assessment, and decision-making category of the practice domain, we noticed improvements among our students in this aspect of discussion orchestration.

The changes associated with our multiple analyses are a sampling of the many changes that resulted in the framework in its current form (see Table 2). We doubt it will ever be totally complete, but we believe it is now sufficient to provide meaningful guidance to our teaching, and perhaps help other mathematics educators examine their teaching as well.

Content of the Framework

The purpose of this section is to introduce the reader to the specific content of the framework as shown in Table 2. Our purpose is not to give a lengthy treatise on the underlying research relating to each outcome, but rather to provide a narrative that highlights the key notions from which the content of the framework is taken. Because much of the framework is grounded in well-accepted mathematics education research, we believe our work may be transferable for use by other mathematics educators. We begin by discussing the content of the beliefs and knowledge outcomes that stem from the four positions (Wood et al., 2001) we described previously in the Positions and Domains of Teacher Learning section. We then discuss the practice outcomes, which represent a synthesis of beliefs and knowledge outcomes across multiple positions.

Key statements relative to the mathematical content position define mathematical work as a sense-making endeavor based upon numerical, spatial, and logical thinking (NCTM, 2000). These statements define mathematical understanding as learning how and why mathematics works (Philipp et al., 2007) within an instructional environment characterized by problem solving, reasoning, gentle argumentation, modeling and real-life application, strategic tool use, attending to precision, and looking for and using structure (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). We knew we wanted the teachers we serve to construct a "Profound Understanding of Fundamental Mathematics" (Ma, 1999, p. 120) in three *cognitive* domains—Conceptual, Procedural & Representational (Fosnot & Dolk, 2001; Hendrickson et al., 2008) and five *mathematical* domains—Number & Operations, Algebraic Reasoning, Geometry, Measurement, & Data Analysis/Probability (NCTM, 2000). We also knew that these teachers' mathematical work, and the work of their students, should be characterized by varying levels of thinking (Bahr & Bahr, 2017).

Teacher learning in the student thinking position is based on a deep appreciation for the way students think. Because children can construct informal mathematical knowledge by interacting with their environment, they "can solve problems in novel ways before being taught how to solve such problems" (Philipp et al., 2007, p. 475). Indeed, "each time one prematurely teaches a child something he [sic] could have discovered himself [sic], that child is kept from inventing it and consequently from understanding it completely" (Piaget, 1970, p. 715). Their thinking is "generally different from the ways adults would expect them to think about mathematics" (Philipp et al., 2007, p. 475). Rather than transmitting information . . ." (Richardson, 2001, p. 279), teachers use the *LES Instructional Model* (Launch-Explore-Summarize [Discuss]); Schroyer & Fitzgerald, 1986) to engage students in legitimate mathematical inquiry. The degree of teacher guidance associated with the use of the *LES Instructional Model* is dependent upon the teacher's assessment of the location of the students' thinking developmentally as described in the *Learning Cycle* (Hendrickson at al., 2008), a generic set of learning phases during which students surface their thinking in a problematic environment, refine that thinking by connecting it other ideas, strategies, and representations, then work to make it more fluent.

A developmental perspective includes the notion that in an equitable learning environment all student can learn and in fact, "teaching should be grounded in how students learn" (Schifter & Fosnot, 1993, p. 193). Learning is a process of "active construction, not merely passive absorption . . . [and in] invention, not imitation" (Baroody & Ginsburg, 1990, p. 52). Mathematical thinking moves generally from the concrete to the abstract (National Gov., 2010; Piaget & Cook, 1952) or from less formal to more formal thinking (Bonotto, 2005). Students' thinking models the actions and contexts inherent in the problems they solve (Carpenter et al., 1999; Fosnot & Dolk, 2001), and serves as landmarks on topical landscapes that define individual learning trajectories. As students make connections within and across landmarks, their understanding moves through a *Continuum of Mathematical Understanding* according to the phases of the *Learning Cycle* (Hendricksonet al., 2008).

Honoring the social nature of learning involves allowing children to think ". . . with others . . . [which] enables children to develop . . . the capacity to think alone"

(Rochat, 2001, p. 139). Thus teaching should involve designing instructional tasks at a level that all children can enter with some degree of success but that will stretch their thinking via discussions with their peers (Vygotsky, 1978). This interaction "... embeds fundamental values about knowledge and authority ..." relating to mathematical correctness (NCTM, 1991, p. 21). Participants in mathematical discussions assume one or more roles—the students who "share" their thinking, the students who listen and interact with the thinking of the sharing students, and the teacher, who orchestrates the discussion and also participates as a listener (Wood & Turner-Vorbeck, 2001). Teachers establish "socio-mathematical norms" (Yackel & Cobb, 1996, p. 458) that make "possible all students' active participation in ... [the] discourse" (Wood & Turner-Vorbeck, 2001, p.192).

The practice domain is divided into two parts—outcomes relating to the planning of lessons and units of instruction prior to instruction and outcomes relating to the questioning, assessment, and decision-making that occur during instruction. Prominent in both parts is knowledge related to all four positions. For example, knowledge of student thinking and how that thinking changes over time informs both planning and instruction. The *LES Instructional Model* (Schroyer & Fitzgerald, 1986) provides a useful framework both for planning lessons and for organizing outcomes relating to instructional implementation. Knowledge of the *Learning Cycle* (Hendrickson et al., 2008) influences tasks design and sequence in unit planning along with the interpretations and decisions associated with instructional implementation. Mathematical content knowledge informs every aspect of planning and the interpretation of student thinking that is so critical to decision making. Finally, the various roles participants play in a mathematical discussion guides the planning of those discussions and the carrying out of those plans as a lesson unfolds.

Conclusions

Our purpose in conducting this self-study was to create a comprehensive and cohesive set of outcomes to guide the teaching and learning of preservice and in-service mathematics teachers with whom we work and to guide our study of that teaching and learning. It resulted in the *Framework of Outcomes for Mathematics Teacher Learning* (FOMTL). As three mathematics educators with views that are both remarkably similar and yet appreciably different, we needed to seek to understand each other's perspectives in order to adequately analyze and synthesize our thinking. Thus, we reconsidered our own perspectives in light of our colleagues' views, a process that resulted in a refinement of our own understandings and in a negotiated shared perspective in the form of our framework.

This work has led us to some generalizations that hold great meaning for us. First, we agree more fully than ever with the *Linking Research and Practice: The NCTM Research Agenda Conference Report* (Arbaugh et al., 2010) regarding the need for a specific set of outcomes to guide mathematics teacher education. We sense that adherence to constructivist principles, principles that we believe in, may have led us to struggle with the call for precise specifications of outcomes that define quality mathematics teaching. We recognize the similarity in our struggle with the debate about the role of outcome or objective specification in teaching children and adolescents. Having a clear sense of where we are leading our students does not mean we ignore our constructivist roots. Rather, we seek to help our students construct outcomes for themselves, as well as the specific beliefs, knowledge, and practices that the outcomes define, without having to construct all of its structure. Our work has enabled us to focus our students more clearly and explicitly on a holistic picture that defines exemplary mathematics teaching and to help them schematize the complex work of teaching mathematics.

Echoing Simon's (1995) statement in the beginning of this paper, not having a clear sense of what good mathematics teaching is, as defined by our framework or other documents, is as dangerous to mathematics teacher education as not having a clear sense of what mathematics is to mathematics teaching. Thus, we honor the NCTM's *Professional Standards for Teaching Mathematics* (1991), along with other landmark documents in the mathematics education reform movement, and hope that our framework will enhance the pioneering work of their authors.

Second, we now have a more refined view of the interaction among the domains of teacher learning—beliefs, knowledge, and practice—and the four theoretical positions that characterize the study of teacher learning—mathematical content knowledge, student thinking, child development, and social interaction (Wood, Nelson, & Warfield, 2001). We have long believed these domains and positions could guide some sort of framework that defines teacher learning. This work has made it clear to us that there are distinct beliefs and knowledge outcomes associated with each position, that beliefs and knowledge outcomes within a position are conceptually related, and that all practices of any grain size require varied syntheses among positions and domains.

Third, we have a sense for the complexities associated with helping novice and inservice teachers conceptualize teaching and learning that aligns with current research, while at the same time using a structure and language that is accessible to them. On the one hand, we wanted to create a framework that was sufficiently comprehensive to guide our students not only in their preservice learning but also in their ongoing professional learning as inservice teachers. On the other hand, we did not want to overwhelm them with excessive amounts of information or with excessively complex information, thus creating negative impressions of the framework and of reform-based teaching in general. Our intent is to help students construct these outcomes as well as the learning that fulfills them rather than our adopting a teaching-as-telling mode. We are also pleased that we have the opportunity to spend considerable time helping inservice teachers pick up where they left off in their preservice learning via our professional development efforts. Indeed, having the same set of outcomes to guide our work in both contexts helps us bridge the chasm, at least for the teachers with whom we work, that exists between preservice and inservice teacher education (Zeichner, 2010). For this reason, among others, we particularly enjoy our professional development efforts when we are privileged to teach our former preservice students.

Based on our most recent teaching, we are convinced that the structure of our framework facilitates instruction that supports the individual construction of networks of connections in our students' learning. Indeed, we see it as a landscape that affords any number of learning trajectories as preservice and inservice teachers make sense of mathematics teaching in their own ways while helping us have a comprehensive and well-defined focus. We believe our framework, although still a work in process, will aid us and our students as we encourage them "in the creation and continued refinement of sophisticated models or ways of interpreting the situations of teaching, learning and problem solving" (Lesh, Doerr, Guadalupe, & Hjalmarson, 2003, p. 227).

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