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Pre-Service Mathematics Teachers' Understanding of Quadrilaterals and the Internal Relationships between Quadrilaterals: The Case of Parallelograms

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Abstract: This study attempts to reveal pre-service teachers' conceptions, definitions, and understanding of quadrilaterals and their internal relationships in terms of personal and formal figural concepts via case of the parallelograms. To collect data, an open-ended question was addressed to 27 pre-service mathematics teachers, and clinical interviews were conducted with them. The factors influential on pre-service teachers' definitions of parallelograms and conceptions regarding internal relationships between quadrilaterals were analyzed. The strongest result involved definitions based on prototype figures and partially seeing internal relationships between quadrilaterals via these definitions. As a different result from what is reported in the literature, it was found that the fact that rectangle remains as a special case of parallelogram in pre-service teachers' figural concepts leads them not to adopt the hierarchical relationship. The findings suggested that learners were likely to recognize quadrilaterals by a special case of them and prototypical figures, even though they knew the formal definition in general. This led learners to have difficulty in understanding the inclusion relations of quadrilaterals.

Keywords: *Internal relationships between quadrilaterals, personal-formal-figural concept, pre-service mathematics teachers.*

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Introduction

The learning of geometrical concepts is a complex process (Kaur, 2015) and as a component of the geometry curriculum, defining and classifying quadrilaterals is considered to be a difficult subject by a lot of learners (Clements & Battista, 1992; de Villiers, 1994; Erez & Yerushalmy, 2006; Fujita & Jones, 2007; Fujita, 2012; Okazaki & Fujita, 2007). This stems from the difficulties encountered while analyzing the properties of various quadrilaterals and distinguishing between their critical and non-critical aspects. Given the difficulties experienced by students in defining and classifying the geometrical concepts (de Villiers, 2004; Monaghan, 2000), studying the learners' understanding is important for mathematics education research. Classification of concepts cannot be considered separate from the definition process of concepts (Tall & Vinner, 1981), both are related processes. Effective learning of the definition and classification of quadrilaterals requires logical reasoning based on establishing appropriate interactions between the concept and images (Fujita & Jones, 2007). While many of the concepts we use cannot be defined completely, we learn to define them through our experiences. These concepts can be developed afterwards and interpreted with more precise definitions. In the process of recalling and directing a concept, there are many methods affecting their meanings and utilization either consciously or unconsciously (Tall & Vinner, 1981). The methods influential on the meanings and utilization of concepts may yield accurate and adequate results whereas it is also possible for them to result in failure. We need to understand the reasons behind this failure in order to establish an effective communication with students in the class. In this sense, exploring how various mathematical concepts exist in students' minds is considered to be an important phase (Vinner & Dreyfus, 1989; Tall et al., 2000). In accordance with given explanation, the purpose of this study is to explore the nature and causes of the "gap" between parallelogram's formal concept and the concept of parallelogram in individuals' minds by revealing the figures and definitions constructed by the individuals regarding the concept.

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Literature Review

Internal Relationships between Quadrilaterals: Partitional versus Hierarchical

De Villiers (1994) states that two types of classifications can be made regarding the relationships between quadrilaterals: hierarchical classification and partial classification. With hierarchical classification, quadrilaterals are associated with one another within the framework of their properties as subsets (de Villiers, 1994). Mentioned classification provides a motivation for further analyses in addition to simple and visual summarization of the information (Craine & Rubenstein, 1993) and, requires establishment of appropriate relationships between concepts and images. It considers shapes as being subsets of other shapes, so that squares are seen as special cases of rectangles and rhombi are included in the set of kites (Forsythe, 2015). In addition to the common approach of hierarchical classification, partial classification is used as an alternative to classify the figures (de Villiers, 1994). In partial classification, quadrilaterals are considered to be independent from each other and, classified according to their properties as separate sets (Erez & Yerushalmy, 2006). Partial classification and definition are not incorrect in mathematical terms. They are simply partial, sometimes necessary and beneficial for a clear distinction between concepts (de Villiers, 1994). However, partitional view can be held very strongly since it has been developed from "an early age", so students often find difficult to accept the inclusion of some classes of shapes within others (Okazaki, 2009). The hierarchical classification involves comprehending the relationships between quadrilaterals, which is a rather difficult activity for many learners (Erez & Yerushalmy, 2006; Fujita, 2012). Young children may find it hard to recognize that a square belongs to the group of rectangles which in turn belongs to the group of parallelograms which in turns belongs to the group of quadrilaterals (Levenson, Tirosh & Tsamir, 2011). It is necessary to provide an insight to eliminate this difficulty regarding the relationships between quadrilaterals.

The use of figural demonstrations of geometric concepts plays an important role in conceptualizing geometric objects. Some features are more characteristic or probable than others and thus some examples are 'better' examples than others. Ideal examples are called prototypes (Levenson, Tirosh & Tsamir, 2011). In terms of geometrical thinking, Hershkowitz (1990) defines prototype phenomenon as the specimen that is usually the subset of the examples that had the longest list of attributes and which has strong visual characteristics. One theme in geometry education where prototype phenomenon has been applied is the classification of geometric shapes (Hershkowitz 1990). This prototype phenomenon has been and continues to be the subject of research and, it is argued to be one of the causes of difficulties in understanding the inclusion relations of quadrilaterals (Fujita, 2012; Sinclair, et al., 2016). Various studies emphasized that it is an obstacle preventing the acquisition and adoption of hierarchical relationships between them, as classifying quadrilaterals (Fujita, 2012; Fujita & Jones, 2007; Okazaki & Fujita, 2007). Perceiving quadrilaterals as different from one another due to prototype phenomenon leads to the partial classification of the relationships between quadrilaterals. Considering from the aspect of the relationships between quadrilaterals, it is believed that the contradictions and wrong generalizations that may be caused by prototype phenomenon may appear in the relevant geometrical thinking levels as well.

Theoretical Opinions Regarding Concept Image and Concept Definition

The importance of geometric concepts` definitions is reflected in the research literature, with many studies on this theme appearing over the past decade: understanding the process of defining (e.g. Zandieh & Rasmussen, 2010) and understanding of quadrilateral definitions (e.g. Govender & De Villiers, 2004; Levenson, Tirosh & Tsamir, 2011). Within the context of the instruction of certain simple geometrical concepts, Tall and Vinner (1981) explained concept definition and concept image regarding the instruction of these geometrical concepts and emphasized the distinction between these two which play a major role in geometric concept formation. Concept image refers to establishing relationships with all the concepts involving cognitive structures (i.e. the entire mental image and relevant properties) and methods. An individual's concept image is created as a result of his/her experience with a situation that either exemplifies the concept or not (Vinner & Dreyfus, 1989; Vinner, 1991). On the other hand, concept definition refers to combination of a series of words to describe the concept and, covers an individual's (i.e. student's) words to explain the image of the concept in his mind. Hence, personal concept definitions can be different from the formal concept definitions that are accepted by mathematical circles later on. Research indicates that although students know the correct definition (inclusive or not) of quadrilaterals, they often prefer to rely on specific examples when identifying shapes (Fischbein & Nachlieli, 1998; Foster 2014; Fujita, 2012). Students, for example, may view the square as having sides which are horizontal and vertical, as well as equal in length, whereas a mathematical definition (four equal sides and two pairs of perpendicular sides) is not dependent on its orientation (Forsythe, 2015).

All the geometrical concepts have visual images representing themselves. In this sense, "Figural Concept Model", which was proposed by Fischbein (1993), deals with the reasoning process in terms of the interaction between geometrical figure and concept, such that the concept of geometrical figure is a "figural concept" having both conceptual and figural aspects. In this bilateral relationship, while Fischbein (1993) regards figural concept as a process in which the harmony between the figural and conceptual aspect develops into the ideal form, he does not address the development of this process in individuals (Erdogan & Dur, 2014). Based on the definitions of concept definition and concept image, Fujita and Jones (2006) reinterpret the figural concept developed by Fischbein (1993). In the case of shapes and their definitions, Fujita and Jones (2007) refers to the “personal figural concept” which is the student’s own personal definition and the “formal figural concept” which is the formal definition used in mathematics. These concepts dealt with the components “image”, “concept”, and “properties”, which are associated with one another, to analyze the development of this complicated process (Fujita, 2012). This theory asserts that each individual has his/her own concept image and definition within his/her geometry learning experiences and figural concepts (Fujita & Jones, 2006; 2007). The relationship networks between these three different theoretical frameworks that we determined are shown in Figure 1. This diagram in Erdogan and Dur’s (2014) study was developed and interpreted in a slightly different manner in our study. While in their work, concept image and concept definition are given in connection with direct figural concept (ibid), we have considered them as two basic concepts covered by other theories.

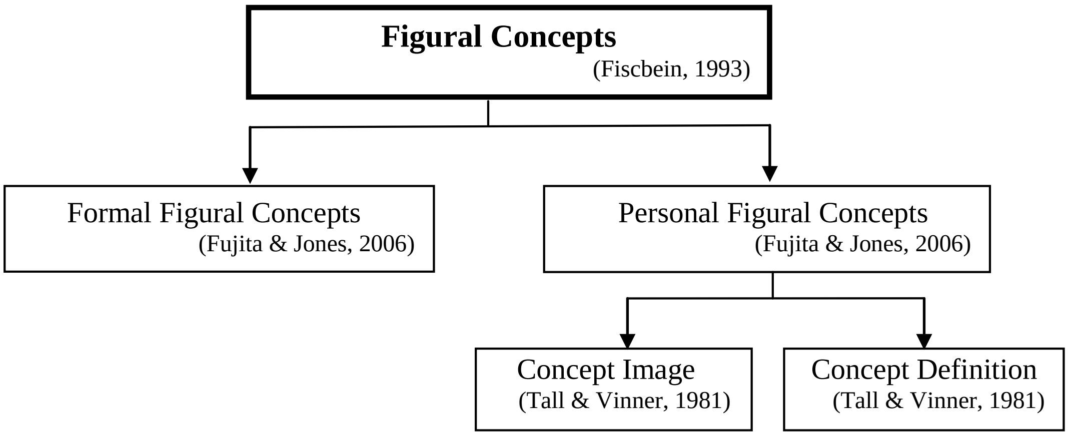


Figure 1: The relationship between theories regarding definition/image of geometric concepts

Although these frameworks’ natures are similar, there are some fundamental differences in the handling of concepts. For example, while Tall and Vinner (1981) discuss cognitive paths to regarding concept, Fischbein (1993) addresses the difficulties between concept and figure. When we interpret Figure 1, a geometric figure has both an image and a definition, so it is a "figural concept". While these definitions, which are included in the scientific sources, educational curriculum and textbooks of the countries, and images based on definition are expressed as "formal figural concept”, they are expressed as “personal figural concepts” in the person's mind (Erdogan & Dur, 2014). That is to say, although concept image and concept definition explained by Tall and Vinner (1981), are directly related to the formal figural concept, they may be compatible with the personal figural concept. In Figure 1, our goal is not to associate three different frames but to show that they are related to each other as well as their different aspects. Thus, we are trying to present this study’s framework – formal and personal figural concept – which are associated with different theoretical frameworks.

In order to move from a partitional manner to a hierarchical one when classifying figures, students need to re-construct how they categorize shapes (Tall et al., 2001). This requires students to give to the figural aspect less importance, but students’ personal figural concepts are so influential that they dominate the way the student defines the properties of shapes (Fujita & Jones, 2007; Okazaki, 2009). For example, when individuals hold a personal figural concept of a rhombus as “a crushed square which has four equal sides”, and a kite as “a shape which must have two smaller sides at the top and two longer sides at the bottom”; they have difficulty accepting that a rhombus is a special case of a kite (Forsythe, 2015). Teachers have a critical role to play in overcoming problems towards hierarchical classifying, which students encounter, or may encounter (Turnuklu, Alayli & Akkas, 2013). Because of this critical role, teachers’ and pre-service teachers’ perceptions regarding the relationships between quadrilaterals are an important area of study. This study investigated to analyze pre-service mathematics teachers’ figural concepts regarding parallelograms (i.e. images and definitions) and reveal their sense of classification of the relationships between parallelograms and other quadrilaterals. The present study is based on the fact that various situations may play a role in detecting the internal relationships between quadrilaterals, which may be caused by various reasons. Hence, a deeper and richer amount of knowledge was needed to reveal understandings regarding a quadrilateral – parallelogram – and internal relationships between quadrilaterals based on parallelograms. It addresses three research questions:

- What are the characteristics of the geometric properties used by pre-service mathematics teachers when defining a parallelogram?
- Which concept definition does have pre-service teachers for a parallelogram and to which concept images do correspond these concept definitions?
- What understanding do pre-service teachers have about the inclusion relations of quadrilaterals?

Methodology

The research is a phenomenological study conducting with a qualitative interpretive paradigm. Phenomenological studies define the meaning of a concept or a phenomenon as a result of experiences regarding such concept or phenomenon and concentrate on what kind of a general understanding people have regarding the phenomenon at hand (Creswell, 2007). The present study dwells on pre-service mathematics teachers' personal figural concepts and their understanding of the relationships between quadrilaterals over the case of parallelogram. It focuses on "what" they experience and "how" they experience regarding the relationships between quadrilaterals as well as the factors that are influential on these experiences. Phenomenological design was preferred to have an in-depth and detailed understanding. We are trying to reveal the understanding of the quadrilaterals and internal relations of the students who have gained experience from the elementary school to the university period on the almost same content education and, the basic factors of the formation of these understandings. For this reason, our focus is not on people but on the people's understandings of conceptions. In order to have an in-depth and detailed understanding, the study was conducted using the phenomenological design.

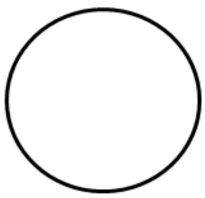
Participants

The data of this study were collected from 27 third-grade pre-service secondary school mathematics teachers studying at a faculty of education in Turkey. An open-ended question was addressed to all the third-year students (43 students). By taking the student responses into consideration, 27 pre-service teachers from various categories were selected as study group based on voluntariness. This study group was preferred because they had taken all the courses regarding geometry (e.g. geometry, dynamic geometry instruction). The course of "Geometry" that the pre-service teachers took in first year of their university program concerns the foundations of the geometry: it is mainly about the reconstruction of Euclid's axiomatic system through definitions, non-defined terms, axioms, postulates, theorems etc. This course aims mainly understanding deductive functioning in an axiomatic system. It is also with the aim of making an introduction for compass and ruler constructions while respecting the constraints of Euclidian geometry's axiomatic system. The "Dynamic Geometry" is another course that the pre-service teachers followed during their university studies, in the second year. This course aimed the use of the dynamic geometry softwares "Cabri-Géomètre and GeoGebra" in the secondary school mathematics teaching. So, in this course, it is on one hand about robust geometrical constructions in Euclidean geometry with the constraints and contributions of the software interface and on the other hand about a didactic look on the dynamic geometry taught in the middle school. Another module of training that the pre-service teachers followed is the "didactics of geometry": this module concerns the teaching of the geometry at the middle school; more precisely the compass and ruler constructions, the geometrical objects in 2D and their geometrical properties through a constructivist approach. So, it was thought that they had high-level knowledge regarding geometry. In addition, they were both a group that would graduate a year later to be teachers and a group of "learners" continuing learning, which was considered to be very beneficial for research. The following section deals with how this process was conducted with the identified study group.

Data Collection Tools

Two different ways that are appropriate in the design of the study were employed for data collection: document analysis through open-ended questions and clinical interviews. The details regarding these data collection tools are given below.

Initially, each pre-service teacher was implemented to an open-ended question to which they would respond individually through reasoning in a written format. The aim was to reveal 27 pre-service teachers' perceptions regarding parallelograms and internal relationships between quadrilaterals (See Figure 2).



Question: Draw a parallelogram within the circle given on the left side with vertices on the circle. Write your steps of drawing the figure in the order you followed.

Figure 2: The open-ended question implemented on the pre-service teachers

This question does not allow drawing a prototype parallelogram within the circle with four vertices on the circle. It is expected from the participants to make use of hierarchical relationships between quadrilaterals and draw either a square or a rectangle. The pre-service teachers were not asked to define the relationships between quadrilaterals, rather they were requested to answer this question requiring reasoning in order to have a deep insight of their knowledge and reveal indirectly their current knowledge. The question we used, see figure 2, was taken from one originally designed by Koseki (1987) to measure the level of understanding of parallelograms by children and then, a study involving this question was conducted by Fujita and Jones (2007). The question was adapted to the study and transformed into an open-ended question. Each pre-service teacher was asked to respond individually. The question involves the steps of how to draw a parallelogram and figure drawing, that reflects both the design in the mind and the steps of drawing. The first of these phases is the drawing phase requiring the spatio-visual elements. The second is the explaining phase of these elements and the Euclid geometry theory.

After the completion of the analysis of the responses given to the open-ended question, clinical interviews were conducted to reveal precisely what each pre-service teacher had in his mind regarding the phenomenon (i.e. parallelogram) and to indicate the mental processes as a result of which the images in their minds were formed. Clinical interviews were employed because they allow using flexible questions to explore the variety of the students' ideas, deal with the main activities, and evaluate cognitive skills (Baki, Karatas & Guven, 2002). Initially, the responses given to the open-ended question were analyzed. Based on the categories obtained from these analyses, semi-structured interview forms were prepared for each pre-service teacher from each category. The personal figural concepts of the pre-service teachers regarding parallelograms were tried to be revealed through clinical interviews.

During the interviews, the participants were asked to define parallelograms and criticize their definitions over their responses to the drawing question. The purpose of these questions was to detect the gap between the formal and personal figural concepts of the learners. Clinical interviews were conducted over the answer texts written by the pre-service teachers. Without directing the pre-service teachers, the researchers asked questions to have a deeper insight such as "What did you pay attention to while drawing the parallelogram?", "How did you draw this?", and "Why did you make this drawing?". Thus, concept definitions, concept images, and the relationships between them were focused on. In this way, what the pre-service teachers thought while solving the question was questioned. The interviews were conducted with the pre-service teachers one-to-one. The interviews took an average of 15 minutes for each pre-service teacher. The pre-service teachers responded the interview items either verbally or in written format (i.e. drawing) depending on the need during the interviews. The pre-service teachers' verbal opinions were recorded via a recorder in order to prevent data loss. The worksheets of the students who resorted to drawing were taken to be examined later on.

Analyzing of Data

The data was obtained from the students' worksheets where written answers are available and from the interviews conducted with students and analyzes were made using these two data sources. Figures drawn in the worksheets by the pre-service teachers were analyzed by considering the related explanations on how they proceeded. The reason why we wanted drawing parts to be explained, is to have a deeper idea about the shape which was drawn, although they can draw it on the circle. Let us explain this case and the analysis method on an example:

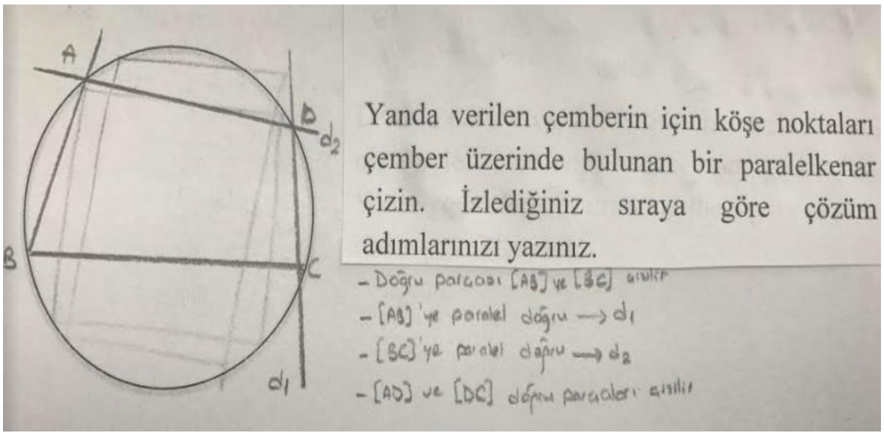


Figure 3: The drawings and explanations made by the pre-service teacher 15 on worksheet

In the solution given by the pre-service teacher 15 (figure 3), the angles seem to draw a figure on the circle. However, this figure looks like a trapezoid, rather than a parallelogram (rectangle or square). When examined the parts of this drawing, which is constructed by erasing repeatedly, a parallelogram was drawn which is parallel to two pairs of opposite sides only by using the parallelism feature. Taking consideration into these two different data, this teacher was evaluated under the category "Drawing a trapezoid following the steps of parallelogram". During the clinical interviews, questions were asked about this situation.

The data were analyzed via "content analysis method", which is a qualitative data analysis method. Basically, content analysis involves coding, categorization of codes, classification, and tabulation of the similar data within the framework of certain concepts and themes (Hancock, 2002; Yildirim & Simsek, 2011). The analysis of the open-ended questions was carried out examining the pre-service teachers' answer sheets. At the end of these analyses, the initial themes regarding the pre-service teachers' figural concept images and definitions regarding the internal relationships between quadrilaterals were formed. For the analysis of the data obtained through clinical interviews, transcriptions of the records and student notes (if there was any) were used. The categories obtained from the analysis of the responses given to the open-ended questions both shaped the interviews and constituted the basis of the data collected from the interviews: Within the scope of the data obtained from the analysis of the clinical interviews, modifications were made in the initial categories according to the information concerning the reasoning processes, and new categories were added whereas certain categories were excluded.

Using different data sources to test the consistency of similar results is a criterion for validity in qualitative studies (Patton, 1990). Data triangulation was made through document analysis of the written responses and data collection carried out via clinical interviews. The data were tried to be collected in similar processes via similar approaches in the study. Particular attention was paid to consistency in data analysis. To ensure reliability, the data collected from the students' answer sheets and interviews and transcribed were ordered and organized. These qualitative data were analyzed and encoded by two researchers independently. Based on these codes, themes and sub-themes were formed. Afterwards, the frequency and percentage values of the themes and sub-themes were calculated and tabulated.

Findings / Results

Findings Regarding the Pre-Service Teachers' Parallelogram Perceptions

The responses given to the open-ended question in relation to drawing a parallelogram were analyzed. Before scrutinizing the responses, the pre-service teachers' prior knowledge regarding parallelograms was tried to be revealed. The parallelogram definitions made by the pre-service students during the interviews were categorized into four sections as given in Table 1.

Table 1: The pre-service teachers' parallelogram definitions

Formal definition	Correct definition	Definition with extra properties	Definition based on prototype figure
Parallelogram is a quadrilateral whose opposite sides are parallel and equal to each other (PST 12, 15, 16, 24, 26).	…whose opposite sides are parallel (PST 3, 6, 7, 11, 14, 19).	A figure whose opposite sides are parallel and equal with successive two angles totaling to 180^0 (PST 9, 17, 18, 20, 22). A figure whose opposite sides are parallel with two successive angles totaling to 180^0 (PST 1). A figure whose opposite sides are parallel with diagonals centering one another (PST, 13).	A figure whose opposite sides are parallel and equal to each other with any angle different from 90^0 (PST 2, 4, 8, 10, 23, 25). A figure whose opposite sides are parallel and equal to one another with different lengths of diagonals (PST 5, 21). A quadrilateral whose opposite sides are oblique, parallel, and equal to each other (PST 27).

The definitions made by the pre-service teachers show that all the pre-service teachers used the expression "opposite sides being parallel". However, some of the pre-service teachers provided an formal definition while using this expression (12, 15, 16, 24, 26), whereas some definitions were categorized under another separate category titled correct definition (3, 6, 7, 11, 14, 19) as their definitions were not like the formal definitions yet they were adequately correct (See Table 2). Some of the pre-service teachers PST 1, 9, 13, 17, 18, 20, 22) made a definition adding extra properties though it included the definition of a parallelogram in it. As seen in the Table 1, some of the pre-service teachers added certain properties that are not included in the actual definition such as "total of the successive two angles corresponds to 180^0" and "diagonals center one another". The remarkable point here is that certain pre-service teachers (PST 2, 4, 5, 8, 10, 21, 23, 25, 27) diverged from the formal definition by adding various conditions taking into account prototype figure (e.g. different diagonal lengths, all the angles different from the right angle, oblique). It was seen that those pre-service teachers made incorrect definitions taking into account the prototype figure. While defining a parallelogram, after stating "a figure whose opposite sides are parallel and have the same length", they added some other expressions based on the prototype parallelogram figure that are not included in the formal definition. One of the pre-service teachers made the following statement in regard to this situation during the interviews:

> PST 8: … Indeed, parallelogram refers to a figure whose opposite sides are parallel and equal. It meets the criteria of rectangles… That means it cannot be rectangle. Then it is wrong.
>
> Researcher: But you have just now said that rectangles meet the criteria of a parallelogram. Why have you given up this idea?
>
> PST 8: Now I am convinced that it is wrong because there is right angle. This is what a parallelogram is (draw a prototype parallelogram). The angles should be different from 90^0.

Based on this dialogue between the researcher and the pre-service teacher, it is seen that the pre-service teacher doubted about the definition of a parallelogram. At first, he gave a formal definition for parallelogram. However, the figure he identified as prototype influenced his perception of parallelogram. This pre-service teacher also drew the prototype figure on a piece of paper to indicate that it was not consistent with his stance. Hence, the pre-service teachers made definitions under three different categories based on the prototype figure as shown in Table 1. As for the incorrect definitions, the pre-service teachers made statements including deficiencies about the properties of a parallelogram as well as incorrect statements such as “with different lengths of diagonals” and “angles different from 90^0”. However, these incorrect definitions have been dealt with under this category as they are based on prototypes. Taking into account the results, it is seen that the definitions of the pre-service teachers regarding geometrical figures were considerably influenced by the images in their personal figural concepts. Almost a majority of the pre-service teachers did not provide any formal definition.

Findings Regarding the Pre-Service Teachers’ Understanding of the Internal Relationships between Quadrilaterals

Another purpose of the study is to reveal the pre-service teachers’ understanding of the relationships between quadrilaterals and the reasons underlying this. To this end, initially the responses given by the pre-service teachers to the open-ended question were analyzed. They struggled draw parallelogram figure repeatedly erasing own figure and, wrote sometimes compatible, sometimes incompatible explanations with their figures. Table 2 shows the analysis of the pre-service teachers’ responses and the categorization of the common responses. It also shows which pre-service teacher gave the relevant response for each category.

Table 2: Pre-analysis of student responses

It cannot be drawn	Drawing a rectangle			Drawing a rectangle following the steps of parallelogram	Drawing a square following the steps of parallelogram	Drawing a parallelogram by leaving one of the vertices inside the circle	Drawing a trapezoid following the steps of parallelogram
	Drawing emphasizing a rectangle	It is not said that it is rectangle. Drawing from vertical lines	It is not said that it is rectangle. Drawing from diagonals				
PST 5, 23	PST 24	PST 11, 13, 19	PST 7, 9, 22	PST 1, 8, 25, 27	PST 14, 16, 20, 21	PST 2, 4, 6, 10, 17, 18, 26	PST 3, 12, 15

In addition to the document analysis, the pre-service teachers were asked to interpret their responses in clinical interviews so as to prevent data loss and incorrect data interpretations. At the end of these interviews, it was seen that certain changes had occurred in the document analysis conducted by the researchers prior to the interviews. The categories that appeared as a result of the researchers’ evaluations changed with the pre-service teachers’ explanations. Table 3 summarizes the analysis of the implicit responses embedded in the pre-service teachers’ written responses.

Table 3: Post-analysis of student responses

No drawing can be made	Drawing a rectangle		Thinking that one is drawing a parallelogram despite drawing a different figure			Leaving one of the vertices inside the circle		Realizing that the drawn figure is incorrect
	Those drawing a rectangle consciously	Those drawing a Rectangle unconsciously	Those drawing a trapezoid	Those drawing a figure that is similar to a square	Those drawing a figure that is similar to a rectangle	Those drawing because four vertices cannot be put on the circle	Those who were not aware of this criterion in the question	
PST 5, 15, 21, 23	PST 9, 11, 13, 19, 22, 24	PST 7	PST 3, 12, 25	PST 14, 16, 20	PST 1, 27	PST 4, 17, 18, 26	PST 2, 6, 10	PST 8

A new categorization was needed because the categories that appeared after the verbal explanations of certain pre-service teachers differed from the researchers' categories prepared based on the written responses. There was also a need to add new categories for certain pre-service teachers.

Certain changes took place in the researchers' initial evaluations. For example, while only 2 students expressed the impossibility of drawing the figure in the written answers, 4 students stated this impossibility during the interviews. 2 pre-service teachers stated that they had given responses that were not intended by them indeed. These pre-service teachers explained their written responses in the interviews. They tried to draw the figure in their written responses; however, they asserted that it was not possible to draw the figure during the interviews. One of the pre-service teachers explained the reason why he changed his response:

> PST 21: If you asked me, I would delete all the things written here. I even wrote these at the last moment. Let me be clear. I thought it was not possible to draw a parallelogram here. … I questioned whether it was really the case. Then, I thought, OK I should not take the risk. If the instructor asks this, then it needs a response. That's why I wrote these. To my way of thinking, I would delete these all. Then I said, anyways and wrote.
>
> Researcher: Do you actually think it is not possible to draw?
>
> PST 21: I think it is not possible to draw. This is because I searched it on the net. There is no such thing.

This dialogue between the researcher and the pre-service teacher indicated that he drew a figure just because "if the instructor asks this, then it needs a response". It is possible to observe the influence of the implicit principles of student-teacher relationship even in an environment where concerns regarding the academic scores are eliminated.

Other categories show that the pre-service teachers drawing rectangles drew the figures intentionally while one of the pre-service teachers realized that the figure he drew was a rectangle during the interviews. Another case is shown in Figure 4 with one of the vertices inside the circle. Four pre-service teachers drew this figure since they could not draw a figure with four points on the circle. Three pre-service teachers stated that they were unaware of this condition. These three pre-service teachers drew the figure again during the interviews taking the condition into account.

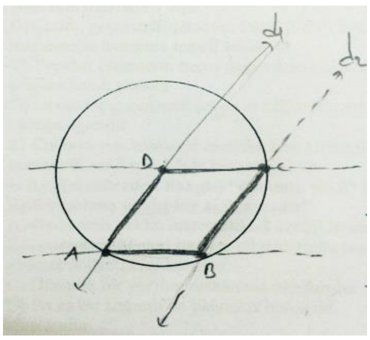


Figure 4: A parallelogram drawing with one vertice inside the circle

In Table 1, 2 and 3, it was given the concept definition in the personal figural concepts related to parallelogram and the answers given to the open-ended question. Accordingly, a new table was needed to show how well the students' definitions and their drawings are compatible. Which drawing was provided, as an answer to the question, by the pre-service teachers who gave the formal parallelogram definition? Did the participants, who answered correctly to the drawing question, have a formal definition? etc. In Table 4, it was tried to find answers to these questions.

Table 4: Comparison of students' definitions and drawings given as answer to the question

Parallelogram definition	Written responses	PST
Formal definition	Drawing a rectangle	24
	One of the vertices inside the circle	26
	No drawing can be made	15
	Parallelogram is drawn	12, 16
Correct definition	Drawing a rectangle	7, 11, 19
	One of the vertices inside the circle - not aware of this	6
	Parallelogram is drawn	9, 13, 22
Definition with extra properties	Drawing a rectangle	1, 20
	One of the vertices inside the circle	17, 18
	Parallelogram is drawn	3, 14
Definition based on prototype figure	One of the vertices inside the circle	2, 4, 10
	Parallelogram is drawn	25, 27
	No drawing can be made	5, 21, 23
	Realizing that the drawn figure is incorrect	8

In Table 4, where the parallelogram definitions and the written answers of the pre-service teachers are compared, some remarkable cases are noticed. Only four of the 12 pre-service teachers (PST 7, 11, 19, 24) who gave formal and correct definition showed that they would draw a rectangle as a correct answer in their written answers or interviews. In addition, some teachers (PST 9, 12, 13, 16, 22) claimed that they could draw the parallelogram shape, which is accepted as a prototype form, although they have made many testing/attempts with questions in the interviews. None of the preservice teachers, who define regarding the prototype shape, can give a satisfactory answer to the question. Three of these pre-service teachers (PST 5, 21, 23) insisted on their prototype personal concept image and claimed that this kind of shape cannot be drawn. When we look at the question in Figure 2, the teachers who are the concept definitions in the personal figural concepts based on prototype and have the "damaged" concept images, have never reached the right conclusion.

After analyzing the responses given by the pre-service teachers to the open-ended question, the reasons underlying their responses and the how they perceived and interpreted the relationships between quadrilaterals were dealt with over those responses. In the tables 1, 2, 3 and 4, using the definitions of the parallelograms and the answers for the drawing question given by the teachers, the concept definitions, concept images and the relation between them were shown. These lead us to focus on how these teachers interpret internal relations between the quadrilaterals when these are considered. For example; how do the pre-service teachers who gave a formal definition of parallelogram but answered the open-ended question wrongly perceive the inclusion among the quadrilaterals? Is it partitional, hierarchical or something else? Also, what are the factors that influence perceptions of these inclusions? In this sense, Table 5 shows the teachers' drawings and their understanding of the relationships between quadrilaterals over the analysis of their interviews.

Table 5: The pre-service teachers' understanding of the relationships between quadrilaterals

Theme	Code		PST	Frequency
Hierarchical Relationships	Reasoning based on definitions/knowledge	Individual reasoning	6, 7, 9, 11, 13 19, 22, 24	8 (30%)
		Directive reasoning	12, 16	2 (7%)
Partial Relationships	Having incorrect properties	The angles of a parallelogram are different from 90^0	8, 10	2 (7%)
		Diagonal lengths of parallelogram are not equal	5	1 (4%)
	Prototype figure mistake		2, 3, 14, 15, 27	5 (18.5%)
	Past experiences (reasoning based on incorrect knowledge)		17	1 (4%)
Special Case Hierarchy	Hierarchical relationship prevented by special case expression	Parallelogram having no axis of symmetry	23	1 (4%)
		Being defined as a special case (Non-internalized knowledge)	4, 18, 20, 25, 26	5 (18.5%)
		Prototype figure	1, 21	2 (7%)

At the end of the analysis, three different themes emerged in regard to the 27 pre-service teachers' understanding of the relationship between quadrilaterals. 37% of the pre-service teachers internalized hierarchical relationship; 33% internalized partial relationship; and 30% partially internalized both the partial relationship and the hierarchical relationship. Eight pre-service teachers drew a "rectangle fulfilling the condition" when the question was asked. Two pre-service teachers (PST 12 and 16) realized the correct response during the interviews as a result of the questions addressed by the researchers and expressed the situation. Thus, those pre-service teachers had hierarchical classification, were categorized under two sub-categories: "individual reasoning" and "directive reasoning". What is meant by directive reasoning is that the pre-service teacher did not consider that the answer could be a rectangle. Therefore, he did not compare the two different geometrical figures (i.e. rectangle and parallelogram) in terms of properties they had. When those pre-service teachers were asked to compare the properties for both quadrilaterals, they noted that every rectangle can also be considered as a parallelogram by their very definitions and expressed that the answer should be a rectangle. With these kinds of directives, the pre-service teachers found the opportunity to reason over themselves and realized the hierarchical relationship. The drawings shown in Figure 5 were drawn by PST 12 during the clinical interviews. He expressed hierarchical relationship based on his reasoning relying on those drawings and definitions.

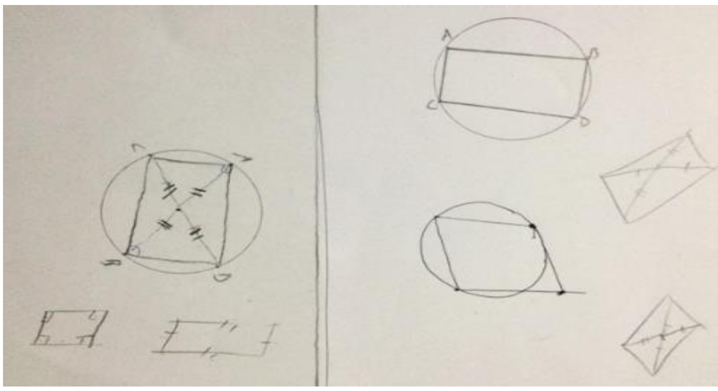


Figure 5: The drawings made by the pre-service teacher 12 during the interviews

It was seen that 9 pre-service teachers internalized partial relationship while classifying the quadrilaterals. Three different factors were influential on their internalization of such relationship. The first factor is "having incorrect properties". It is believed that this category was formed due to the influence of the parallelogram definitions given in Table 1. There were 2 pre-service teachers expressing that the angles of a parallelogram are different from 90^0. Another pre-service teacher stated that the lengths of diagonals are different from one another. It can be said that due to these incorrect properties in their personal figural concepts, those pre-service teachers could not provide formal definitions. Those incorrect definitions led them to consider a parallelogram as a quadrilateral completely different from a rectangle (or a square). Another sub-category under this theme is internalization of partial relationships as a result of "prototype figure mistake". Five pre-service teachers deduced over the prototype figure that a parallelogram needs to be oblique. The dialogue exemplifying this situation is as follows:

> PST 14: My figure may look like a square. But I thought if we drag it, it will be a parallelogram.
>
> Researcher 1: Does it look like a square or is it a square?
>
> PST 14: It is not a square. It looks like a square. This is because I tried to draw it like that (draw a prototype parallelogram). But it did not work. It looked like a square any way.
>
> Researcher 2: In other words, this is a parallelogram, but it looks like a square due to your drawing?
>
> PST 14: Yes. Its angles will not be similar perpendicularly. It only needs to keep its parallelism here. Their lengths need to be close to each other. It is close to square but not equal.
>
> Researcher 2: How close?
>
> PST 14: Very close but not equal because it is a parallelogram.
>
> Researcher 1: What would happen if it was equal?
>
> PST 14: If it was equal, it would be a square because it would not be oblique; it would be perpendicular. I think it cannot be a parallelogram.

The pre-service teacher based his explanations on the idea that the figure did not comply with the prototype figure in his figural concept. Hence, he tried to draw an "oblique" drawing. The first solution he gave is shown in Figure 6a. The figure was drawn to put vertices on the circle. Though his drawing looked like a square, the pre-service teacher stated that it was not a square. It was seen that he did not consider drawing squares in any way based on the steps followed for the drawing in his written answer. In those steps, any three points were put on the circle to draw line segments, and lines parallel to those were drawn. Figure 6b showed another trial showing this pre-service teacher randomly taking any two points on the circle closer to each other. However, he gave up that drawing as well stating that it became a trapezoid. On the other hand, five pre-service teachers did reasoning taking into account the prototype figure. Actually, in their figural concepts, they internalized the prototype figure of parallelogram. As that figure did not comply with a square (or a rectangle), they established partial relationships between quadrilaterals.

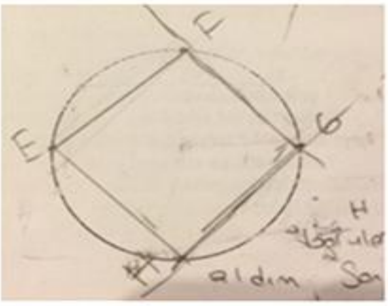


Figure 6a: The drawing made by PST 14

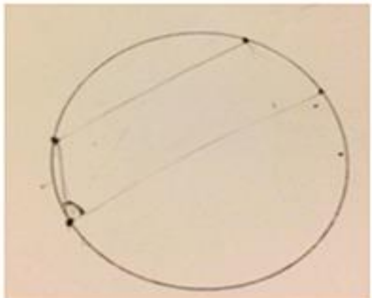


Figure 6b: The drawing made by PST 14

The last sub-category under this theme with partial relationships is "past experiences (reasoning based on incorrect knowledge)". This category includes a pre-service teacher who still keeps the partial relationships that he internalized in his previous learning life. A part of the dialogue between the pre-service teacher and the researcher exemplifying this sub-category is as follows:

> PST 17: Maybe we learnt it this way in the early years. Maybe it settled in my mind that way, and I find it difficult because of that. I do not know. This is because back then, they used to say that parallelogram is this. It was never mentioned that a rectangle is a parallelogram. Maybe that is why I have a dilemma right now.

As expressed in the quotation above, the pre-service teacher internalized partial classification in his previous learning life. Although he experienced the establishment of hierarchical relationships between quadrilaterals in his later school life, he could not overcome this situation that he created in his figural concept. Internalizing partial classification due to previous learning experience was mentioned by many pre-service teachers during the interviews.

The third theme summarized in Table 5 that was constituted at the end of content analysis is "special case hierarchy". Researchers have decided to name this theme as special case hierarchy which refers to being in between partial and hierarchical classifications. Nearly 30% of the pre-service teachers who could not internalize the relationship between quadrilaterals due to various factors experienced this situation, and their answers were classified under this theme. "Parallelogram having no axis of symmetry" and "prototype figure" is factors that prevent the three pre-service teachers from internalizing the hierarchical relationship. All three pre-service teachers reasoned in relation to the statement "a rectangle is also a parallelogram" based on the definition of parallelogram. However, the prototype figure given in Table 5 for those pre-service teachers and parallelogram "having no axis of symmetry" prevented them from responding the parallelogram drawing question with a rectangle. They stated that they were not sure and could not give a clear response. All the 5 pre-service teachers in the category "defining rectangle as a special case", which is included in this theme, could not give a clear response similarly. They displayed a rather different approach than the other participants of the study. A dialogue between one of the pre-service teachers in this category and the researchers is as follows:

> PST 18: If vertices were on the circle, it would be a rectangle or a square. These are also parallelograms. But they constitute a special group of parallelograms. That is, the figure that I drew should have been a parallelogram as well. No matter what are the lengths or measures, there should be a consistence with the whole parallelogram. Squares and rectangles are special. I tried to draw an ordinary parallelogram.
> …
> Researcher: According to you, is not it possible to draw a parallelogram in response to this?
>
> PST 18: No. It is not possible to draw a parallelogram that is not a special case. I drew this not to leave it blank. I worried. I did not want to say it is impossible in a clear-cut way. I thought the instructor should know something for asking this to us.
>
> Researcher: Do you think squares and rectangles are parallelograms?
>
> PST 18: Yes, I agree. They are special parallelograms. They comply with the criteria of parallelogram. However, they cannot be answers to draw a parallelogram as they are special.

As we seen from the quotation above, these pre-service teachers coded rectangles and parallelograms as "rectangle is a special kind of parallelogram" in their own figural concepts. Hence, they claimed that a special concept cannot be given as a response to a parallelogram drawing question. When they compared the definitions of parallelograms and rectangles, they asserted that rectangles comply with parallelogram criteria. However, coding them as "special case parallelograms" influenced their reasoning process and led to incomprehensibility.

Discussion and Conclusion

This study sought to reveal pre-service mathematics teachers' definitions and understanding in relation to internal relationships between quadrilaterals in terms of their personal and formal figural concepts in the case of parallelograms. It was seen that the definitions made by the pre-service primary school mathematics teachers regarding quadrilaterals are generally within the framework of their own perceptions. Personal definitions were more prominent in the findings of this study rather than formal definitions. The pre-service mathematics teachers' concept images in their personal figural concepts manifest themselves in the definitions. The study results indicate that these images involve dominant prototype figures. There are similar studies in the literature reporting this finding (Erdogan & Dur, 2014; Fujita & Jones, 2007; Fujita, 2012; Okazaki & Fujita, 2007). Some of the pre-service teachers defined a parallelogram as "a quadrilateral whose opposite sides are parallel to each other" (definition) while a great majority of them added certain incorrect conceptual properties as a possible result of the unchangeable prototype figure of a parallelogram (Erez & Yerushalmy, 2006; Fujita, 2012). It was seen in this study that the pre-service teachers defined the common prototype figure of a parallelogram with expressions like "a parallelogram does not have right angles", "a parallelogram is an oblique quadrilateral", and "the lengths of diagonals are not equal in a parallelogram". One of the reasons underlying this obstacle is prototype figure (the images in learners' personal figural concept) – the prototype figure of parallelogram in the present study. Prototype figures emerge and are formed probably when learners first encounter the object of parallelogram (Fujita, 2012). As stated in the study conducted by Linchevski, Vinner and Karsenty (1992), these teachers could not eliminate this prototype figure which they internalized in their previous learning lives.

It was also seen that in addition to prototype figures, certain situations that appear while defining a parallelogram are also influential on the interpretation of the relationships between quadrilaterals. The expression "having no axis of symmetry" is given as a property of a parallelogram in school mathematics. While giving geometrical figures with an axis of symmetry, the fact that a parallelogram does not have an axis of symmetry is particularly emphasized. However, such an instruction is based on partial classification. Therefore, hierarchical properties between a rectangle and a square are ignored in terms of symmetry. Pre-service teachers, who conceived that the parallelogram has not a symmetric axis, were confronted with another situation: "While square and rectangle have a symmetric axis, parallelogram does not have a symmetric axis. For this reason, they cannot be a parallelogram." They mentioned the existence or nonexistence of axis of symmetry, which might have made it more difficult for the pre-service teachers to understand hierarchical relationships. This finding, which is not mentioned in the literature shows that the previous instruction influences on students' understanding negatively and, is considered to involve a factor preventing the relationships between quadrilaterals.

Our research dwelt on the difficulties stemming from pre-service teachers’ comprehension regarding the hierarchical relationship between quadrilaterals. It showed that the participants perceive the relationships between quadrilaterals in three different ways: These are “hierarchical classification”, “partial classification”, and “special case hierarchy”, which refers to a category in between partial and hierarchical classifications. Though the participants of the present study were pre-service mathematics teachers attending a university and they had received courses such as geometry and dynamic geometry instruction, it was seen that hierarchical classification was not internalized by them widely. The mistaken or unnecessary properties added by the learners to the definitions are considered as one of the main cause of difficulties encountered (Fujita, 2012), while understanding the relationships between quadrilaterals. Therefore, they supported the idea that as a parallelogram cannot have right angles, a square cannot be a parallelogram. It is clear that they had a tendency to interpret the relationships partially. Some of the participants managed to establish hierarchical relationships between squares, rectangles, and parallelograms: they defined square and rectangle as a special case of parallelogram and made hierarchical classifications. However, as a different point from literature, it was seen in the present study that for some of the pre-service teachers “a special parallelogram” is a factor that restricts them. As stated Fujita and Jones (2006) and, Okazaki and Fujita (2007), there is a gap between learner’s personal figural concepts and formal figural concepts that causes the difficulties in classification of the quadrilaterals. However, as different from the literature review, it was seen in the present study that one of the reasons for this gap is the case expressed as “special case hierarchy”, which refers to definition of the relationships between quadrilaterals as a special case. In order to teach hierarchical relationships in teaching geometry, our explanations e.g. "square is a special form of rectangle” has caused to create a different perception in the pre-service teachers’ minds. Evaluating shapes as special has led to a tendency to find a "general" result, due to the so-called special rather than establishing a relationship between quadrilaterals. This last finding can be regarded as a strong result that emerged as a result of the teaching process in school mathematics. It is a kind of obstacle that prevents one from seeing the hierarchical relationships between parallelogram and other quadrilaterals. Perhaps complicating matters even further is the issue of “naming”. How can one shape have two different names (Levenson, Tirosh & Tsamir, 2011)? Learners can suppose that a given object have only one name. This supposition may cause difficulties in accepting the hierarchical inclusion of geometric figures (Kaur, 2015). Similarly, Hershkowitz (1989) showed that learners imposed properties either to accept or reject the categorization of a given geometric figure into a named class of shapes.

Some different situations were also encountered in the present study, though they were not part of the focus of it. As the pre-service teachers made random freehand construction drawings without using drawing rules in geometry, they were not sure about the accuracy of the drawn figures or what kind of a figure would come up in the end. Hence, it was deduced that they did not engage in an adequate reasoning process regarding the properties of the figure to be formed. In addition, some of the pre-service teachers tried to make a drawing on the paper despite their belief in that there was no response to the question, which was observed during the interviews. The pre-service teachers adopting the principle that if an instructor is asking something, then there should surely be an answer had the tendency to respond due to such justification or principle. The situation mentioned here is expressed by Brousseau (1988) as “didactic contract”. It refers to a group of implicit behaviors most of which are expected by teachers from students and by students from teachers. This result, which has not been expressed in studies on defining and classifying quadrilaterals, has emerged through both written answers and interviews. This led to noticing the didactic contract and observing the actual opinions of the pre-service teachers. As a consequence of the fact that the effect of the didactical contract was overcome by interviews, some teachers changed their written answers as mentioned in the findings. Thus, this situation, which we did not foresee at the beginning, has been interpreted as a remarkable result as well as enhancing the validity of the study.

As revealed by this study and suggested by Pusey (2003), geometry education in early years constitutes the basis of either a success or a failure in the upcoming years. Though the participants of the study were third year university students, they were still under the influence of figural concepts emerging through their learning experiences from earlier school lives. This result is in agreement with the result of Erdogan and Dur (2014)’s research. Therefore, it is important to detect these difficulties in early phases and contribute to enriching mathematical concepts by reviewing the school mathematics during teacher training programs in line with the developing pedagogical concept knowledge (Akkoc, 2008). Considering school mathematics, other types of representations of geometrical objects should be used in addition to prototype figures while instructing quadrilaterals to ensure variety. An environment should be created in which learners can reach necessary and adequate definitions (De Villiers, 1998) rather than memorizing the definitions and properties and compare the quadrilaterals as well as their properties. Hence, the learners will find the chance to access hierarchical relationships between quadrilaterals when they face various difficulties.

This study is limited to a specific number of pre-service mathematics teachers and only pre-service secondary school teachers. It may not produce definitive and generalizable results in accordance with the nature of qualitative studies. However, for quadrilaterals and their internal relationships, it provides results that can be better recognized by the pre-service teachers’ understandings, and reveals examples, explanations and experiences. We think that our research in this direction may bring both contributions to the field and practice. In addition, this study draws attention to the existence of different perceptions related to quadrilaterals and the relationship between quadrilaterals for various levels of learners or teachers. It is believed that the pre-service mathematics teachers' perceptions and the mistakes and difficulties on which these perceptions are based are important for directing the geometry education for each level of education (i.e. from primary school to university).

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