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Error Patterns in Ordering Fractions Among At-Risk Fourth-Grade Students

Amelia S. Malone, PhD[1] and Lynn S. Fuchs, PhD[1]

Abstract

The three purposes of this study were to (a) describe fraction ordering errors among at-risk fourth grade students, (b) assess the effect of part-whole understanding and accuracy of fraction magnitude estimation on the probability of committing errors, and (c) examine the effect of students' ability to explain comparing problems on the probability of committing errors. Students (*N* = 227) completed a nine-item ordering test. A high proportion (81%) of problems were completed incorrectly. Most (65%) errors were due to students misapplying whole number logic to fractions. Fraction-magnitude estimation skill, but not part-whole understanding, significantly predicted the probability of committing this type of error. Implications for practice are discussed.

Keywords

elementary age, at risk/prevention, mathematics

Many students persistently misunderstand fractions and have difficulty assessing fraction magnitude (e.g., Brown & Quinn, 2007; U.S. Department of Education, 2008). For example, on the 2013 National Assessment of Educational Progress (NAEP) assessment, 40% of fourth-grade students could not determine that thirds were bigger than fourths, fifths, and sixths. Similarly, 65% of eighth-grade students could not explain that 1/2 + 3/8 + 3/8 was greater than one. One reason students struggle with assessing fraction magnitude is that fractions have different properties than whole numbers. Whereas whole numbers are successive (e.g., the distance between 1 and 2 is the same as the distance between 543 and 544) and discrete (e.g., there is only one whole number between 3 and 5), fractions are not successive (e.g., the distance between 1/4 and 1/5 is greater than the distance between 1/6 and 1/7) and are not discrete (e.g., there is an infinite number of fractions between 1/4 and 1/5; Vosniadou, Vamvakoussi, & Skopeliti, 2008). In addition, fractions have different calculation properties than whole numbers (e.g., multiplying two positive whole numbers yields a greater value whereas multiplying two positive proper fractions yields a smaller value).

One common error students make is misapplying whole number logic to fractions; that is, they incorrectly apply whole number counting properties to fraction concepts (e.g., Ni & Zhou, 2005). This is referred to as *whole number bias* (e.g., incorrectly assuming that 1/12 is larger than 1/2 because the whole number 12 is larger than 2; Ni & Zhou, 2005). Until the upper elementary grades, instruction primarily focuses on the one-to-one counting properties of whole numbers. The introduction of fractions poses newfound difficulties because rational numbers do not have the same properties as whole numbers. That is, when students encounter fractions, they fall back on whole number knowledge (which is the focus of the curriculum up until fourth grade) to make sense of magnitude (e.g., Geary, 2006; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010; Vosniadou et al., 2008). Because fractions have different properties than whole numbers, students may have difficulty expanding their understanding of number representation when fractions are introduced in the upper elementary grades.

Including rational numbers and whole numbers into a single numerical framework is "a process of progressively broadening the class of numbers that are understood to possess magnitudes and of learning the functions that connect that increasingly broad and varied set of numbers to their magnitudes" (Siegler, Thompson, & Schneider, 2011, p. 275). Students likely experience difficulty with fractions because they have difficulty in simultaneously considering the numerator and denominator when assessing magnitude. In this way, Siegler et al. (2011) hold that whole number knowledge does not necessarily interfere with fraction learning. But because magnitude understanding unites both

[1]Department of Special Education, Vanderbilt University, Nashville, TN, USA

Corresponding Author:
Amelia S. Malone, PhD, Department of Special Education, Vanderbilt University, 110 Magnolia Cir., Box 228, Nashville, TN 37203-5721, USA.
Email: amelia.malone@vanderbilt.edu

whole numbers and fractions, students should learn that fractions have magnitude and can be ordered, compared, and placed on the number line. Without explicitly teaching how to assess magnitude, students will likely struggle to consolidate properties of whole numbers and fractions.

Typical instruction in the United States fails to emphasize how to conceptually assess fraction magnitude. As a result, students rely on whole number properties to determine fraction value and thus struggle to expand their concept of number to include fractions, which helps explain why misapplying whole number logic to fraction magnitude comparisons is so common. In the following sections of this introduction, we describe common error patterns when students assess fraction magnitude, and describe how the present study extends prior work.

Common Error Patterns with Assessing Fraction Magnitude

It is well documented that individuals have substantial difficulty assessing fraction magnitude (e.g., Bonato, Fabbri, Umilt, & Zorzi, 2007; Schneider & Siegler, 2010; Siegler et al., 2011). We located three studies that investigated the accuracy with which individuals compare fractions and the comparing error patterns on which problem solvers rely. Stafylidou and Vosniadou (2004) investigated explanatory frameworks for comparing and ordering fractions among 200 average-performing middle and high school students. One common misconception was the belief that a fraction is composed of two independent whole numbers (e.g., "the value of a fraction increases when the numbers that comprise it increase"; p. 507). That is, as the whole number values in the numerator or denominator increases, the value of the fraction also increases. Accordingly, many students ordered or compared fractions based on the whole number value in the numerator or the denominator (e.g., 2/6 > 1/2, which is incorrect).

Another common misconception Stafylidou and Vosniadou (2004) found among students in their sample was the belief that the value of a fraction increases when the whole numbers that comprise it decrease. This somewhat more advanced misconception reflects some rudimentary understanding that fractions with smaller denominators have bigger parts, but lacks understanding that the numerator and denominator operate synergistically to determine fraction value. That is, the numerator and denominator are not independent whole numbers, but rather two numbers that form a single value, which can be represented on a number line. Authors found that these misconceptions were less common among the eighth-grade and high school students in their study than they were among the fifth-, sixth-, and seventh-grade students.

Similar misconceptions about fraction values were found by DeWolf and Vosniadou (2011) among 28 adult undergraduate students enrolled at a prestigious university. The participants were presented with 40 fraction pairs on the computer and asked to determine which fraction was bigger. Of the fraction pairs, 20 were consistent with whole number ordering. That is, the larger of the two fractions was also the one with larger whole numbers in the numerator and denominator (e.g., 5/6 > 1/2). The other 20 fraction pairs were inconsistent with whole number ordering. That is, the larger of the two fractions had smaller whole numbers in the numerator and the denominator (e.g., 4/6 > 5/10). Accuracy and speed of response were superior for items that were consistent with whole number ordering than for items that were inconsistent with whole number ordering, which suggests that estimating fraction magnitude is difficult even for advanced students. This speaks to the difficulty of thinking about the numerator and denominator concurrently to assess a fraction's value when the comparison is inconsistent with whole number knowledge.

Like DeWolf and Vosniadou (2011), Meert, Grégoire, and Noël (2010) found that students were more accurate and quicker to respond when the fraction pairs were consistent with whole number ordering than inconsistent with whole number ordering. Average-achieving fifth- (n = 24) and seventh-grade (n = 44) students identified the larger value within 64 pairs of fractions. Of the pairs, 32 had the same numerator (i.e., inconsistent with whole number ordering) and 32 had the same denominator (i.e., consistent with whole number ordering). Seventh-grade students were more accurate and quicker to respond for both problem types than were fifth-grade students. However, the same numerator problems (i.e., inconsistent with whole number ordering) were difficult for both fifth- and seventh-grade students, as indicated by slower response times and decreased accuracy.

The Present Study

Across the age range studied in these studies, students demonstrated consistent whole number ordering errors when assessing fraction magnitude. Yet we identified no prior study that investigated error patterns among fourth-grade students, examined the error patterns of students at risk for mathematics difficulties, or used different types of fraction knowledge to predict the probability of committing these errors. Previous intervention research (e.g., Cramer, Post, & delMas, 2002; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs, Malone, et al., 2015; Fuchs et al., in press) has shown that teaching students how to assess fraction magnitude significantly improves their conceptual understanding of fractions, but none of these studies investigated how specific types of fraction knowledge predict at-risk students' error patterns. As indicated in NAEP (U.S. Department of Education, 2013) data and as revealed in the studies described above (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004), many students struggle with assessing fraction magnitude. Describing the frequency and types of errors among at-risk fourth-graders has important implications for designing curriculum to improve students' conceptual understanding of fractions.

Hecht and Vagi (2010) found that fourth- and fifth-grade students with mathematics difficulties (i.e., scoring below the 25th percentile on a standardized mathematics test) had lower than expected procedural and conceptual knowledge about fractions compared to average-achieving students. This is probably true among the present study's at-risk sample. Lower conceptual knowledge likely leads to higher frequencies of errors than the average-achieving samples just reviewed (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004). In the present study, risk was operationalized as scoring below the 35th percentile on a measure of whole number knowledge at the start of fourth grade (when fraction knowledge is insufficiently developed to screen on). Because the curriculum primarily focuses on whole number knowledge in grades K–3, students who continue to demonstrate difficulty with whole numbers in fourth grade will likely have difficulty understanding fraction concepts as they attempt to expand their concept of number to include rational numbers.

In the present study, we had three purposes. The first was to describe fraction ordering error patterns among fourth-grade students identified as at risk for developing mathematics difficulties. The sample included students from three different academic years over the period in which the school district moved toward implementation of the Common Core State Standards (CCSS; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2013). This shift in time provides an interesting opportunity for description, because the CCSS emphasize fraction magnitude (e.g., ordering fractions from least to greatest), a skill with which many students struggle with (DeWolf & Vosniadou, 2011; Meert et al., 2010; NAEP, 2013; Stafylidou & Vosniadou, 2004). Despite this move toward implementation of CCSS, we hypothesized minimal improvement in fraction magnitude judgments among this at-risk sample. This expectation is based on findings by Fuchs, Fuchs, et al. (2015), who found that, across 3 years of intervention research, the achievement gap on fractions performance widened between at-risk students and their not-at-risk classmates in the same district where the present study took place moved toward implementation of CCSS. This widening gap is likely due to the rigor of CCSS, affording greater learning outcomes for not-at-risk students, even as it posed specific difficulty for at-risk students who already struggle with foundational skills (Powell, Fuchs, & Fuchs, 2013). Specifically, low-performing students tend to struggle with magnitude estimation (e.g., Fuchs et al., 2013; Fuchs, Fuchs, et al., 2015; Mazzocco & Devlin, 2008), a core tenet of fraction learning outlined in the CCSS.

Bolstering understanding of fraction magnitudes at fourth grade is important because fraction knowledge has been found to be a unique predictor of future performance in algebra (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Brown & Quinn, 2007; Siegler et al., 2012). In this study, we refer to errors in which students order fractions from least to greatest based on the whole number values in the numerator and/or the denominator as *whole number ordering errors* (WN-ORs). We hypothesized that these errors were the most common ordering error among at-risk fourth-graders and that the next most common pattern of error involved judgments about fraction magnitude based solely on the value of the denominator (i.e., students indicating that the fraction with the smallest denominator is the largest fraction and the fraction with the largest denominator is the smallest fraction without concurrently considering the value in the numerator; e.g., $5/8 < 3/4 < 1/2$). In this study, we refer to this type of error as a *smallest denominator–biggest fraction ordering error* (SDBF-OR). This error is more advanced than WN-ORs, because it indicates rudimentary knowledge that the smaller denominator means the fraction has bigger parts. This misconception exemplifies the lack of appreciation that the numerator also factors in to the value of the fraction. We also expected to find random error. That is, errors that did not fit into a pattern. We refer to these errors as *no-category ordering errors* (NC-ORs).

The second purpose was to assess the effect of students' part-whole understanding and accuracy of fraction-magnitude estimation (i.e., placement of fractions on the number line) on the probability of committing errors. These constructs were chosen as predictors of errors because each represents a key form of fraction understanding (e.g., Hecht, Vagi, & Torgesen, 2007; Kilpatrick, Swafford, & Findell, 2001) and help explain individual differences in fraction competence (Hecht, Close, & Santiso, 2003; Hecht & Vagi, 2010; Siegler et al., 2011). Each ordering problem could only have one outcome (i.e., each problem was correct or incorrect). If incorrect, only one error type was possible. Therefore, each of the error types (i.e., WN-OR, SDBF-OR, NC-OR) was analyzed with a separate statistical model. Note that the focus of the article is on identifying pervasive *patterns of errors* among at-risk fourth-grade students. Therefore, we focus our discussion on patterns of error (i.e., WN-ORs and SDBF-ORs). We include an analysis of accuracy and NC-ORs for the readers' reference.

In this study, part-whole understanding was indexed by performance on a subset of released fraction items from the NAEP (U.S. Department of Education, 2010). Part-whole understanding emphasizes part to whole relationships. This type of fraction knowledge is often represented as shaded parts of a shape or as pieces of pizza (e.g., Charalambous & Pitta-Pantazi, 2007). The curriculum in the district where this study took place primarily focuses on part-whole understanding, even as CCSS (National Governors Association

Center for Best Practices, Council of Chief State School Officers, 2013) implementation occurred. Such understanding is foundational but not sufficient for developing an understanding of fraction magnitude (e.g., Charalambous & Pitta-Pantazi, 2007). Focusing solely on part-whole understanding does not allow students to expand their understanding of fractions to include the following: (a) fractions as division, (b) fractions as a unit of measure, (c) fractions as operators, and (d) fractions as ratios (Kieren, 1993). As C. L. Smith, Solomon, and Carey (2005) showed, students could not reliably use part-whole representations to judge the size of nonunit fractions. Therefore, focusing on part-whole understanding only partially addresses the fractions as units of measure construct.

For example, C. L. Smith et al. (2005) documented that students could accurately judge the relative size of unit fractions (a fraction with 1 in the numerator) using part-whole explanations but were unable to accurately answer other questions about fraction-magnitude properties (e.g., when you multiply two fractions, the amount gets smaller). Because part-whole understanding provides an inadequate basis for succeeding with comparisons among nonunit fractions (C. L. Smith et al., 2005), we hypothesized that part-whole understanding is not a significant predictor of committing an ordering error when students' accuracy of fraction-magnitude estimation is controlled for in the model.

Fraction-magnitude estimation was indexed by the accuracy with which students placed fractions on the 0 to 2 number line. The ability to accurately place fractions on the number line indexes students' understanding that fractions have magnitude and can be ordered and compared (e.g., Siegler et al., 2011; Wu, 2008). We expected superior accuracy on number line to significantly predict the probability of committing SDBF-ORs. In the first year of a strong focus on fraction learning (i.e., fourth grade), at-risk students likely do not have advanced understanding of fraction magnitude. Therefore, an increase in SDBF-ORs concurrent with an increase in accuracy of fraction-magnitude estimation may represent "a transitory phase in the process of fraction learning" and reflect the difficulties students have when they take on new information about fractions (Stafylidou & Vosniadou, 2004, p. 512). By contrast, we expected an inverse relation between fraction-magnitude estimation and the probability of committing a WN-OR. That is, the greater students' accuracy with placing fractions on the number line, the less likely they were to commit a WN-OR.

To extend the focus further, our third purpose was to describe how students' verbal explanations related to error patterns. Students in Fuchs, Malone, et al. (2015) completed an assessment that probed their understanding of fraction magnitude more deeply, by asking them to provide explanations for how they compared nine pairs of fractions. We expected students' ability to write sophisticated explanations to decrease the probability of committing WN-ORs and SDBF-ORs. This hypothesis is consistent with previous research that the ability to construct sound explanations is correlated with higher mathematics achievement, thus decreasing errors (e.g., Rittle-Johnson, 2006; Wong, Lawson, and Keeves, 2002). In addition, Fuchs, Malone, et al. found that a 10-min self-explaining condition, which provided explicit instruction on providing explanations on fraction comparison problems, significantly increased students' fraction comparison accuracy compared to the other intervention condition (*ES* = 0.43). In this study, Fuchs, Malone, et al. compared two intervention conditions—both conditions received the same multicomponent fraction intervention focusing on the measurement interpretation of fractions. The only difference between the two conditions was the 10-min activity: One condition received supported self-explaining and the other learned to solve multiplicative word problems.

Method

Participants were 227 at-risk fourth-grade students. Risk was defined as scoring below the 35th percentile on a measure of whole number knowledge (*Wide Range Achievement Test–4* [WRAT-4]; Wilkinson, 2008). The 227 students were from three cohorts within three large randomized-control trials, in which these students were randomly assigned to intervention or control. The present study relied exclusively on the control group (i.e., those that did not receive intervention) because intervention was designed to disrupt the developmental pattern expected for these at-risk students. Cohorts 1, 2, and 3 included 84, 72, and 71 students, respectively. The present study added the analysis of error types to previous reports based on the full sample (Fuchs et al., 2014; Fuchs et al., in press; Fuchs, Malone, et al., 2015), and results presented here do not overlap with those described in previous reports.

Screening Measure

WRAT-4 Arithmetic includes 40 problems of increasing difficulty. At fourth grade, it primarily assesses students' proficiency with whole number computation skill and includes 37 computation problems (addition, subtraction, multiplication, and division) and 3 estimation problems. Alpha for the present sample was .76. (Estimated on a larger sample drawn from the same population, alpha was .85.)

Fraction Measures

Ordering fractions. Ordering Fractions is a subtest from the *2012 Fraction Battery* (Schumacher, Namkung, Malone, & Fuchs, 2012) and requires students to order nine sets of three fractions from least to greatest. Three problems have the same numerator (and different denominators) and six have different numerators and different denominators. Alpha for the present sample was .69. (Estimated on a larger sample drawn from the same population, alpha was .84.)

Table 1. Ordering Errors Coding Form.

Student/Teacher/Cohort

Problem		Answer		Fraction understanding error						
		Correct = 1	Incorrect = 0	WN	WN-D	WN-N	SDBF	NC-WF	NC	DNA
A. *	$\frac{1}{2}$ $\frac{5}{8}$ $\frac{2}{6}$	0	1	1	2	3	4	5	6	7
B.	$\frac{1}{2}$ $\frac{1}{5}$ $\frac{1}{12}$	0	1	1	2	3	4	5	6	7
C. *	$\frac{3}{10}$ $\frac{7}{12}$ $\frac{1}{2}$	0	1	1	2	3	4	5	6	7
D. *	$\frac{3}{4}$ $\frac{1}{2}$ $\frac{2}{8}$	0	1	1	2	3	4	5	6	7
E.	$\frac{1}{8}$ $\frac{1}{3}$ $\frac{1}{2}$	0	1	1	2	3	4	5	6	7
F.	$\frac{5}{4}$ $\frac{9}{8}$ $\frac{7}{6}$	0	1	1	2	3	4	5	6	7
G.	$1\frac{4}{6}$ $1\frac{4}{12}$ $1\frac{4}{10}$	0	1	1	2	3	4	5	6	7
H. *	$\frac{7}{12}$ $\frac{10}{6}$ $\frac{3}{2}$	0	1	1	2	3	4	5	6	7
I. *	$1\frac{7}{8}$ $1\frac{2}{10}$ $1\frac{1}{2}$	0	1	1	2	3	4	5	6	7

Note. WN = whole number; WN-D = whole number (denominator specific); WN-N = whole number (numerator specific); SDBF = smallest denominator–biggest fraction; NC-WF = no category errors where students wrote the wrong fractions or the same fraction twice; NC = no category errors (i.e., could not determine an error pattern); DNA = did not attempt. If an error is gray, the particular error could not occur on the problem.
*Problems marked with an asterisk were coded as SDBF ordering errors only *if* students also got Problems B, E, and D correct (i.e., the same numerator problems). Answering these problems correct indicates that students understand that as the denominator gets smaller, the parts get bigger. However, students failed to correctly address the value in the numerator for problems that have fractions with different numerators and different denominators. All errors were entered as a 1 for present and a 0 for absent.

Coding of ordering errors. Each item on each student's test was first coded for accuracy. If the student answered a problem incorrectly, their answer was assigned an error type. See Table 1 for the coding sheet and error possibilities by problem.

There were three examples of WN-ORs depending on the problem. The first example included incorrectly ordering the fractions from least to greatest based on the whole number value in the numerator and denominator. That is, students would get the same incorrect answer whether they ordered the three fractions from least to greatest based on the whole number in the numerator or the denominator (e.g., 1/2 < 2/6 < 5/8). The second example included incorrectly ordering the fractions from least to greatest based on the whole number value in the denominator (i.e., same numerator problems). Students ordered the whole numbers in the denominator from least to greatest (e.g., 1/2 < 1/5 < 1/12). Students did not demonstrate an understanding that when a unit is divided into more parts (i.e., a bigger denominator), each of the parts gets smaller. The third example included incorrectly ordering the fractions from least to greatest based on the whole number value in the numerator (e.g., 1/2 < 2/8 < 3/4).

SDBF-ORs occurred when students incorrectly ordered the three fractions from least to greatest considering the denominator (i.e., how many equal parts the unit is divided into) but not the numerator (i.e., how many equal parts are in the fraction). For example, for 3/10, 7/12, and 1/2, students would incorrectly order them 7/12 < 3/10 < 1/2. That is, students assumed that the smaller the denominator, the bigger the fraction magnitude. A fraction with a small denominator is divided into bigger parts, but a student must also consider how many parts (i.e., the numerator) the fraction has to determine value. This error could occur on five of the nine problems (see Table 1).

To ensure that SDBF-ORs were consistent with how students understood fractions, this error was only coded as present *if* students also got all three of the same numerator problems (different denominators) correct on the test. Accuracy on the same numerator problems indicated that students understood that as the denominator gets smaller, the size of the parts gets bigger (e.g., 1/2 > 1/5 because halves are bigger than fifths). However, they could not transfer this knowledge to a set of fractions that had different numerators and different denominators and failed to account for the number of parts when ordering the fractions. If students did not get the same numerator problems correct, the coder could not assume that students had an understanding about how the denominator affects the size of the fraction.

If a student's answer did not fit into any category, his or her error was coded as NC-OR. No category errors typically included students writing the wrong fractions in the blanks, or writing the same fraction twice (therefore, an error type could not be assigned). If students did not attempt the problem, this was also coded. Two independent coders scored all of the tests and entered the data. On the first scoring attempt, coders scored the tests with 98.65% agreement. All discrepancies were discussed and resolved.

Part-whole understanding. To index part-whole understanding, we relied on eight released fourth- and eighth-grade items from the NAEP (U.S. Department of Education, 2010). Seven questions ask students to identify or write fractions using a picture, and one question assesses students' part-whole understanding with a word problem. The maximum score for the part-whole items is 9. Alpha for the present sample was .65. (Estimated on a larger sample drawn from the same population, alpha was .71.)

Fraction-magnitude estimation. *Fraction Number Line* (Hamlett, Schumacher, & Fuchs, 2011, adapted from Siegler et al., 2011) requires students to place proper fractions, improper fractions, and mixed numbers on a 0 to 2 number line. Students are presented with a number line on a computer screen and instructed to place the fraction where they estimate it goes. The following fractions are presented in random order: 12/13, 7/9, 5/6, 1/4, 2/3, 1/2, 1/19, 3/8, 7/4, 3/2, 4/3, 7/6, 15/8, 1 1/8, 1 1/5, 1 5/6, 1 2/4, 1 11/12, 5/5, and 1. Scores reflect students' percentage of absolute error: the difference between where the student placed the fraction and where it actually goes, averaged across the items and then multiplied by 100. Because it was a 0 to 2 number line, we then divided scores by 2 to obtain percentage of absolute error. Lower scores indicate greater accuracy. Therefore, students' scores were multiplied by –1 so a positive relation between number line and the outcome reflected superior performance. Alpha was .85, and test–retest reliability estimated with Cohort 2 was .80.

(Note that in the initial analysis, we also used a subset of magnitude items from NAEP as a predictor of errors. This subset of 11 items require students to order fractions, compare fractions, write equivalent fractions, and identify a fraction on the number line. However, because these items were closely aligned with the outcome measure (i.e., ordering fractions), we decided to drop it as a predictor in this report. Results paralleled those from the number line.)

Verbal explanations. Explaining Comparing Problems (Cohort 3 only), from the *2013 Fraction Battery* (Schumacher, Namkung, Malone, & Fuchs, 2013), requires students to explain why one fraction is greater than or less than another fraction. Students place the greater than or less than sign between the two fractions and then write an explanation for and draw a picture to show how to think about the comparison in values and why their answer makes sense. There are nine items: Three have the same numerator (and different denominators), three have the same denominator (and different numerators), and three require students to compare a fraction to 1/2. We chose these items because they represent a range of comparing types (including comparing to a benchmark). Scoring awards credit for four components of sound explanations. Each component is either present or absent, and each component is weighted to account for the sophistication of the explanation.

The first component indexes the accuracy of sign placement between the two fractions being compared (e.g., 4/5 > 4/8). Students earn 1 point if the sign is correct. The second and third components assess whether students demonstrate an understanding of how the numerator and denominator work together to make an amount. Because these components indicate a more advanced understanding of fractions, they are weighted more heavily. Students must demonstrate that (a) the numerator indicates the number of parts and (b) the denominator indicates the size of the parts. These two components are scored independently, and students receive 2 points if the component is present and 0 points if the component is not present. For example, when comparing 4/5 and 4/8, we awarded 4 points for the following: "Both fractions have the same number of parts, but fifths are bigger than eighths so 4/5 is greater than 4/8."

The last component assesses the tenability of a picture the student draws to support the verbal explanation. A correct picture includes drawing two units the same size (e.g., rectangles that start and end at the same place), correctly dividing the unit into equal parts (the denominator), and correctly shading in the appropriate number of parts (the numerator). This component is awarded 1 point if present. The maximum score of each of the nine items is 6 points, for a maximum total score of 54. Two independent coders scored all of the tests with 98.08% agreement. All discrepancies were discussed and resolved. Estimated on a larger sample drawn from the same population, alpha was .91.

Fraction Instruction in the Classroom

The district uses enVision Math, which includes two units on fractions (Scott Foresman-Addison-Wesley, 2011). Fraction content includes adding and subtracting, constructing equivalent fractions, word problems, and explaining concepts using words and pictures. Instruction largely relies on part-whole understanding and procedures. Based on a curriculum analysis (Malone & Loehr, 2015), fourth grade enVision Math primarily teaches comparing magnitudes using equal shares (with pictures) and comparing rules (e.g., if two fractions have the same denominator, the one with a bigger numerator is the bigger fraction). We only found one instance where the instructional manual used number lines to teach comparing fraction magnitudes.

Supplemental survey data about classroom instruction were also collected for Cohorts 2 and 3. We first determined what teachers relied on for fraction instruction (the mean percentage of instructional time reported by Cohort 2 and Cohort 3 is provided in parentheses): the textbook only (2% and 0%), a combination of the textbook and CCSS (72% and 85%), CCSS only (26% and 13%), or other (0% and 2%). We then asked teachers what percentage of instructional time they spent teaching how to compare fractions using the following (the mean percentage of instructional time reported by Cohort 2 and Cohort 3 is provided in parentheses): (a) cross multiplying (12% and 21.5%), (b) number lines (14% and 12%), (c) benchmark fractions (10% ad 13%), (d) finding common denominators (27% and 21%), (e) drawing a picture of each fraction (19% and 15%), (f) reference manipulatives (7% and 7%), (g) thinking about the meaning of the numerator and denominator (8% and 8%), and (h) other (3% and 2.5%).

Procedure

Students in Cohorts 1, 2, and 3 were screened with the WRAT-4 in the fall of fourth grade (2012, 2013, and 2014, respectively). They completed the Ordering Fractions, *Fraction Number Line*, NAEP, and Explaining Comparing Problems (Cohort 3 only) measures in the spring of fourth grade, after fraction instruction in their classroom had occurred. Students in Cohorts 1 and 2 took the tests in a whole-class setting; Cohort 3 took the tests in small groups of two to six students.

Testers were graduate research assistants employed by a local university, all of whom received training on testing during two 4-hr sessions. Research assistants practiced administering the tests and passed a fidelity check before administering tests in schools. For each test in each year of the study, two independent research assistants scored and entered all data. As previously indicated, all scoring discrepancies were discussed and resolved.

Data Analysis

Data analysis included three steps. The first was to determine how accurately students ordered fractions and how frequently they committed errors. The second step included running three sets of cross-classified random-effects models (CCREM). (Note that we include analyses of accuracy and NC-ORs, but the focus of the article is assessing individual differences in patterns of error among at-risk fourth-grade students.) The logistic regression CCREMs were run using the lme4 command in R (R Core Team, 2013). The data structure included three levels. Problem-by-problem

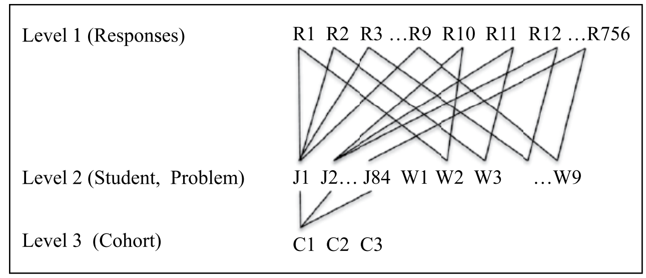

Figure 1. Cross-classified hierarchical data structure (depicted for Cohort 1).

student responses were at Level 1. Because each student answered the same nine problems, responses were cross-classified by student and problem at Level 2. Students were also nested within cohort at Level 3. See Figure 1 for a depiction of the data structure.

Students are likely to demonstrate a consistent error pattern throughout the test, and including the cross-classified term accounts for this fact. The intraclass correlation coefficient for CCREMs indicates the degree to which student variance is conditional on problem variance, and vice versa (Cho & Rabe-Hesketh, 2011). Baayen, Davidson, and Bates (2008) provide a detailed analysis of how CCREM are advantageous for considering both student-specific and item-specific effects. By using a student/item cross-classified term, we can control the fact that an individual student is more likely to answer similarly across items than another student. The authors point out that "no two brains are the same, and that different brains have different developmental histories" (p. 407). Without including a cross-classified student-by-item term, we run the risk of overinflating the variance components due to both student and item (Baayen et al., 2008). Note that all of the analyses reported are *student-level* analyses. We simply take item-level variance into account in each model. That is, including the item and student cross-classified term accounts for the fact that no two brains are the same. This type of model has been used successfully in the reading research literature, with similar data structure (e.g., Gilbert, Compton, & Kearns, 2008). We calculated the intraclass correlation coefficient (ICC) for the empty model for accuracy, using the following formula (Snijders & Bosker, 2012):

$$\rho_I = \frac{\tau_0^2}{\tau_0^2 + \frac{\pi^2}{3}}$$

The ICC for student was .58, which means that a substantial proportion of variance in responses is due to between-student differences. The ICC for problem was < .001, which means that virtually no variance in responses is due to between-problem differences. Note that the ICC for problem without accounting for between-student differences was .14, meaning a substantial proportion of between-problem differences is due to student variance.

We also conducted a comparison of two empty models for accuracy. Model 1 accounted for the cross-classified person-level random effects term. Model 2 did not account for the cross-classified term. The decrease in deviance between the two models was 36.44, which is significant ($p < .001$) on a chi-squared distribution with $df = 1$. We therefore included the cross-classified term. Because each student answered the same set of 9 problems (at the same level), this is included as a person-level crossed random effect.

The first set of CCREMs assessed the effect of cohort membership (Cohort 2 served as the referent group for this analysis) on accuracy, WN-ORs, SDBF-ORs, and NC-ORs. The second set of CCREMs assessed whether part-whole understanding (i.e., NAEP part-whole items) and accuracy of fraction-magnitude estimation (i.e., number line) predicted accuracy and the probability of committing a WN-OR, SDBF-OR, or NC-OR. Because the two constructs are related ($r = .32$, $p = .01$), we controlled for both domain-specific tasks in the models to assess the contribution that one type of fraction understanding makes above and beyond the other. To make the intercept interpretable, all of the student-level predictors were centered at their respective means. Therefore, the intercept could be interpreted as the probability of committing an error at the predictors' respective means.

The full sample included 2,043 responses. However, because the purpose of analyzing errors was to compare the probability of committing one error over another error, analysis of these errors excluded correct answers from their respective models. Of the 2,043 responses that could have WN-ORs or NC-ORs, 379 were correctly answered. Therefore, the total sample for the WN-OR and NC-OR outcomes included 1,664 responses. That is, for example, students either committed a WN-OR or some other error (i.e., SDBF-OR, NC-OR, or did not attempt the problem). Of the 1,135 responses that could have SDBF-ORs (this error could only occur on five of the nine ordering problems), 85 were correctly answered. That is, students either committed a SDBF-OR or some other error (i.e., WN-OR, NC-OR, or did not attempt the problem). Therefore, the total sample for SDBF-ORs included 1,050 responses.

The third set of CCREMs assessed whether students' ability to explain comparing problems (Cohort 3 only; $n = 71$ students) significantly predicted the probability of committing a WN-OR ($n = 513$ responses), a SDBF-OR ($n = 322$ responses), or a NC-OR ($n = 513$ responses).

To contextualize results, the final (third overall) step was to convert all significant logit coefficients to probabilities using the following logit link formula (Snijders & Bosker, 2012):

$$P = \frac{1}{1 + e^{-\operatorname{logit}\left(\pi_{i[jk]}\right)}}$$

We calculated these probabilities for the student-level predictors at the following values: one standard deviation below the mean, at one standard deviation above the mean, and two standard deviations above the mean. This allowed us to assess how the probability of correctly answering a problem or committing an error changed as a function of an increase or decrease in part-whole understanding and accuracy of fraction-magnitude estimation across the distribution of scores.

Results

Of the 2,043 responses, students correctly answered 379 of the items (i.e., 19%). The most common ordering error was WN-ORs. Of the 1,664 responses with errors, 1,078 had WN-ORs (i.e., 65%). The next most common error was SDBF-ORs. Of the 1,050 responses that could have this error (this error could only occur on five of the nine ordering problems), 190 had SDBF-ORs (i.e., 18%). Students also committed random errors: They wrote the incorrect fraction or the same fraction twice (i.e., 5% of errors) or committed a NC-OR that did not fit a pattern (i.e., 17% of errors). See Table 2 for frequencies of correct answers and errors by cohort and across the sample. See Table 3 for proportions and standard deviations of correct answers and errors for students by problem.

Effect of Cohort on Accuracy and Errors

There were no significant differences among cohorts on accuracy or the frequency of errors. See Table 4 for a summary of the CCREM results. Therefore, cohort membership (i.e., Level 3) was dropped from subsequent analyses.

Effect of Part-Whole Understanding and Accuracy of Fraction-Magnitude Estimation on Accuracy and Errors

See Table 5 for a summary of the CCREM results. Controlling for number line performance, performance on NAEP part-whole items ($M = 6.33$, $SD = 1.50$) failed to predict the probability of committing a WN-OR, $p = .079$. By contrast, performance on number line ($M = 0.52$, $SD = 0.14$), while controlling for performance on the NAEP part-whole items, significantly predicted the probability of committing a WN-OR, $p < .001$. To gain a better understanding of this significant effect, we calculated the probability of committing a WN-OR when a student performed one standard deviation below the mean (0.954), one standard deviation above the mean (0.242), and two standard deviations above the mean (0.038) on number line while holding part-whole understanding constant at the mean.

Performance on NAEP part-whole items and number line did not significantly predict the probability of committing a SDBF-OR or NC-OR. Because none of the predictors was statistically significant, we did not calculate probabilities.

Table 2. Frequency of Correct Answers, Whole Number Ordering Errors, Smallest Denominator–Biggest Fraction Ordering Errors, No Category Errors, and Did Not Attempt Errors.

Group	Correct	%	WN[a]	%	SDBF[b]	%	NC[a]	%	DNA[a]	%
Cohort 1, 2011–2012	116	15	417	65	52	13	148	23	23	4
Cohort 2, 2012–2013	137	21	318	63	75	22	114	22	4	1
Cohort 3, 2013–2014	126	20	343	67	63	20	98	19	9	2
Across cohorts	379	19	1,078	65[c]	190	18	360	22	36	2
	N = 2,043		n = 1,664		*n* = 1,050		*n* = 1,664		*n* = 1,664	

Note. WN = whole number ordering errors; SDBF = smallest denominator–biggest fraction ordering errors; NC = no category errors (i.e., students wrote the wrong fractions or the same fraction twice or could not determine an error pattern); DNA = did not attempt.
[a]Indicates the frequency of errors of the responses that had errors (i.e., subtracting correct answers from the denominator). [b]Indicates the frequency of SDBF ordering errors of the responses that had errors (i.e., subtracting correct answers from the denominator). This error could occur on only five of the nine ordering problems. [c]Of whole number ordering errors, 46% included ordering the three fractions from least to greatest based on the whole number value in the denominator (e.g., 1/2 < 1/5 < 1/12); 33% of whole number ordering errors included ordering the three fractions from least to greatest based on the whole number values in the numerator and denominator (e.g., 1/2 < 2/6 < 5/8); 21% of whole number ordering errors included ordering the three fractions from least to greatest based on the whole number values in the numerator (e.g., 1/2 < 2/8 < 3/4).

Table 3. Proportions and Standard Deviations by Problem of Correct Answers, Whole Number Ordering Errors, Smallest Denominator–Biggest Fraction Ordering Errors, No Category Errors, and Did Not Attempt Errors.

	Correct		WN		SDBF[a]		NC		DNA	
Problem	Proportion	*SD*	Proportion	*SD*	Proportion	*SD*	Proportion	*SD*	Proportion	*SD*
A	0.08	0.02	0.55	0.03	0.23	0.03	0.25	0.03	<0.01	<0.01
B	0.34	0.03	0.77	0.03			0.28	0.03	0.02	0.01
C	0.07	0.02	0.52	0.03	0.25	0.03	0.25	0.03	0.01	0.01
D	0.10	0.02	0.59	0.03	0.14	0.02	0.28	0.03	0.01	0.01
E	0.37	0.03	0.81	0.03			0.20	0.03	0.03	0.01
F	0.27	0.03	0.82	0.03			0.19	0.03	0.02	0.01
G	0.31	0.03	0.81	0.03			0.24	0.03	0.04	0.01
H	0.05	0.01	0.60	0.03	0.13	0.02	0.31	0.03	0.02	0.01
I	0.07	0.02	0.53	0.03	0.17	0.02	0.35	0.03	0.03	0.01

Note. WN = whole number ordering errors; SDBF = smallest denominator–biggest fraction ordering errors; NC = no category errors (i.e., students wrote the wrong fractions or the same fraction twice or could not determine an error pattern); DNA = did not attempt. The proportion of correct answers was calculated by dividing the number of items correct by the number of students in the sample. To be consistent with all other analyses, the proportion of each *error* was calculated by dividing the number of times the error occurred on an item by the number of students in the sample, minus the number of problems correctly answered. This was to reflect the proportion of each error type of those responses that had errors. The standard deviation of the proportion was calculated using the following formula: $SD_{proportion} = \sqrt{\frac{sample\ proportion \times (1 - sample\ proportion)}{sample\ size}}$
[a]SDBF errors could occur on only five of the nine ordering problems.

Effect of Students' Explanations (Cohort 3 Only) on Errors

See Table 6 for a summary of the CCREM results. The component scores for the explanation test were as follows: (a) correct sign placement ($M = 5.28$, $SD = 2.07$); (b) number of parts ($M = 0.45$, $SD = 1.18$); (c) size of parts ($M = 0.34$, $SD = 0.96$); and (d) correct drawing ($M = 0.61$, $SD = 1.74$). Students' ability to explain comparing problems ($M = 6.97$, $SD = 4.00$) significantly predicted the probability of committing a WN-OR. To gain a better understanding of this significant effect, we calculated the probability of committing a WN-OR when students performed one standard deviation below the mean (>0.999), one standard deviation above the mean (0.269), and two standard deviations above the mean (0.001) on explanations.

By contrast, students' explanations did not significantly predict the probability of committing a SDBF-OR ($p = .065$) or a NC-OR ($p = .78$). Therefore, we did not calculate probabilities.

Discussion

There were three purposes to the present study. First, we described fraction ordering error patterns among fourth grade students at risk for developing mathematics difficulties. Second, we assessed whether part-whole understanding and fraction-magnitude estimation significantly

Table 4. Effect of Cohort on Accuracy, Whole Number Ordering Errors, and Smallest Denominator–Biggest Fraction Ordering Errors.

Outcome	Fixed effects	γ	*SE*	*z*	*p*	Random effects	Variance
Accuracy	(intercept)	−3.97	0.64	−6.23	<.001		
	Cohort 1	0.96	0.54	1.78	.075	Student	7.43
	Cohort 3	0.78	0.54	1.43	.152	Problem	1.89
WN ordering errors	(intercept)	1.00	0.69	1.45	.148		
	Cohort 1	−0.76	0.96	−0.78	.435	Student	23.98
	Cohort 3	0.54	0.97	0.58	.581	Problem	0.51
SDBF ordering errors	(intercept)	−10.41	1.08	−9.61	<.001		
	Cohort 1	0.74	1.12	0.66	.509	Student	134.89
	Cohort 3	0.47	1.16	0.41	.685	Problem	0.72
NC ordering errors	(intercept)	−2.22	0.31	−7.34	<.001		
	Cohort 1	−0.07	0.35	−0.20	.844	Student	3.08
	Cohort 3	−0.48	0.36	−1.32	.186	Problem	0.26

Note. WN = whole number; SDBF = smallest denominator–biggest fraction; NC = no category. Cohort 2 was the referent group.

Table 5. Effect of Part-Whole Understanding (i.e., NAEP Part-Whole Items) and Accuracy of Fraction-Magnitude Estimation (i.e., Number Line) on Accuracy and Errors.

Outcome	Fixed effects	γ	*SE*	*z*	*p*	Random effects	Variance
Accuracy	(intercept)	−3.53	0.55	−6.15	<.001		
	NAEP_PW	0.61	0.15	4.17	<.001	Student	5.07
	NL	7.27	1.48	4.91	<.001	Problem	2.06
WN ordering errors	(intercept)	0.95	0.47	2.00	.046		
	NAEP_PW	−0.50	0.29	−1.76	.079	Student	24.69
	NL	−15.07	3.53	4.26	<.001	Problem	0.55
SDBF ordering errors	(intercept)	−9.74	0.94	10.37	<.001		
	NAEP_PW	0.58	0.42	1.39	.164	Student	113.47
	NL	5.08	3.91	1.30	.194	Problem	.66
NC ordering errors	(intercept)	−2.40	0.24	−9.96	<.001		
	NAEP_PW	−0.05	0.10	−0.53	0.60	Student	3.10
	NL	2.01	1.15	1.75	0.08	Problem	0.26

Note. WN = whole number; SDBF = smallest denominator–biggest fraction; NC = no category; NAEP_PW = National Assessment of Educational Progress part-whole items; NL = number line.

Table 6. Effect of Students' Explanations (Cohort 3 Only; *n* = 71) on Errors.

Outcome	Fixed effects	γ	*SE*	*z*	*p*	Random effects	Variance
Accuracy	(intercept)	−2.91	0.59	−4.97	<.001	Student	4.39
	Explanation	0.32	0.08	4.00	<.001	Problem	1.53
WN ordering errors	(intercept)	4.74	2.09	2.27	.023	Student	111.23
	Explanation	−1.44	0.45	−3.24	.001	Problem	3.38
SDBF ordering errors	(intercept)	−8.13	1.178	−4.56	<.001	Student	85.62
	Explanation	0.66	0.36	1.84	.065	Problem	0.91
NC ordering errors	(intercept)	−2.91	0.42	−6.87	<.001	Student	4.53
	Explanation	0.02	0.08	0.28	0.78	Problem	0.15

Note. WN = whole number; SDBF = smallest denominator–biggest fraction; NC = no category.

predicted the probability of committing an error. (Note that the focus of the article was on pervasive *patterns of errors* among at-risk fourth grade students. Therefore, we focus our discussion on these patterns of error identified in the sample [i.e., WN-ORs and SDBF-ORs].) Third, we explored whether students' verbal explanation of comparing problems predicted the probability of committing an error.

Consistent with previous research (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004), students in the present study demonstrated substantial difficulty assessing fraction magnitude, with no significant differences among cohorts on accuracy or error patterns for these low performers. Across years, after their fourth grade year of focused classroom instruction on fractions, this sample of at-risk fourth-grade students correctly ordered fractions on only 19% of problems. We identified two error patterns. WN-ORs occurred when students ordered the fractions from least to greatest based on whole number values. This error was pervasive among the sample of at-risk fourth grade students. The majority of students ordered fractions from least to greatest based on the whole number values in the numerator or denominator, and there were no significant differences among cohorts on the frequency of this error. Of the errors made, 65% were WN-ORs.

SDBF-ORs occurred on problems with different numerators and different denominators, in which students ordering the fractions from least to greatest based on the size of the parts (i.e., denominator), without concurrently considering the number of parts (i.e., numerator) to determine the fraction's value. This error, which was more advanced than simply attending to the whole number values in the numerator or denominator, did not occur as frequently as expected. Of the errors made on the five problems with potential for this error (i.e., 1,050 responses), 18% (i.e., 190) were SDBF-ORs. The low frequency of this error may help to explain why accuracy of fraction-magnitude estimation failed to predict the probability of committing SDBF-ORs. However, this type of error may represent a transitional phase as students attempt to assimilate fractions into their numerical framework (Stafylidou & Vosniadou, 2004). Because students made very few of these errors, we could not examine the tenability of this hypothesis. Future research should investigate whether this type of error represents a transition from operating with whole number bias to assimilating fractions into a single numerical framework.

Because WN-ORs were the most pervasive error in the sample, we center the majority of the discussion on how accuracy of fraction-magnitude estimation (i.e., number line performance) relates to the probability of committing WN-ORs. We then discuss practical implications and curriculum recommendations as they relate to remediating WN-ORs, followed by a discussion of study limitations.

As expected, part-whole understanding failed to predict the probability of committing WN-ORs when accuracy of fraction-magnitude estimation was controlled. Increased part-whole understanding did not significantly decrease the probability of committing a WN-OR. Based on these results, it appears that only focusing on teaching part-whole understanding will not significantly decrease the likelihood of operating with whole number bias.

Accuracy of fraction-magnitude estimation, on the other hand, did significantly predict the probability of committing a WN-OR. Superior number-line performance dramatically decreased WN-ORs. When students performed one standard deviation below the mean on number line, the probability of committing a WN-OR was 95%; the probability of committing this error decreased to 4% when students performed two standard deviations above the mean (holding part-whole understanding constant at the mean). In all, 20 students in the sample performed one standard deviation below the mean and 7 performed two standard deviations above the mean.

It should be noted that students who tended to operate with whole number bias on ordering also had both poor part-whole understanding ($r = -.23$, $p = .01$) and poor skill in fraction magnitude estimation ($r = -.13$, $p = .01$). That is, students who committed WN-ORs (a systematic rather than random error) demonstrated a lack of fraction understanding across both constructs. But, as students' performance on both tasks increased, the likelihood of correctly answering an ordering problem also increased. It is possible that inhibiting whole number bias is key to increasing fraction magnitude understanding, and increasing accuracy across a range of fraction tasks. To assist students with this, effective fraction instruction should emphasize magnitude understanding (e.g., Booth & Siegler, 2006; Siegler et al., 2011). This is corroborated by 4 years of intervention research (Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in press; Fuchs, Malone, et al., 2015) and an error analysis of whole number bias with calculations (Schumacher & Malone, 2015) that increasing students' understanding of fraction magnitude (i.e., measurement understanding) with explicit instruction significantly improves performance across a range of fraction measures and decreases whole number bias.

Students' ability to explain comparing problems also significantly decreased the probability of committing WN-ORs. For explaining comparing problems (Cohort 3 only), the probability of committing a WN-OR was 99.99% when students performed one standard deviation below the mean; the probability decreased to 0.1% at two standard deviations above the mean. Seven students performed one standard deviation below the mean, and five students performed two standard deviations above the mean. Results suggest the importance of constructing sophisticated explanations for comparison problems for decreasing the probability of operating with whole number bias when ordering fractions from least to greatest.

Although a few students were capable of writing sophisticated explanations, the majority of students in this sample (Cohort 3 only) had substantial difficulty verbally explaining how to compare fractions as indicated by the mean on this measure. As reported in the literature (e.g., Stigler & Perry, 1988; Yang & Cobb, 1995), many American students (not just low-performing students) struggle to construct verbal explanations. For example, in a descriptive study comparing American and Chinese students' learning of place-value, Yang and Cobb (1995) found that Chinese students constructed far superior verbal explanations about mathematics concepts than their American counterparts. In a similar descriptive study comparing mathematics classroom discourse, Stigler and Perry (1988) showed that Japanese teachers devoted the majority of class time on modeling how to explain the problem-solving process. Teachers stressed the importance of thinking through the correct answer. By contrast, American teachers spent little time on explanations. Results from the present study reflect this, as indicated by the low scores on explaining comparing problems. Of a possible 54 points on the test, the average explanation score was only 7 points. There is a long history of achievement discrepancies between the United States and Asian countries in mathematics (e.g., Provasnik et al., 2012). The ability to explain concepts and connect ideas in mathematics may be a key factor explaining this achievement gap (e.g., Stigler & Perry, 1988).

Not only did students fail to provide satisfactory verbal explanations, many of the explanations provided were based on whole number knowledge. In fact, 38% of the explanations provided on the explaining comparing problems measure included incorrect whole number explanations. These problematic whole number explanations included examples such as stating that "3/6 is bigger than 3/4 because 6 is bigger than 4" or cross-multiplying and comparing whole number values to assess magnitude. The prevalence of these types of whole number explanations indicates students have deep misconceptions about fraction properties. In the next section, we discuss recommendations for mitigating these misconceptions.

Instructional Implications

One of the CCSS (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2013) fraction standards requires students to "extend core understanding of fraction equivalence and ordering" (www.corestandards.org). Results from this study suggest that accurately estimating fraction magnitude and providing sophisticated verbal explanations may be key components in increasing students' conceptual understanding of fractions. This is corroborated by previous research highlighting the importance of explicitly teaching how to assess fraction magnitude (e.g., Cramer et al., 2002; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in press; Fuchs, Malone, et al., 2015; Siegler et al., 2011; J. P. Smith, 2002) and using the number line as a representational tool for teaching fraction-magnitude topics (e.g., Keijzer & Terwel, 2003).

In 4 years of intervention research, the series of intervention studies conducted by the Fuchs et al. (2013, 2014, in press; Fuchs, Malone, et al., 2015) research group indicated that at-risk fourth-grade students who received intervention emphasizing fraction magnitude (i.e., ordering fractions, comparing fractions, and placing fractions on the number line) significantly outperformed control students on these topics. In fact, the treatment students from these cited studies made very few ordering errors. Given there were no differences among cohorts on accuracy or error patterns in the present study as the district moved toward CCSS (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2013) implementation, it appears that the CCSS are not being effectively implemented for students with risk for mathematics difficulty. This present sample of at-risk students, who received their fraction instruction within general education classrooms, had limited understanding of fraction magnitude.

This has important implications. Ramping up mathematics standards does not ensure higher achievement, especially for low performers. In fact, these standards may pose substantial difficulty for at-risk students because these students lack the foundational skills necessary to achieve the depth of knowledge the CCSS (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2013) attempts to instill (Powell et al., 2013). In the era of CCSS, educators must be aware that at-risk students may need more specialized and individualized intervention to fulfill these more rigorous standards (Fuchs, Fuchs, et al., 2015; Powell et al., 2013). Otherwise, at-risk students may be left behind (Fuchs, Fuchs, et al., 2015).

Before CCSS (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2013) were implemented in the district where this study took place, Fuchs et al. (2013, 2014) reported that the achievement gap between a sample of at-risk control students (similar to those included in the present study's sample) and a sample of low-risk students (i.e., average-achieving classmates) was approximately one standard deviation on a set of 18 items from the NAEP that probed both part-whole and fraction-magnitude understanding. (Average-achieving student data were not collected for number line performance.) This gap widened for at-risk children who received fraction instruction in their classrooms as the district moved toward implementation of the CCSS. This widening achievement gap stands in stark contrast to the not-at-risk students' fraction understanding (indexed by NAEP), which substantially increased each year (Fuchs, Fuchs, et al., 2015).

Moreover, as at-risk students who did not receive specialized fraction intervention fell further and further behind each year the standards became more rigorous, the achievement gap was nearly eliminated for at-risk children who received specialized fraction intervention specifically designed to disrupt the developmental pattern of mathematics difficulty. In these studies, intervention included explicitly teaching conceptual comparing strategies, providing adequate scaffolding for teaching these concepts, and ensuring there was adequate support for developing and maintaining foundational mathematics skills such as addition and multiplication. Part-whole understanding was a component of instruction, but was not the focus.

On the other hand, instruction in the general education classroom (as indicated by teachers' report on how they teach fractions) primarily focuses on part-whole understanding and procedural methods for assessing magnitude. Results from the present study suggest that part-whole understanding is insufficient for reducing the likelihood of operating with whole number bias. Without emphasizing that fractions have magnitude and can be ordered, compared, and placed on the number line, students resort to the only knowledge they have: whole number knowledge. Operating with whole number bias can have a substantial negative effect on students' ability to succeed in higher-level mathematics courses such as algebra (Bailey et al., 2012; Booth & Newton, 2012; Brown & Quinn, 2007; Siegler et al., 2011).

It could be that American teachers struggle to teach how to assess fraction magnitude because they themselves have insufficient fraction understanding. For example, in a descriptive study comparing American and Japanese fourth-grade teachers' understanding of rational numbers, Moseley, Okamoto, and Ishida (2007) found that American teachers relied almost exclusively on part-whole relationships to describe fraction concepts. But they often inaccurately described fraction concepts and struggled to rationalize how to apply part-whole explanations to more difficult concepts like proportional reasoning. Like teachers in the Moseley et al. study, teachers in this study's district may also struggle to rationalize how to apply part-whole explanations to more difficult fraction topics such as ordering fractions with different numerators and different denominators. By contrast, Japanese teachers' explanations of fraction concepts focused more on quantity relationships, which allowed them to more accurately explain more difficult mathematics concepts. In addition, Japanese teachers spend considerably more time on problem-solving procedures (i.e., time per problem) than U.S. teachers, providing in-depth explanations for both concepts and procedures (U.S. Department of Education, National Center for Education Statistics, 2003).

Moseley et al.'s (2007) descriptive study helps to explain why many teachers in this study reported that they focused on procedural methods for assessing magnitude (e.g., cross-multiplying). However, teaching procedures without concepts likely leaves students confused and unable to judge the accuracy of their answers (Kilpatrick et al., 2001; Rittle-Johnson & Siegler, 1998), as both conceptual and procedural knowledge are important for developing an understanding of fraction magnitude (e.g., Rittle-Johnson, Siegler, & Alibali, 2001). This is especially true for students with risk for mathematics difficulties since these students struggle with foundational mathematics skills such as addition and multiplication (e.g., Powell et al., 2013). When students lack both procedural competence and conceptual knowledge, understanding fraction principles becomes substantially difficult. Based on evidence from the present study, we believe that the U.S. curriculum needs a stronger focus on quality explanations of both concepts and procedures and procedural explanations should be deeply rooted in concepts (rather than presented as mathematics "rules") (Fuchs et al., 2008).

Although the pattern of errors is unknown for a sample of average-achieving students (as these data were not collected), previous research (e.g., DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004) suggests that even average-achieving students struggle to assess fraction magnitude. This speaks to the importance of improving fraction instruction to benefit all students (Siegler et al., 2011). It is possible to reduce students' tendency to operate with whole number bias (i.e., Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in press; Fuchs, Malone, et al., 2015). However, more work must be done to achieve the standards goals of the CCSS (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2013) and improve students' conceptual understanding of fractions, especially for those at risk for mathematics difficulties.

Limitations

Of course, this study must be viewed in light of limitations. First, this study examined only the differential predictability of two types of fraction understanding—part-whole understanding and accuracy of fraction-magnitude estimation. Although both represent important components of conceptual understanding of fractions (e.g., Kilpatrick et al., 2001) and help to explain individual differences in fraction skills (Hecht et al., 2003), there are also other important ways to interpret a fraction. These include defining a fraction as a ratio, which is a labeled relationship between two quantities (e.g., 3 out of 4 cars are blue): a quotient, which represents the decimal value of *a* divided by *b* (e.g., 3/4 = 0.75), or an operator, which represents the multiplicative properties of fractions and how a fraction can operate on another quantity (e.g., 3/4 of 100 can represent the expression $\frac{100x3}{4}$ or $\left(\frac{100}{4}\right)x3$). The present study did not analyze these constructs because they are not typically assessed or taught in fourth grade. This is not to say that these constructs are not important. But laying a foundation for students to understand fraction magnitude in the first year of focused instruction on fractions will likely better prepare them to consolidate the many interpretations of fractions in the later grades.

Second, we reported how teachers taught students fractions using a self-report survey. We did not conduct live observations in the classroom to determine whether what they reported was in line with how they were actually teaching fraction concepts. Therefore, these data must be viewed as an approximation of how these teachers taught fractions in their classroom.

Despite these limitations, results demonstrate the substantial difficulty at-risk students have with estimating fraction magnitude. Findings also shed light on the importance of incorporating fraction magnitude instruction into the curriculum. Previous intervention research (e.g., Cramer et al., 2002; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs, Malone, et al., 2015; Fuchs et al., in press) supports teaching students how to assess fraction magnitude with comparing strategies and the number line to significantly improve students' conceptual understanding of fractions. In these cited studies, students were explicitly taught how to compare fractions, order fractions, and place fractions on the number line. Instruction also focused on constructing conceptual verbal explanations for how to complete these tasks. In the present study, the ability to accurately estimate fraction magnitude and explain comparison problems had a dramatic effect on reducing WN-ORs (i.e., whole number bias) among fourth grade students at risk for developing mathematics difficulties.

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