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Cover Page Footnote
The author would like to acknowledge the outstanding work of the students and support of the principal,
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Polygons, Pillars and Pavilions: Discovering Connections between Geometry and Architecture

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Crowning the second semester of geometry, taught within a Catholic middle school, the author's students explored connections between the geometry of regular polygons and architecture of local buildings. They went on to explore how these principles apply famous buildings around the world such as the monuments of Washington, D.C. and the elliptical piazza of Saint Peter's Basilica at Vatican City within Rome, Italy.

Keywords: Catholic middle school mathematics, geometry, architecture, education

Connecting the beautiful, though often abstract, ideas of geometry with tangible human experience is one of our goals as mathematics teachers. To that end, I would like to share a series of lessons I have developed, intended to foster my students' discovery of the geometric principles used by architects to enhance the aesthetic appeal of the buildings they design. This unit begins with an exploration of the properties of regular, convex polygons and culminates with an analysis of symmetry exhibited by buildings as mundane as gazebos found in parks and as sublime as the monuments found in great cities like Washington, D.C. and Rome, Italy.

Excellent line drawings of regular polygons are easily downloaded from internet sites or from clip art libraries like the one found within Microsoft Word. I print multiple copies of equilateral triangles, squares, and other regular polygons such as pentagons, hexagons and septagons, and distribute these to my students at the beginning of the unit. I expect my students to collaborate in pairs or groups of three to answer the following questions:

- What are the lengths of the sides of each polygon?
- What are the measures of the central, interior, and exterior angles?
- What patterns do you notice?

Of course, answering these questions requires students to make a variety of measurements.

I provide protractors and rulers if students don't bring their own. I don't make the assumption that all students will be skilled in the use of these instruments and am always prepared to demonstrate their use. However, I find that this exploration activity retains the most intrigue for students if I provide only the most necessary instructions to get them started. As they work, students will naturally ask questions such as:

- "How do I measure the central angles of a polygon?"

- Suggested answer: use a straight-edge to draw a set of greatest diagonals between vertices. These will intersect at the center. The central angles will be those with a vertex at this intersection, whose rays pass through adjacent vertices of the polygon. (See Figure 1).

- "How do I measure the exterior angles?"

- Suggested answer: Moving in one direction along the outside of the polygon, extend the sides with a straightedge until you've got rays large enough to measure with a protractor. (See Figure 1).

- "How do I measure the interior angles?"

- Suggested answer: No additional work required here, the sides of the polygon represent the rays of the angle you want to measure.
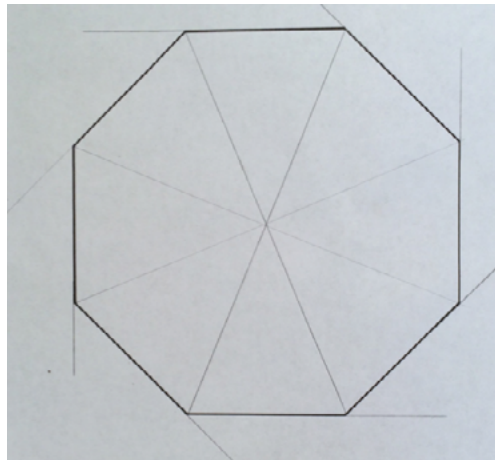

Figure 1. A line drawing showing how to find the central and exterior angles of a polygon.

As they work, I ask my students to record their data in a table such as that in Figure 2.

The data students gather from this initial exploration invites observations, which, if they are not voiced by the students themselves, should be raised by the teacher. Among the conclusions they will inevitably draw are:

- The central angles follow the pattern 360°/n, where n is the number of sides.
- The interior angles follow the pattern (n − 2)(180°)/n, where n is the number of sides.
- The exterior angles follow the same pattern as the central angles.

Polygon	# of sides	Side length	Central ∠	times sides	Interior ∠	times sides	exterior ∠	times sides
Regular Equilateral Triangle	3	16cm	120°	360°	60°	180°	120°	360°
Square	4	10.5cm	90°	360°	90°	360°	90°	360°
Regular Pentagon	5	9.4cm	72°	360°	108°	540°	72°	360°
Regular Hexagon	6	7.3cm	60°	360°	120°	720°	60°	360°
Regular Septagon	7	7.5cm	approx. 51.5°	360°	approx. 128.5	900°	approx. 51.5°	360°
Regular Octagon	8	6.5cm	45°	360°	135°	1080°	45°	360°

Figure 2. One student's table capturing data on the measurement of regular polygons.

In addition to creating the table, students prepare a summary explaining the process by which they drew their conclusions. (See Figure 3.)

The pattern I noticed was that the exterior angles and the central angles were the same. I think this is because all the sides are equal, and the angles to the center were the same as the angles between the lines you drew out, and the actual side. I also saw that when you multiplied the central and exterior angles by the number of sides it had, the number was always 360°. Also, the interior angle increased by 180°.

The pavilions we saw at Lincoln Park, Sherwood Park, and the Currier Inn were all built with regular polygon floor plans. These buildings were regular. I know this because all the sides were equal, and the angles were all the same. They all had an even number of vertices. This is true because they all had an even number of sides, eight.

Figure 3. One student's summary of her observations from the first two parts of our unit on polygons and architecture.

Aside from the numbers in the table, I ask my students to examine the rotational symmetry of each polygon and to look for differences in the rotational symmetry of polygons with an odd number of vertices compared to those with an even number of vertices. In other words, I ask them to rotate their polygons about the center counting how many times the image superimposes itself. I also ask them to observe how many sets of parallel sides their polygons contain and whether they contain any parallel diagonals. Figures 4 and 5 illustrate this idea for the regular octagon.
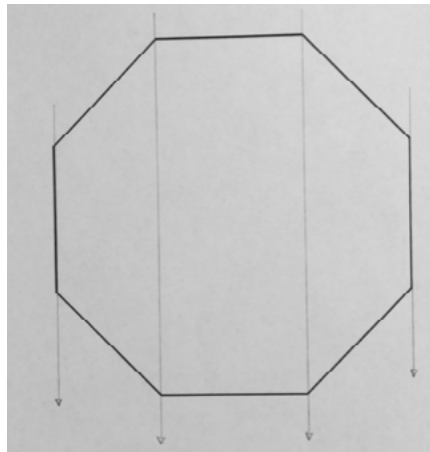

Figure 4. A line drawing demonstrating one manner in which columns which are part of parallel sides or minor diagonals align.
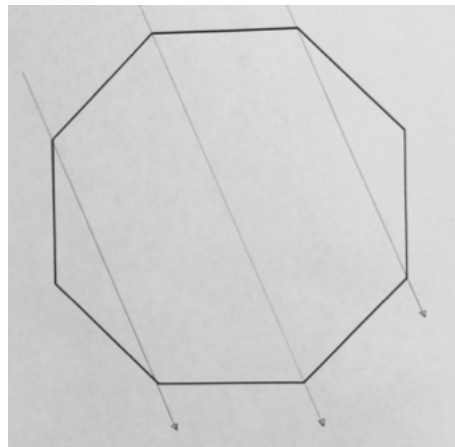

Figure 5. A line drawing demonstrating a second a manner in which the columns standing at parallel diagonals align with one another. Note that the outer columns are symmetrically spaced from the center as well.

Their observations might also be recorded in a table such as Table 1 below. Alternatively, the teacher might simply present this data and ask students to look for patterns.

Table 1

Sample Polygon Observation Table

Polygon	Odd or even # of vertices	Degree of rotational symmetry	Degrees per turn	Sets of parallel sides	Sets of parallel diagonals?
Equilateral triangle	Odd	3	120	0	0
Square	Even	4	90	2	0
Regular pentagon	Odd	5	72	0	0
Regular hexagon	Even	6	60	3	3 sets of 2
Regular septagon	Odd	7	≈51.4	0	0
Regular octagon	Even	8	45	4	4 sets of 2 4 sets of 3

Patterns I point out from this analysis include:

- Polygons with an odd number of vertices never form diagonal parallels or parallels between sides.

- Polygons with an even number of vertices form at least two sets of parallel sides and beginning with a hexagon, at least three sets of parallel diagonals.

- As the number of sides in even sided polygons increases, the number of parallel sides and diagonals increases as well.

- Beginning with the octagon, parallel diagonals include sets that contain the diagonal through the center, which are distinct from sets of parallel diagonals that do not include the diagonal through the center. (See Figures 4 and 5.)

Teachers may want to encourage motivated students to explore these patterns further as an enrichment activity. Nevertheless, these observations become important when analyzing architecture based on polygons.

I next ask my students to imagine that the line drawings of these polygons were the floor plans of pavilions, temples, monuments, or gazebos. I ask them to place dry erase markers at the vertices of each polygon as if they were columns supporting a roof. Next, I ask them to walk around this miniature model of a building, keeping the columns at their eye level. The artistic concept of vanishing point becomes important now. Students must maintain some distance between themselves and their models in order to witness columns that are parts of parallel pairs of line segments align with one another. This is an important point for teachers to make, if students don't discover it for themselves: As our point of perspective moves further away from the center of the polygon, the columns (or dry erase markers) standing at vertices will "line up" with respect to the parallel sides and diagonals of which they are a part. The photo below shows two of my students examining their model pavilion in the classroom.
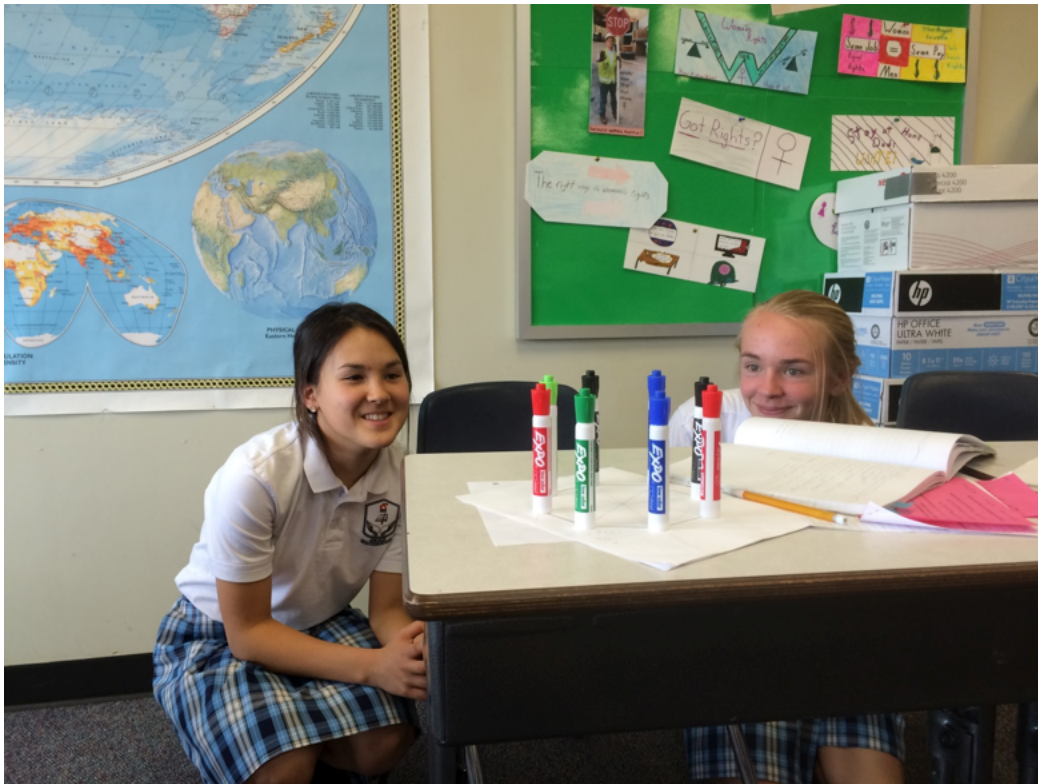
Students examining the properties of polygonal models, like this one, based on the regular octagon. (Photograph taken by author; used with permission).

While students are engaged in this task I ask them questions such as:

- "Do you notice anything about the behavior of the columns as you walk around your dry erase marker building?"

- "How does your model building demonstrate rotational symmetry as you walk around it?"

- "How do the model buildings with an odd number of vertices differ in their behavior from those with an even number of vertices?"

I find that my students discover the answers to these questions quite readily and often express their ideas elegantly. .For those who don't make the connection on their own, I invite them to follow me around their models as I point out the following:

- For odd polygons, though we can witness rotational symmetry as we walk around our model pavilions, no more than two columns ever align with one another.

- For even polygons, we see rotational symmetry as we walk around the exterior of the model. Moreover, we also see groups of columns align in two ways.

  - When we stand at a point along the center line through a great diagonal and,

  - When we stand at a point along the center line between two adjacent columns.

Having progressed from making measurements on line drawings of polygons to making observations on classroom models of columns placed at the vertices of these polygons, the next step is to ask, "Do architects really make use of these principles when designing buildings?" To answer this question I schedule a field trip in which my students and I observe and make measurements on gazebos, pavilions and other local buildings. Our list includes gazebos in the yards of local residents near our school, as well as local landmarks. At each of these buildings students are able to take measurements with protractors and string to verify the lengths of the sides and the measures of the central, interior, and exterior angles of the structures. Finding agreement between these measurements and those data collected in the classroom provides powerful reinforcement of the lesson. The following photos show my students making measurements on some of these buildings.

Students preparing to measure the geometric features of a local bed and breakfast built on an octagonal floor plan. (Photograph taken by author; used with permission).

Students measuring interior angles of a regular octagonal building. (Photograph taken by author; used with permission).

The next phase of the unit addresses the question: "What sorts of examples of regular-polygonal architecture exist outside of our city?" Using the internet or examples from their own travel experience my students and I have captured images that demonstrate not only that there are many beautiful polygonal buildings in the world, but that they exhibit the same principles of symmetry and alignment of columns we initially discovered in the classroom. These include the District of Columbia War Memorial., the Gazebo on Turret Hill, Dromoland Castle in Ireland, and the Tomb of Andrew Jackson. Based on their classroom observations, students can easily imagine the parallel pairs of columns of these buildings snapping into alignment from a vantage point outside the structure.

Capstone Activity

Many monumental buildings visually interact with their visitors. This behavior is a testament to the genius of their architects. Examples include the Washington Monument and its Reflecting Pool, the Jefferson Memorial (also located in the United States capital), as well as the colonnade enclosing the elliptical piazza of Saint Peter's Basilica in the Vatican City, within Rome, Italy. As a capstone to this unit on geometry and architecture I lead my students on a discussion of the geometry of Saint Peter's Square, which was designed in the sixteenth century by Gian Bernini. For those not fortunate enough to visit Italy in person, a virtual tour of the piazza may be made through Google Earth (http://www.google.com/earth/.)

The plan of the piazza is an ellipse, at the center of which stands an Egyptian obelisk. Each focus of this ellipse is marked by a brass plate laid into the cobblestone. Surrounding the piazza are four concentric (or more accurately, con-elliptical) colonnades. Each column weighs many tons and is several feet in diameter. Together, this forest of stone columns towers over the visitors below and seems to create an enclosed park. When an observer moves toward one of the foci, however, this forest of stone columns snap into place. When standing at one focus the outer columns are completely obscured by the inner most ring of columns. One can then clearly see between the columns to the city streets beyond, which seem to disappear again when stepping away from the focus. This dramatic effect is still delightful to witness hundreds of years after the square was built and is a consequence of the fact that the columns have been carefully and equidistantly placed along lines of sight emanating from the two foci.

Visitors to Saint Peter's Square in Rome, seemingly enclosed by a wall of columns in the background. (Photograph taken by author.)

Standing at one of the foci of the elliptical colonnade, a visitor sees only the columns in the inner ring (the others are standing behind this first set along the line of sight). The city beyond is clearly visible. (Photograph taken by author.)

To end this unit I ask my students to recreate this effect by building a scale model of Saint Peter's Square in the school parking lot using common materials found in the classroom or brought from home. We begin in the classic manner, by tracing a line on the pavement using a piece of chalk held by string stretched across two nails acting as the foci. This process is repeated with different lengths of string to create four sets of ellipses using the same foci. We then draw lines of sight emanating from the foci and mark the points of intersection with the ellipses. At these points of intersection students stand as if they were the columns of Saint Peter's Square. They take turns at the focus viewing this human model as their classmates step in and out of the lines of sight, thus recreating the visual effect one might witness in the piazza itself.
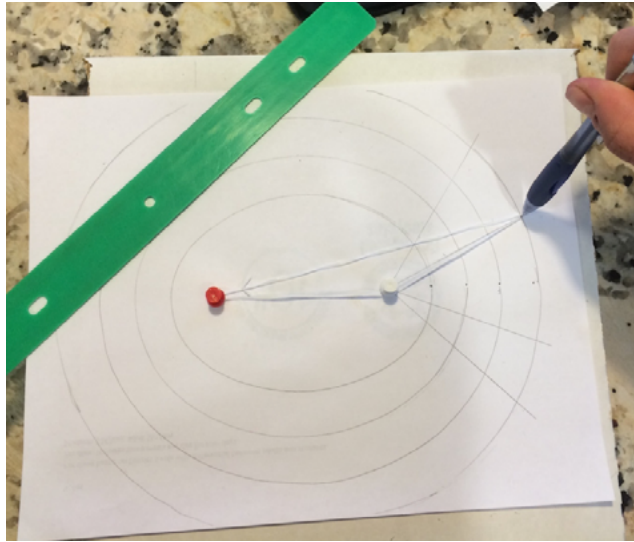
Demonstrating the classic technique of drawing ellipses. (Photograph taken by author; used with permission).

Students using the classic technique to trace a set of ellipses in the school play-ground. (Photograph taken by author; used with permission).

Students standing in for their model colonnade. (Photograph taken by author; used with permission).

Students stepping out of line, thus demonstrating the design of the colonnade at Saint Peter's Square. (Photograph taken by author; used with permission).

Concluding Thoughts

My hope in sharing the lesson described in this paper is that teachers and students who engage in it will appreciate that mathematics finds expression in architecture. James S. Ackerman has written, "Expression in architecture is the communication of quality and meaning. The functions and the techniques of building are interpreted and transformed by expression into art, as sounds are made into music and words into literature" (https://www.britannica.com/topic/architecture/Framed-structures#toc31842).

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