Fraction Multiplication and Division Models: A Practitioner Reference Paper

Heather K. Ervin
Bloomsburg University

To cite this article:

Fraction Multiplication and Division Models: A Practitioner Reference Paper

Heather K. Ervin

Article Info	Abstract
	It is well documented in literature that rational number is an important area of understanding in mathematics. Therefore, it follows that teachers and students need to have an understanding of rational number and related concepts such as fraction multiplication and division. This practitioner reference paper examines models that are important to elementary and middle school teachers and students in the learning and understanding of fraction multiplication and division.

Introduction

According to Rule and Hallagan (2006), multiplication and division by fractions are two of the most difficult concepts in the elementary mathematics curriculum and many teachers and students do not seem to have a deep understanding of these concepts. Achieving a conceptual understanding of models may help people to learn fraction multiplication and division more effectively. Models can aid in the discussion of mathematical relations and ideas and help teachers to gain a better understanding of individual students' understandings (Goldin & Kaput, 1996). Models can help people to develop, share, and express mathematical thinking. Teachers may be able to use students' work with models in order to create more student-centered classrooms because teachers may be able to gain a better understanding of students' ideas (Kalathil & Sherin, 2000). Models are an important piece of mathematics education because they not only aid in the study of mathematics, but they also aid in the study of learning mathematics.

Definition of Model

Models can be of numerous forms and often the definition of a representation depends on the context in which the representation is being used. A representation is a configuration that "…corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else" (Goldin & Kaput, 1996, p. 398). According to NCTM, the term representation denotes processes and products where the process refers to the capturing of a particular concept or idea and the product is the form of representation that is chosen to represent the concept or idea (Goldin, 2003). Models can be personal and do not occur alone; understandings of other concepts and ideas influence the formation of representations. Representations are structured around a person's existing beliefs and knowledge and may change or be adapted as new knowledge is gained and experiences are translated into a model of the world (Bruner, 1966; Goldin & Kaput, 1996). Sometimes the terms 'representation' and 'model' are used interchangeably. According to Van de Walle et al. (2008), a model "…refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed" (p. 27).

Models can be viewed as a means of communication. Zazkis and Liljedahl (2004) described models as helping in the communication of ideas and in communication between individuals, creating an environment ripe for mathematical discourse. Models can also help with the manipulation of problems in that students can concentrate on the manipulation of symbols then later determine the meaning of the result.

The NCTM (2000) recommends that students in prekindergarten through grade twelve be prepared to "organize, record, and communicate mathematical ideas; select, apply and translate among mathematical representations to solve problems; and to use representations to model and interpret physical, social and mathematical phenomena" (p. 268). The NCTM's recommendations are very useful in the study of students' learning and understanding. Models are useful only if students are able to make connections between the ideas that are actually being represented and the ideas that were intended to be represented (Zazkis & Liljedahl, 2004). Modeling is an important step in the learning process before computational algorithms are examined. "Using models to highlight the meaning of division should precede the learning of an algorithm for division involving fractions" (Petit et al., 2010, p. 8) because computational algorithms can be easily forgotten. Models that are anchored in deep understanding, however, are much more likely to be recalled by students at a future point in time.

Types of Models

There are many types of models that may contribute to learning and understanding fraction multiplication and division. Before detailing models for fraction multiplication and division, it may be useful to explore general fraction models. *Bits and Pieces* (2006; 2009a; 2009b) is a sixth grade mathematics series that focuses on fractions, fraction operations, decimals, and percents and poses questions throughout the series that involve various fraction models. Not only are students given the opportunity to choose their own models in this series, but many examples of models are explained in detail and presented in a context that would be conducive to learning with understanding. Van de Walle et al. (2008) agree that models are important in the learning and understanding of fractions and fraction operations. Models can be used to help clarify ideas that may be confusing when presented only in symbolic form. Also, models can provide students with opportunities to view problems in different ways and from different perspectives and some models may lend themselves more easily to particular situations than others. For example, an area model can help students differentiate between the parts and the whole, while a linear model clarifies that another fraction can also be found between any two given fractions. Van de Walle et al. consider three particular types of models: region/area, length, and set, as being important in the learning and understanding of fractions.

Area Model

According to Van de Walle et al. (2008), the idea of fractions being parts of an area or region is a necessary concept when students work on sharing tasks. These area models can be illustrated in different ways. Circular fraction piece models are very common and possess an advantage in that the part-whole concept of fractions is emphasized as well as the meaning of relative size of a part to a whole. Similar area models can be constructed of rectangular regions, on geoboards, of drawings on grids or dot paper, of pattern blocks, and by folding paper (Figure 1). This figure illustrates how Van de Walle et al. (2008) explain the different forms of area models (p. 289).
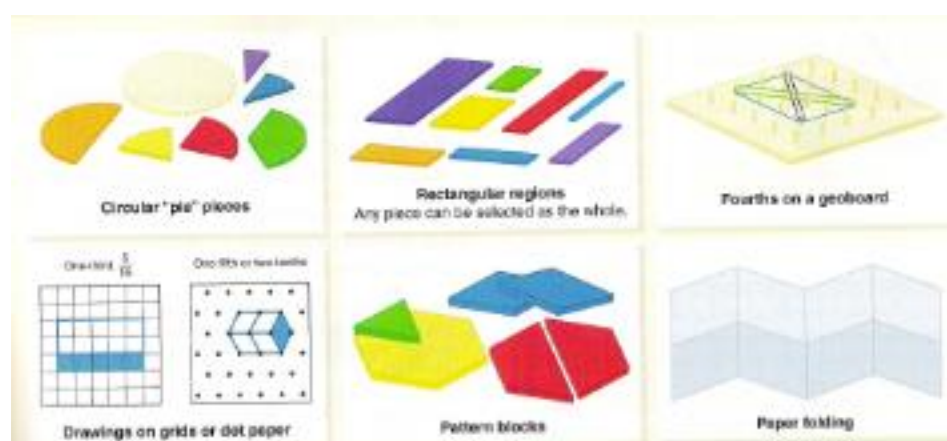

Figure 1. Region/area models

As the focus shifts from fractions to decimals and the relationships between these concepts, tenths grids are often introduced as area models. A tenths grid is a square fraction strip divided into ten equally sized pieces (Figure 2). The tenths grid is used to help students make sense of place value as well as conversions from

fractions to decimals and vice versa. Figure 2 shows a tenths grid and the equivalence of $\frac{1}{10}$ and 0.1 (Lappan et al., 2006, p. 36).
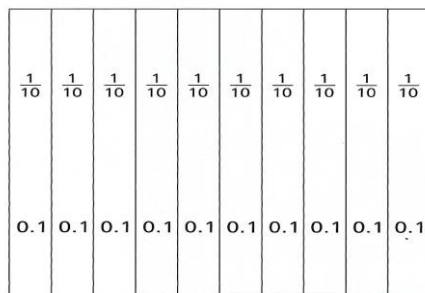

Figure 2. Tenths grid

Hundredths grids are used to help students make connections between fractions and decimals. Hundredths grids are created by further dividing a tenths grid into one hundred equally sized pieces (Figure 3). Both tenths grids and hundredths grids are pictorial representations of place value. Hundredths grids can be used to give a pictorial representation of decimal multiplication. An example of such a problem is 0.1 x 0.1 $= \frac{1}{10}$ x $\frac{1}{10} = \frac{1}{100}$ because a student can look at a hundredths grid and see that $\frac{1}{10}$ of $\frac{1}{10}$ is one square out of the total of one hundred squares. Figure 3 shows a hundredths grid, which is a tenths grid cut horizontally into ten equally sized horizontal pieces (Lappan et al., 2006, p. 37).
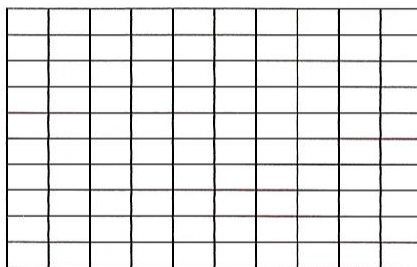

Figure 3. Hundredths grid

Progression to percents leads to an introduction of percent bars (Lappan et al., 2006). Percent bars are area bars divided into percents. One whole percent bar typically represents 100%, which is one whole unit. Percent bars are used in the same way that fraction bars are used. Percent bars are primarily used to show relationships between percents, to examine magnitude, and to compare different ratios, where ratio is defined to be a comparison of two quantities usually expressed as '*a* to *b*' or *a:b* and sometimes expressed as the quotient of *a* and *b* (p. 59). Connections are then established between percent bars and fractions. For example, students may be asked to estimate the fraction benchmark nearest to the given value on a percent bar. As students' understanding progresses, percent bars may be extended to represent values greater than 100% (Figure 4). This figure illustrates how students may use a percent bar to convert percentages to fractions (Lappan et al., 2006, p. 67).
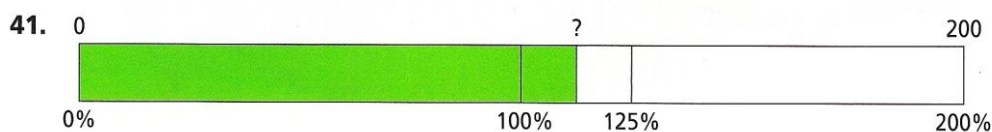

Figure 4. Percent bar

Fraction Multiplication

The area model (Figures 5 and 6) of fraction multiplication seems to be the most fruitful for many reasons. It allows students to see that the multiplication of fractions results in a smaller product and helps to build fractional number sense, number sense related to fractions as opposed to whole numbers (Krach, 1998). This model can also show a visual for two fractions being close to one resulting in a product close to one. Finally, the area model "…is a good model for connecting to the standard algorithm for multiplying fractions" (Van de Walle et al., 2008, p. 320). The area model is the most popular model for teaching fraction multiplication (D'Ambrosio & Mendonga Campos, 1992). Typically, area models are shown using rectangles and squares, but fraction circles (Figure 7) are common as well (Taber, 2001). The unit can be in any shape or size as long as the unit is well defined. Figures 5 illustrates how an area model in the form of a rectangle representing a unit may be used to solve a fraction multiplication problem (Van de Walle et al., 2008, p. 320).
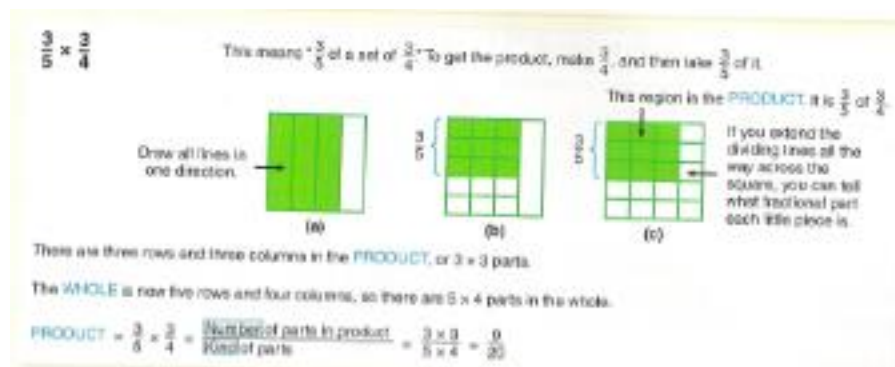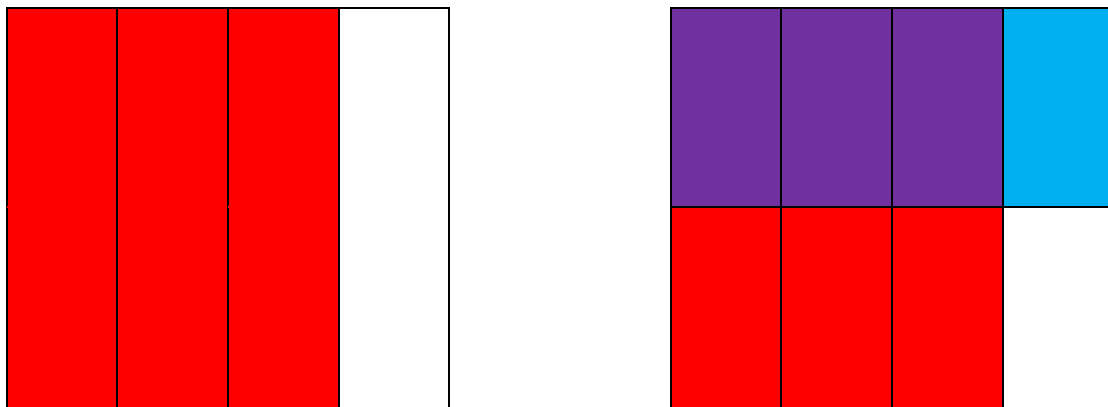

Figure 5. Area model for multiplication: rectangle

Figures 6 illustrates how an area model in the form of a rectangle representing a unit may be used to solve a fraction multiplication problem.

What is $\frac{1}{2}$ x $\frac{3}{4}$?

Partition one unit into four equal pieces. Then shade three pieces to represent $\frac{3}{4}$.

Partition each of the four pieces into two equal parts and shade one of each of the two parts. There are three double-shaded areas out of eight total pieces, thus, $\frac{1}{2}$ x $\frac{3}{4}$ = $\frac{3}{8}$.

Figure 6. Area model for multiplication: Rectangle.

Figures 7 illustrates how an area model in the form of a circle representing a unit may be used to solve a fraction multiplication problem.
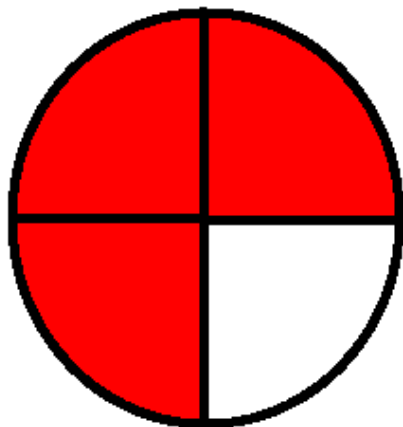
What is $\frac{1}{2} \times \frac{3}{4}$?

Partition one unit into four equal pieces. Then shade three pieces to represent $\frac{3}{4}$.

Partition each of the four pieces into two equal parts and shade one of each of the two parts. There are three double-shaded areas out of eight total pieces, thus, $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.
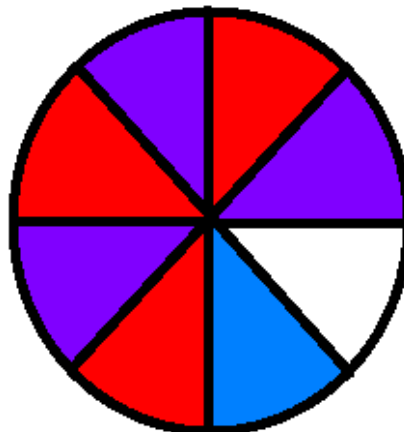
Figure 7. Area model for multiplication: Circle

Many textbooks and curriculum materials encourage students to multiply mixed numbers using improper fractions. However, the area model can also be used for mixed number problems (Figure 8) and can help students to generalize the computational algorithm. This is efficient and can lead to class discussions about the distributive property when students discover that a fraction such as $3\frac{1}{4}$ can be written as $3 + \frac{1}{4}$. It would follow that $\frac{1}{2} \times 3\frac{1}{4} = (\frac{1}{2} \times 3) + (\frac{1}{2} \times \frac{1}{4})$. Area models can also be used for the multiplication of two mixed numbers. There would be four partial products instead of two, as with a problem containing only one mixed number. Using the distributive property to work fraction multiplication problems can tend to be more conceptual and may encourage students to practice estimation, building and making use of number sense (Tsankova & Pjanic, 2009; Van de Walle et al., 2008). Figure 8 illustrates how an area model may be used to complete fraction multiplication for a mixed number and a fraction less than one (Van de Walle et al., 2008, p. 321).
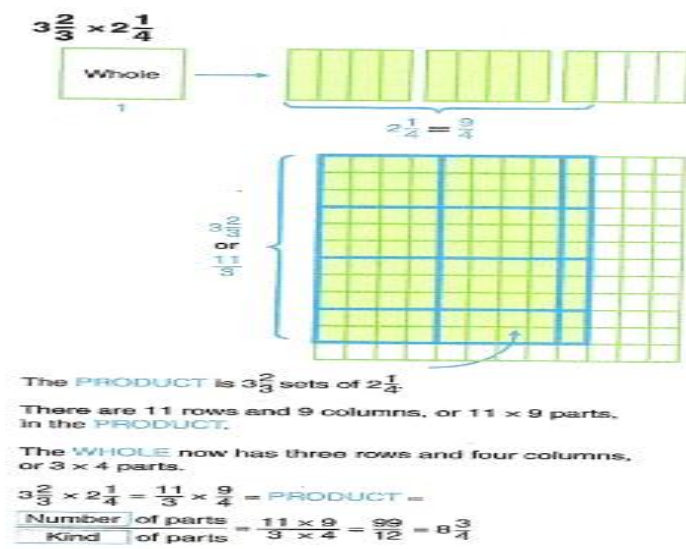
Figure 8. Area model for mixed number multiplication

The area model can also be used to help students make connections to the algorithm for fraction multiplication (Figure 9). Figure 9 considers $2\frac{1}{2}$ x $3\frac{1}{5}$, which can be rewritten as $\frac{5}{2}$ x $\frac{16}{5}$. If we want to take five halves of sixteen fifths, we draw the sixteen fifths first. We will need five halves, so we draw three complete sets ensuring that we have at least five halves (in this case six halves). After drawing three sets of sixteen fifths, we cut each set in half. We will need five halves, so we circle five of the halves and count what we have circled. We have three one-halves and one-tenth contained in each circle. There are five circles, so we have 5 x ($\frac{1}{2}$ + $\frac{1}{2}$ + $\frac{1}{2}$ + $\frac{1}{10}$) = 8. Figure 9 illustrates how an area model may help to make sense of the fraction multiplication algorithm.

What is $2\frac{1}{2}$ x $3\frac{1}{5}$?

$2\frac{1}{2}$ x $3\frac{1}{5}$ = $\frac{5}{2}$ x $\frac{16}{5}$
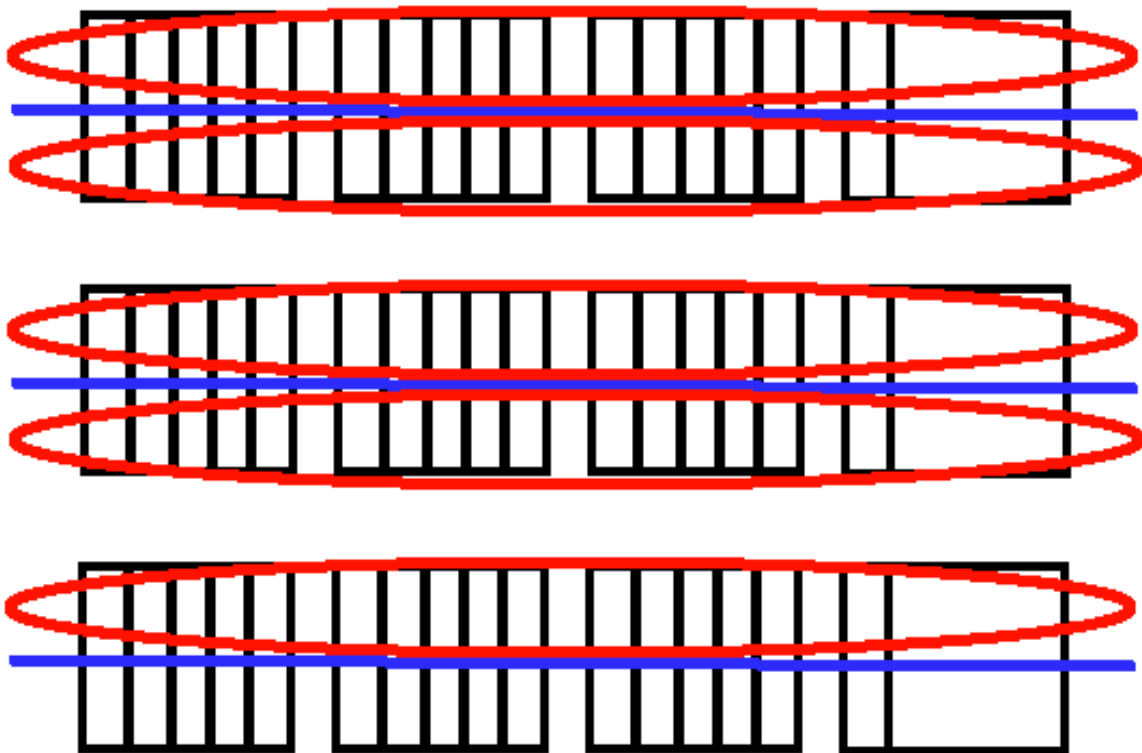
$5\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{10}\right) = 5\left(1\frac{6}{10}\right) = 5\left(\frac{16}{10}\right) = \frac{80}{10} = 8$

Figure 9. Area model with connections to algorithm

Fraction multiplication can also be modeled using a sheet of paper as a manipulative (Figure 10). This representation is created by folding a piece of paper into equal size pieces according to the problem under examination (Taber, 2001; Tsankova & Pjanic, 2009; Van de Walle et al., 2008). Equal size pieces are important to solving fraction multiplication problems so that relationships between the two different fractions can be compared. It should be noted that this model is the same general concept as the area model for multiplication, but instead of being drawn, paper is physically folded. It is important for preservice teachers to be able to both draw and fold area models as folding paper provides students with a tactile experience. Folds do not have to be horizontal and vertical. Students can subdivide parts as illustrated by Van de Walle et al. (Figure 11). Figure 10 illustrates how paper may be folded to solve a fraction multiplication problem using horizontal and vertical folds.

What is $\frac{1}{3}$ x $\frac{3}{5}$?
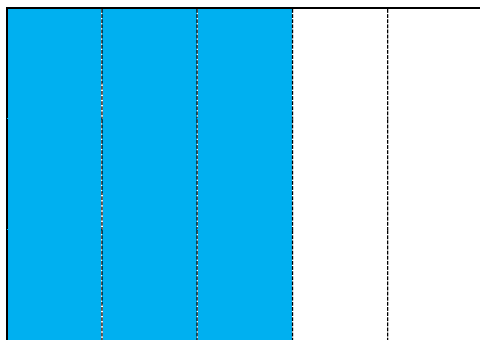
Fold the sheet of paper vertically into five equal pieces. Then shade three pieces to represent $\frac{3}{5}$.
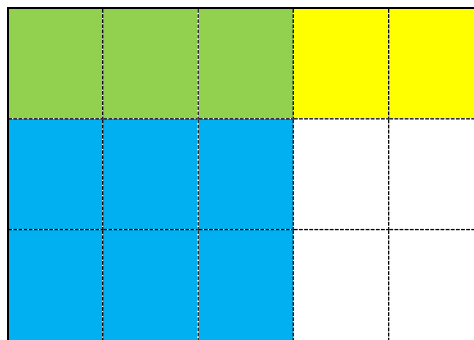
Fold the sheet of paper horizontally into thirds. Then shade one piece of each third. There are three double-shaded areas out of fifteen total pieces, thus,
$\frac{1}{3} \times \frac{3}{5} = \frac{3}{15}$ or $\frac{1}{5}$.

Figure 10. Paper folding for multiplication

Figure 11 illustrates how paper may be folded horizontally to solve a fraction multiplication problem (Van de Walle et al., 2008, p. 319).
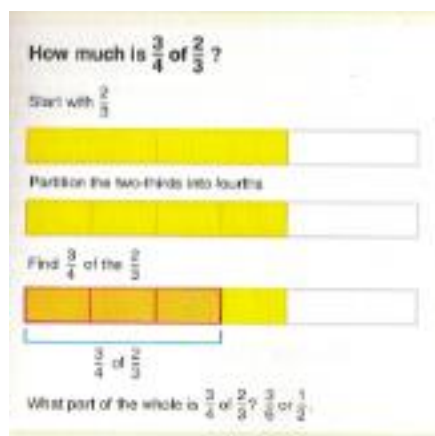

Figure 11. Paper folding for multiplication 2

Fraction Division

There are two ways to view division: partitive and measurement. Partition problems are classically viewed as sharing problems (i.e. You have ten candy bars to share with five friends, how many will each get?). But rate problems (i.e. You drive one hundred miles in two hours, how many miles do you drive per hour?) are also partitive because you are still trying to obtain the value of "one", whether it be the amount for one friend or for one hour. Fractions come into play in partitive division problems in two ways; a fraction may be the dividend or the divisor. If the fraction is the dividend, these problems can still be looked at from a sharing perspective. For example, if Molly has $2\frac{2}{3}$ yards of wrapping paper and needs to know how much she can use per gift if she needs to wrap four gifts, Molly is 'sharing' eight thirds and will have two-thirds for each. Fractional divisors may be easier if viewed more from the perspective of 'how much is one' as opposed to sharing.   For example, it is $3.50 for a $2\frac{1}{3}$ pound cake, how much is each slice if slices are sold by the pound? There are seven thirds total in the cake, which is $3.50. So one-third would be $0.50. There are three thirds in one pound so each

pound is $1.50. In this problem, we partitioned to find the amount of one-third, then we iterated to find the value of one whole (Van de Walle et al., 2008).

Measurement problems are also referred to as repeated subtraction or equal group problems (i.e. equal groups are repeatedly taken away). According to Van de Walle et al. (2008), students tend to be able to solve problems such as these more easily in context (Gregg & Gregg, 2007; Perlwitz, 2005). For example, if Billy buys six jars of paint for a craft project and each person will need $\frac{2}{3}$ of a jar to complete the craft, students do not typically struggle drawing six shapes to represent jars of paint, cutting each into thirds, and determining how many portions of $\frac{2}{3}$ can be found. Students may have issues when trying to make sense of this problem written as $6 \div \frac{2}{3}$. Measurement problems that focus on 'servings' (Figure 12) allow students to use their knowledge of whole numbers to begin to build a better understanding of fraction division (Gregg & Gregg, 2007). This figure illustrates how servings can be used to build student understanding of fraction division (Gregg & Gregg, 2007, p. 491).
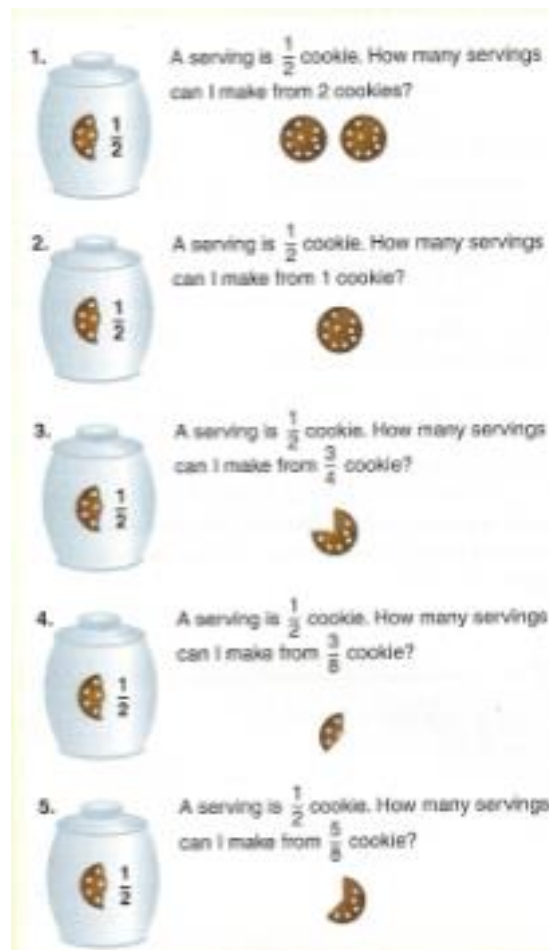

Figure 12. Measurement problems as servings

The area model for division of fractions is a representation that allows students to visualize this process. This model may help students build fractional number sense by showing that the quotient can be larger than the dividend (unlike whole number division) (Wentworth & Monroe, 1995). In area models for these problems, the unit is divided by making horizontal cuts to represent one divisor and vertical cuts to represent the other divisor. This type of set up can aid in solving problems where the equal size pieces may be difficult to construct and promotes the common-denominator algorithm (Van de Walle et al., 2008). Approaching problems in this way will ensure that the entire unit is cut into equal sized pieces (common denominator) so that when fractions are divided, only the numerators need to be divided (Figure 13). However, this process of cutting does not always lead to the least common denominator (Figure 14). In the example shown, the unit could have been cut into fourths with the same result.
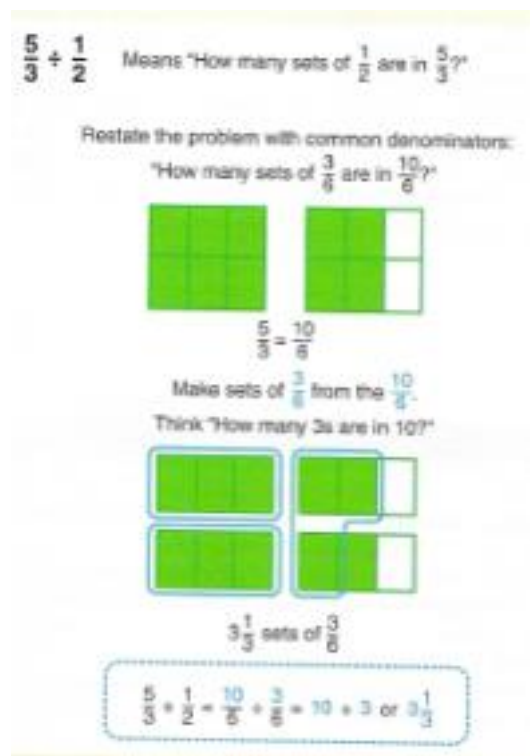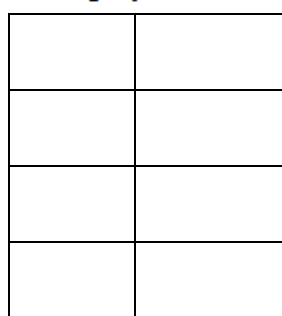
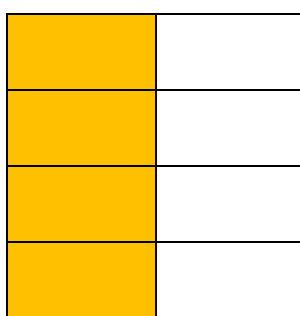Figure 13. Area model for division with least common denominator

This figure illustrates how an area model may be used to solve a fraction division problem where horizontal cuts and vertical cuts resulted in the least common denominator (Van de Walle et al., 2008, p. 325). In this case, the area model shown represents the least common denominator. Figure 14 illustrates how an area model may be used to solve a fraction division problem where horizontal cuts and vertical cuts did not result in the least common denominator.

What is $\frac{1}{2} \div \frac{1}{4}$?

Whole Unit

How many one-fourths will fit into $\frac{1}{2}$?   Two squares make up $\frac{1}{4}$.
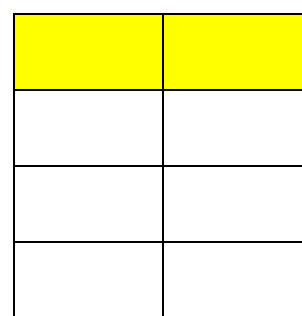
Two sets of these two squares will fit into the squares that represent $\frac{1}{2}$. Thus, $\frac{1}{2} \div \frac{1}{4} = 2$.

Figure 14. Area model for division without least common denominator

As previously stated, area models can be any shape or size. A circular area model for division is shown in Figure 15. Consider $\frac{1}{2} \div \frac{3}{4}$. The circle is the unit. One-half is represented in one circle, while three-fourths is represented in the other circle. How many three-fourths can we fit into one-half? We can see that exactly two of the three pieces from the three-fourths will fit into the half. Thus, $\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$. Figure 15 illustrates how an area model in the form of a circle representing a unit may be used to solve a fraction division problem.

What is $\frac{1}{2} \div \frac{3}{4}$?
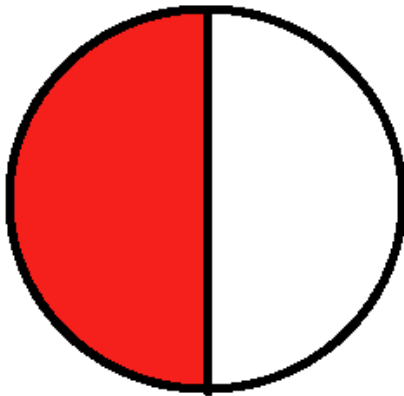
Partition one unit into two equal pieces. Then shade one piece to represent $\frac{1}{2}$.
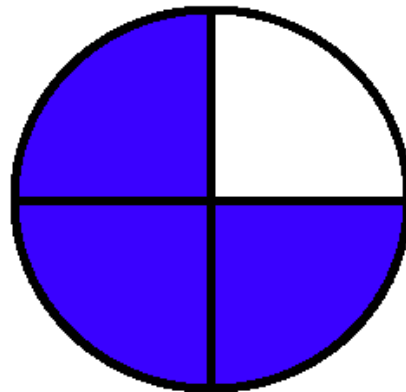
Partition one unit into four equal parts and shade three of the pieces to represent $\frac{3}{4}$.
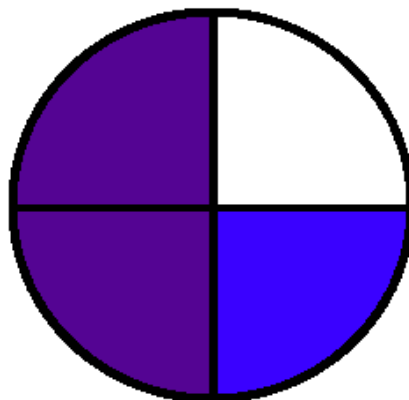
Two of the three blue pieces will cover the red pieces. Thus, two of three, or $\frac{2}{3}$ of the $\frac{3}{4}$ will fit into $\frac{1}{2}$.

Figure 15. Area model for division: Circle

Fraction division can be modeled using a sheet of paper (Figure 16) as a form of the area model, just as fraction multiplication was modeled with paper. This model is created in a similar fashion to the multiplication example by folding a piece of paper into equal size pieces according to the problem under examination (Taber, 2001). It should be noted that this representation is the same general concept as the area model for division, but instead of being drawn, paper is physically folded. Whether drawn or represented in a more concrete way such as folding paper, "modeling plays an important role in students' understanding and visualizing what a division problem is enacting…" (Johanning & Mamer, 2014, p. 350). Through modeling, students may be better able to view problems in symbolic form through a lens that emphasizes the magnitude of the dividend and divisor and be able to better judge whether their solution is reasonable. Figure 16 illustrates how paper may be folded to solve a fraction division problem.

What is $\frac{3}{5} \div \frac{1}{3}$?

$$\frac{3}{5}$$

$$\frac{1}{3}$$
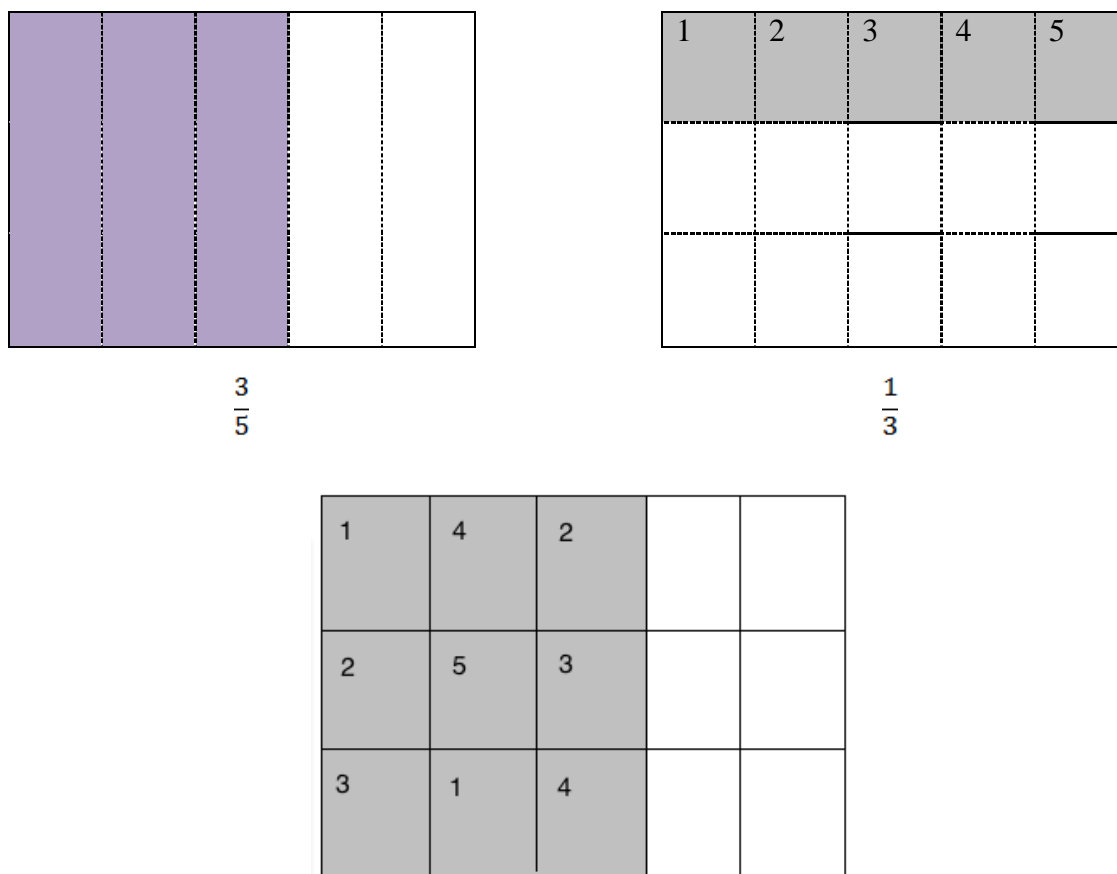
How many one-thirds will fit into $\frac{3}{5}$? Five gray blocks make up $\frac{1}{3}$ of the entire unit. One entire set of these gray blocks and four out of five of a second set of gray blocks will fit into the purple area if we consider the purple area to consist of nine gray blocks. Thus, $\frac{3}{5} \div \frac{1}{3} = 1\frac{4}{5}$.

Figure 16. Paper folding for division

Length Model

Length models differ from area models in that measurements or lengths are compared as opposed to areas. These models aid students in making connections to problems that are linear in context. Cuisenaire rods or strips of paper are often used as length models because different lengths can be identified with different colors and any length can represent the whole (Van de Walle et al., 2008). Also referred to as fraction bars, these models are often utilized to compare fractions (Figure 17). One of the main ideas expressed through the use of fraction bars is that of the unit and how students can compare fraction bars whose entire length is the same and represents the same unit. Fraction bars are an example of models that allow students to clearly see the part in relation to the whole (Lappan et al., 2006). Figure 17 illustrates how Lappan et al. (2006) used fraction bars to compare fractions (p. 9).
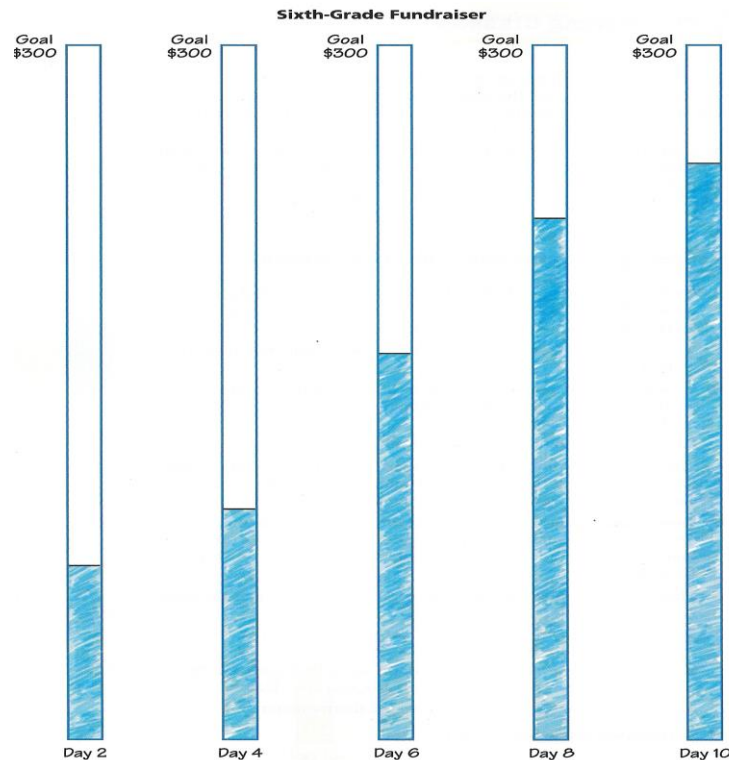
Sixth-Grade Fundraiser

Figure 17. Fraction bars

Fraction bars and fraction strips serve many of the same purposes. Fraction strips are folded strips of paper in which the entire strip represents the whole. These models can be used to show relationships between fractions, compare lengths, and examine equivalent fractions (Figure 18). Students can be asked to 'imagine' folding a strip of paper instead of actually having to do so. This figure shows an example asking students to find equivalent fractions using fraction strips (Lappan et al., 2006, 27).

In Exercises 36–38, paint has spilled on the page, covering part of the fraction strips. Use what is showing to reason about each set of strips. Find the equivalent fractions indicated by the question marks.
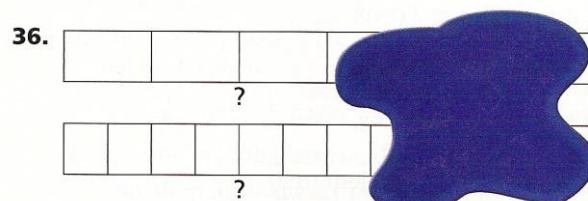
Figure 18. Fraction strips

A number line is another example of a length model. Number lines can be used to demonstrate many operations and should be emphasized in the teaching and learning of fractions (Van de Walle et al., 2008). The number line lends itself nicely to measuring and illustrates that a fraction is a number itself while at the same time showing students its relative size compared to other numbers and sometimes help students 'see' multiplication. For example, the number line can be used to illustrate $\frac{2}{3}$ of $\frac{3}{4}$, as demonstrated by Lannin et al. (2013) (Figure 19). The number line can also be used to show that there is always another fraction between any two given fractions. An advantage of number lines is the ability to deal with real-world situations because measurement is something that students are familiar with and use in their everyday life. Figure 19 illustrates how Lannin et al. (2013) explain the number line model as helping students to 'see' fraction multiplication (p. 139).

Figure 19. Number line model

Fraction Multiplication

Lee et al. (2011) explain that fraction multiplication on a number line can be demonstrated through a comparison to fraction subtraction in an effort to show students how to correctly view units in each problem (Figure 20). In this example, teachers were given a subtraction problem, $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$, and a multiplication problem, $\frac{1}{4} \times \frac{1}{5} = 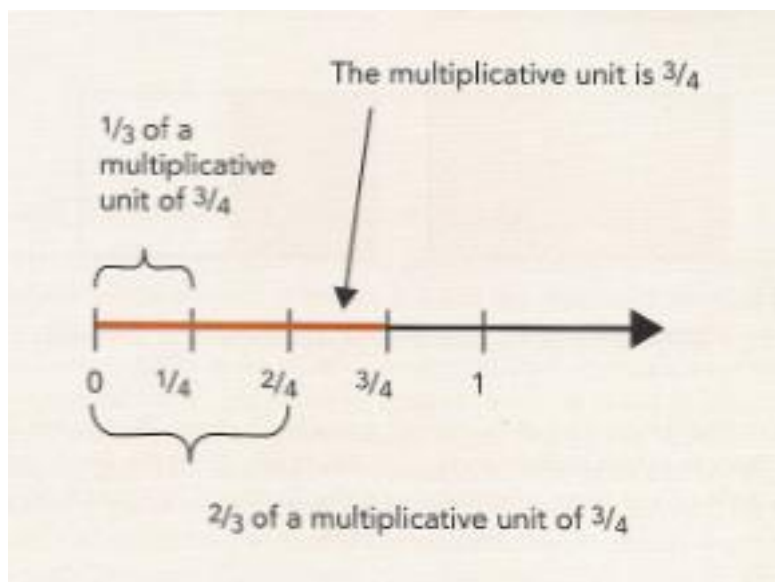\frac{1}{20}$, on separate number lines and asked which number line correctly represents the multiplication problem. In the subtraction number line, part a, the unit for one-fifth was the whole instead of the one-fourth. The number line for fraction multiplication is different because this operation examines a part of a part of a whole. The number line shows one-fourth of a whole, then the one-fourth is divided into five equal size pieces. One of those five pieces is highlighted to represent one-twentieth. Figure 20 illustrates how Lee et al. (2011) explain the use and understanding of number lines for fraction multiplication (p. 209).

FIGURE 1 a. Number-line model of 1/4 – 1/5 = 1/20; b. Number-line model of 1/4 × 1/5 = 1/20.

Figure 20. Number line model for multiplication

Fraction Division

The number line can be used to model measurement fraction division. Lee et al. (2011), found that teachers were not familiar with models for fraction division and were inclined to use the invert-and-multiply algorithm to select a model from given examples that demonstrated the algorithm instead of selecting a model to illustrate the division problem (Figure 21). In their example, $\frac{2}{3} \div \frac{1}{4}$ was viewed as four groups of two-thirds because the problem was changed to multiplication and interpreted as 'groups of'. If the problem had been kept as a division problem, a perspective of 'fit into' may have helped the number line model make more sense (Figure 22). In this case, two-thirds of one unit is shown on the number line. We would be interested in finding how

many one-fourths fit into the two-thirds. So using the same unit, we would divide the unit into quarters and see how many quarters 'fit into' the two-thirds. We can see that two whole quarters and two-thirds of a quarter fit. Thus, $\frac{2}{3} \div \frac{1}{4} = 2\frac{2}{3}$. The number line model here shows students that the resulting unit is different from the starting unit. Figure 21 illustrates how Lee et al. (2011) explain the use and understanding of number lines for fraction division when reliance is placed on the invert-and-multiply algorithm (p. 214).
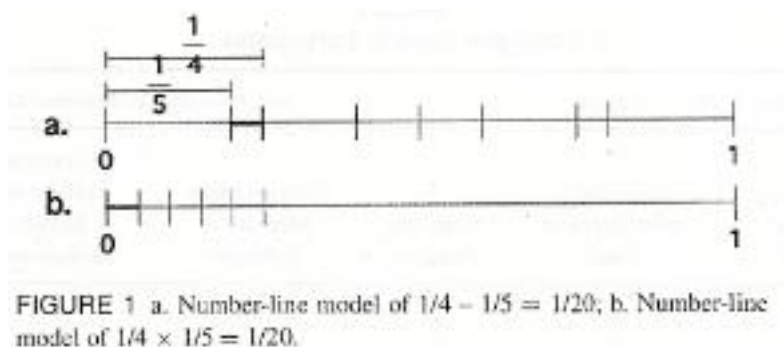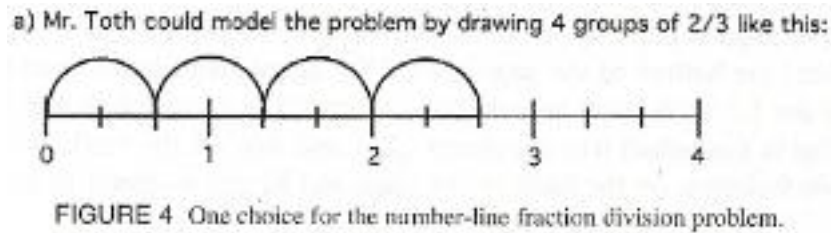

Figure 21. Number line model for division

Figure 22 illustrates another way that the number line can be used for fraction division.
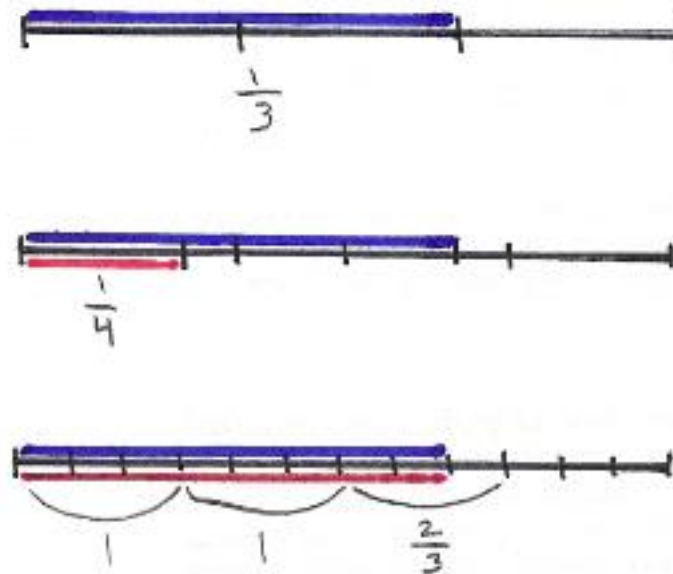
Figure 22. Number line model for division

Set Model

Set models consist of a set of objects where subsets of the whole set represent fractional parts of the whole (Figure 23). For example, if there are ten apples, two of those apples would make up one-fifth of the set of apples. The entire set represents one, the whole. Sometimes using a set of counters or concrete materials to represent one is a difficult concept to grasp. Despite this disadvantage, set models can be very useful when trying to make connections with real-world applications and ratio concepts. It can be helpful to present set models in two colors to show fractional parts (Van de Walle et al., 2008). Figure 23 illustrates how Van de Walle et al. (2008) explain the use of set models (p. 291).
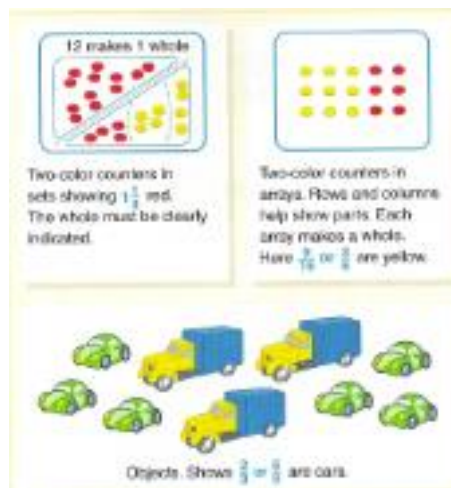
Figure 23. Set models

Fraction Multiplication

Counters, a form of set model, can be used to model fraction multiplication (Figure 24). These models can be especially useful if students are used to using counters, however, these models can cause difficulties. One of the struggles students have with counter models is understanding what is considered to be the whole. Van de Walle et al. (2008) recommend that students not be discouraged from using counter models, but that teachers should be ready to aid students when they are trying to determine the whole. In Van de Walle's example, two red counters and one yellow counter show two-thirds. Since the numerator, two, cannot be partitioned into five parts, we can use more than one group of these counters. If we have groups of three and also need to look at fifths, a multiple of three that is also divisible by five is fifteen. So we can use five groups of these counters to represent one set, or one whole unit. The red counters represent the two-thirds. Three-fifths of the total red counters is six red counters. Six out of fifteen counters total show that $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$. Figure 24 illustrates how counters may be used to solve a fraction multiplication problem (Van de Walle et al., 2008, p. 319).
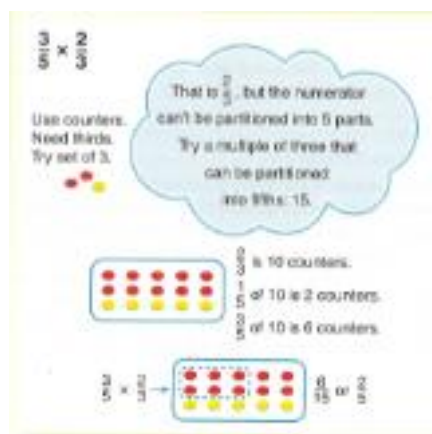
Figure 24. Counter model for multiplication

Invert-and-Multiply Algorithm

Computational algorithms are sometimes not taught in a way that encourages students to think about operations and what is actually happening as the problems are completed. "When students follow a procedure they do not understand, they have no means of assessing their results to see if they make sense" (Van de Walle et al., 2008, p. 310). Memorizing the computational algorithms for fraction multiplication and division does not necessarily lead to learning with understanding and often times these algorithms are forgotten. For example, students ask whether or not a common denominator is needed or which fraction needs to be inverted. Van de Walle et al. suggest students may build number sense and learning for understanding by learning fractions through contextual tasks, connecting the meaning of fraction computation with whole number computation, estimation

and informal methods, and exploration of operations using models. Using models will hopefully provide students with a solid background as they make the progression to computational algorithms.

As shown in the many examples of models for fraction division, most models demonstrate the common denominator algorithm. The invert-and-multiply algorithm can be more difficult to understand using these models. The invert-and-multiply algorithm can be explained through the use of an area model. If we consider $4 \div \frac{2}{3}$, we can imagine that a good first step would be to cut each of our four wholes into thirds. We would then need to consider how many sets of two can be counted (Figure 25). Instead of considering four pieces, we would really be looking at twelve (4 x 3 = 12), then we would find that there are six sets of two (12 ÷ 2 = 6). So we were essentially multiplying by the denominator and dividing by the numerator, hence, the invert-and-multiply algorithm. The area model is a very useful tool for students to see how the invert-and-multiply algorithm works and "highlights the role that meanings for whole-number operations play in developing understanding of a computational algorithm that traditionally has been taught with no justification" (Cavey & Kinzel, 2014, p. 383). Figure 25 illustrates how an area model may be used to demonstrate the invert-and-multiply algorithm.
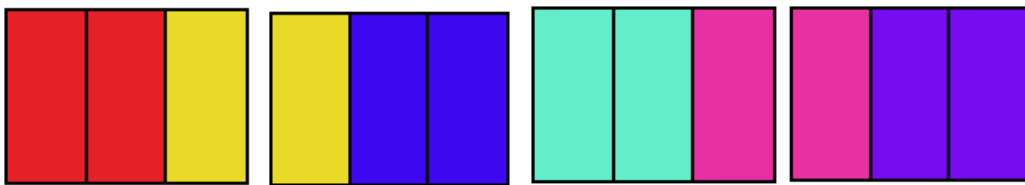
Figure 25. Area model for invert-and-multiply algorithm

We can also illustrate the invert-and-multiply algorithm using a unit rate interpretation of multiplicative inverses. $1 \div \frac{1}{4} = 4$ because there are four quarters in one unit. It follows that $4 \times \frac{1}{4} = 1$. The multiplicative inverse of a natural number n is $\frac{1}{n}$, which is how much of n there is in one unit. Cavey and Kinzel (2014) explain that we can use this to consider fraction division. For example, if we need to find how many $\frac{2}{3}$ yard bows we can make using 15 yards of ribbon, $15 \div \frac{2}{3}$ can be viewed through the lens of multiplicative inverse. The multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$. So for each unit (i.e. yard), there are $1\frac{1}{2}$ strips of ribbon that are $\frac{2}{3}$ yard in length (Figure 26). Since there are 15 units total and $1\frac{1}{2}$ strips of ribbon in each, we can simply multiply $15 \times 1\frac{1}{2}$ to obtain the solution (Figure 27). In viewing fraction division from this perspective, "…students have developed meaning for invert-and-multiply that is based on reasoning with unit rates" (p. 382). Figure 26 illustrates how Cavey and Kinzel (2014) explain multiplicative inverse (p. 382).

Figure 26. Multiplicative inverse

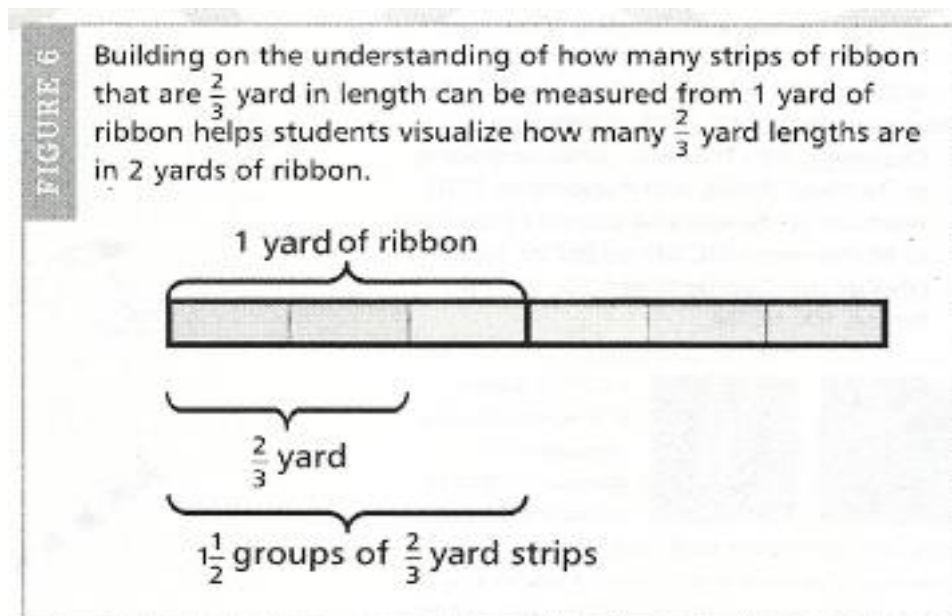Figure 27 illustrates how a number line model may be used to demonstrate the invert-and-multiply algorithm (Cavey & Kinzel, 2014, p. 382).
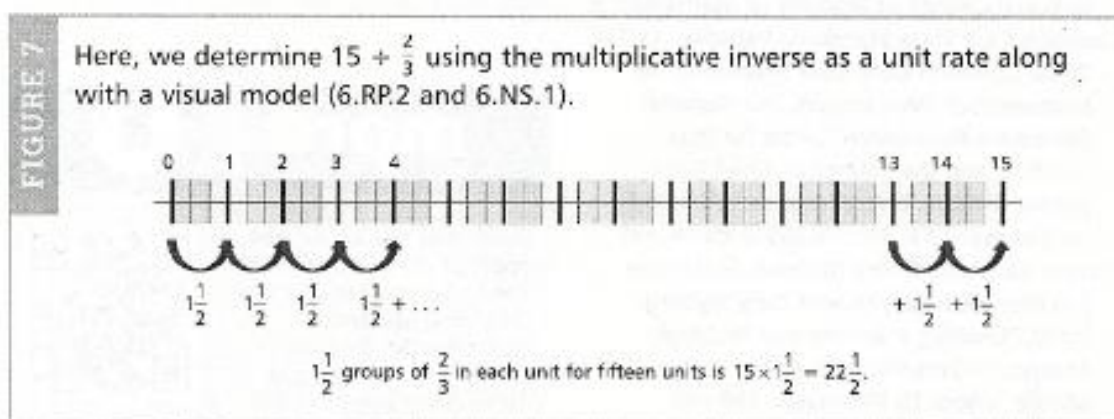
Figure 27. Number line model for invert-and-multiply algorithm

As shown by the various examples above, there are many different models that can be used to help students and teachers, at the elementary/middle and college levels, gain a deeper understanding of fraction multiplication and division. Area models in particular seem to help students be able to make sense of fractions, fraction operations, and commonly used algorithms. The area model allows students to build understanding of fraction multiplication based on prior knowledge of multiplication with whole numbers (Tsankova and Pjanic, 2009) and encourages students to think about the process of fraction multiplication (Pagni, 1999). In looking at fraction multiplication and division using the area model, students may become more flexible in their thinking and be able to apply their understandings in problem solving. However, it should be noted that the area model may not provide the best illustration for the invert-and-multiply algorithm. The invert-and-multiply algorithm is probably better represented using a length model such as the number line.

Why Are These Models Important?

According to Siegler et al. (2010), many students in the United States do not possess the mathematical skills necessary to pursue a career in the science, technology, engineering, or mathematics (STEM) fields and this may be attributed in part to a poor understanding of fractions. Fractions are essential for the understanding of algebra (Brown & Quinn, 2007; Siegler et al., 2010; Son, 2011). In particular, a solid understanding of fractions

through the use of diagrams and other visual representations can be important in ratio, rate, and proportion problems, which are important when learning algebra (p. 9). Son (2011) highlights the importance of fractions in elementary school mathematics by explaining that fractions enable students to perform computations but more importantly, fractions allow students to later work with rates, percents, slope, and other topics in secondary school. Prediger (2011) suggests that knowledge of fractions and fraction operations may help students overcome challenges related to word problems and increase their sense-making abilities.

Algebra

Much of algebra is based on rational number concepts and the ability to work with and manipulate fractions. For instance, proportion is elemental to the rational number concept. There are many mathematical topics that are related to proportions in addition to those previously mentioned, such as decimals, ratios, probability, and linear functions. All of these topics depend on the knowledge of fractions. If students have a gap in knowledge regarding rational numbers, they will likely have gaps that become greater and more obvious in courses like algebra (Brown & Quinn, 2007). Algebra instruction typically includes fractional notation to indicate a quotient. Due to this aspect, Wu (2001) argues that fractions are the best pre-algebra practice available and that fluency in fractions is of paramount importance to gain a meaningful understanding of algebra. Wu stresses that rational expressions could cause great difficulty in algebra if students do not have a strong foundation in fractions. A study by Laursen (1978) showed that first-year algebra students tended to make errors that could be attributed to an incomplete understanding of fraction operations and algorithms (Brown & Quinn, 2007). Siegler et al. (2010) believe that this lack of conceptual understanding may be a combination of different misunderstandings. The inability to view fractions as numbers, focusing on numerators and denominators separately, and confusing fraction properties with whole number properties (i.e. there is no other fraction between $\frac{3}{5}$ and $\frac{4}{5}$ because there is no other whole number between 3 and 4) are a few examples of common misconceptions (p. 7). Correcting these misunderstandings through a more careful and in-depth concentration on fractions and fraction operations could help to alleviate some student struggles in algebra. "From linear equations to completing the square, from solving systems of linear equations to solving rational equations,…algebra is replete with examples that are directly and indirectly related to fractions" (Brown & Quinn, 2007, p. 29). Thus, fractions are an important building block for student success in algebra.

Teacher Effectiveness

Understanding of fraction models is important to teachers and students alike. If teachers struggle with a concept, they are not likely to be able to teach that topic effectively. According to Ball (1990), preservice teachers have great difficulty understanding division of fractions, as this is a topic that is generally not taught conceptually. In Ball's study most participants "…were able to consider division in partitive terms only; forming a certain number of equal parts. This model of division corresponds less easily to division with fractions…" (p. 140) than a measurement model. Tirosh and Graeber (1990) obtained similar results in a study that considered preservice teachers' misconceptions of division. This limitation in flexibility can challenge preservice teachers when confronted with a task that requires more than reproduction of previously taught material. Thus, a shallow understanding of fraction multiplication or division may allow preservice teachers to correctly solve a problem using a procedure, but they will probably not be able to generate an accurate model for the statement. Ball (1990) refers to this type of knowledge as "rule-bound and compartmentalized" (p. 141). A deeper understanding of these concepts through models can increase understanding and better teachers' teaching.

Making Connections

There are many concepts that comprise fraction multiplication. For example, teachers must make sure that students have the opportunity to make connections with whole number concepts. It may be helpful for teachers to take time to explore whole number multiplication before moving on to fraction multiplication. For example, one meaning of 2 x 3 is that there are two groups of three. Teachers could use this understanding as a starting point for working on problems such as $2\frac{1}{2}$ x $3\frac{1}{5}$ (Figure 28) (Van de Walle et al., 2008, p. 317). This figure illustrates how $2\frac{1}{2}$ x $3\frac{1}{5}$ can be viewed as $2\frac{1}{2}$ groups of $3\frac{1}{5}$. There are two whole groups of $3\frac{1}{5}$. The third set of

circles shows another group of $3\frac{1}{5}$ where each piece is cut into two pieces. This represents one half of a group if we count only one of every two pieces. Thus, there are $3\frac{1}{5} + 3\frac{1}{5} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{10} = 8$. So $2\frac{1}{2}$ x $3\frac{1}{5} = 8$.
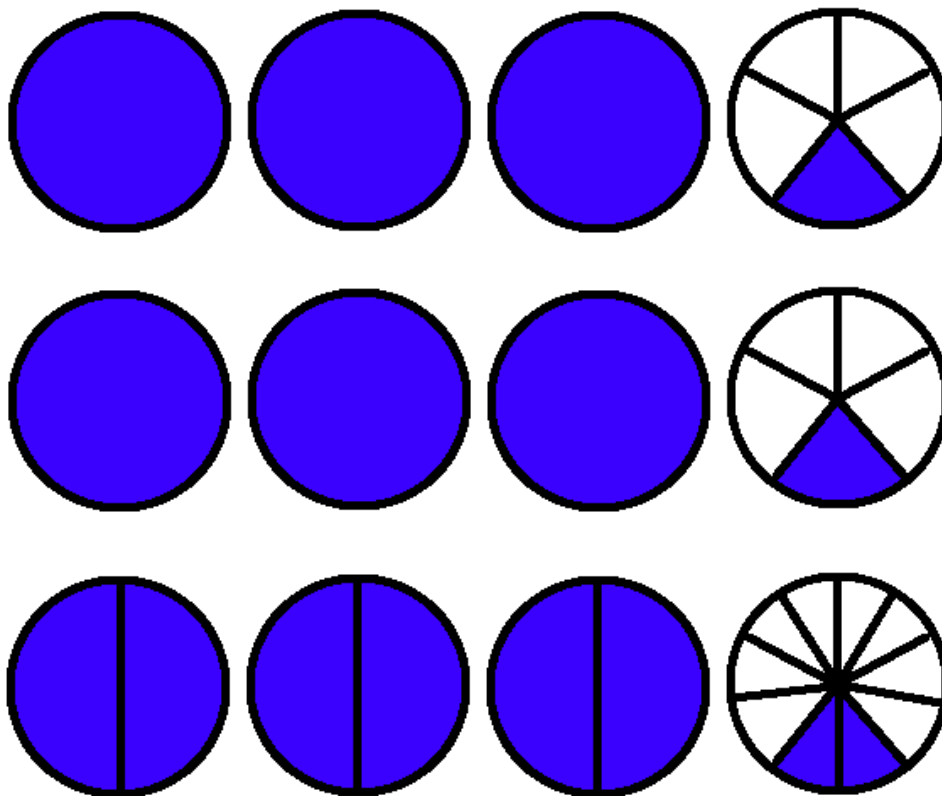

Figure 28. Fraction multiplication as groups (set model)

When teaching fraction multiplication, different types of models can, and should, be utilized. Area bars are good models to demonstrate multiplication because they are two-dimensional representations, but they are not the only models that can be used. Area models, discrete models, and number lines are just a few examples of different models that can be used for multiplying fractions (Van de Walle et al., 2008). Teachers should make an effort to make connections between these models and the standard algorithm for fraction multiplication so that students will be able to eventually work problems without having to draw a picture. In examining different ways to represent this operation, teachers will be better able to understand students' comprehension of fraction multiplication.

"Invert the divisor and multiply is probably one of the most mysterious rules in elementary mathematics. We want to avoid this mystery at all costs" (Van de Walle et al., 2008, p. 321). It is ideal for preservice elementary and middle school teachers to be comfortable viewing fraction division problems as problems that are asking 'how many will fit into'. For example, if Lily needs $2\frac{1}{2}$ cups of sugar but only has a $\frac{1}{4}$ cup measuring cup, how many $\frac{1}{4}$ cups will she need to get the desired amount? In other words, how many $\frac{1}{4}$ cups will fit into $2\frac{1}{2}$ cups? Although this method helps students to gain a better understanding of one of the interpretations of what division by fractions is, it is still not enough to provide a deep understanding of fraction division. Teachers should be able to find patterns and draw conclusions based on what they see. What is different about $5 \div \frac{1}{4}$ and $5 \div \frac{3}{4}$? It is helpful for preservice elementary and middle school teachers to be able to clearly explain that there is a shift in how you view the unit in the second problem (Figure 29). The group of three is the reconceptualized unit. Barnett-Clarke et al. (2010) highlight in Big Idea 2 that being able to make sense of the multiple interpretations of rational number depends on the ability to identify the unit. Knowledge of this type will enable teachers to have a good foundation for the reasoning behind the invert-and-multiply algorithm. My example shows the multiplication pictorially for both problems, however, it would be useful to see if students evolve and can solve $5 \div \frac{3}{4}$ differently after solving $5 \div \frac{1}{4}$ pictorially. Figure 29 illustrates how a problem with 'one' as the numerator in the divisor differs from a division problem with a different number as the numerator in the divisor.

What is $5 \div \frac{1}{4}$?
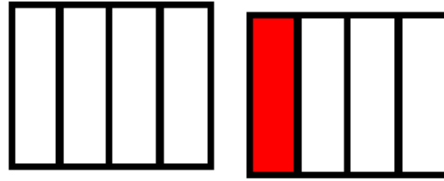
Partition one unit into four equal pieces because we want to 'fit' $\frac{1}{4}$ into a unit of this size. Then shade one of the four pieces so we can see what $\frac{1}{4}$ looks like in comparison to the unit.

The problem asks for $5 \div \frac{1}{4}$ so we need to consider five whole units and see how many one-fourths can fit into five units. If four pieces fit into one unit, then we can reason that twenty pieces will fit into five units. Thus, $5 \div \frac{1}{4} = 20$.

What is $5 \div \frac{3}{4}$?

Partition one unit into four equal pieces because we want to 'fit' $\frac{3}{4}$ into a unit of this size. Then shade three of the four pieces so we can see what $\frac{3}{4}$ looks like in comparison to the unit.
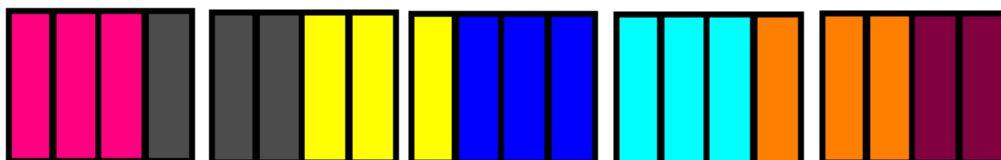
The problem asks for $5 \div \frac{3}{4}$ so we need to consider five whole units and see how many three-fourths can fit into these five units. Now the group of three pieces is one unit to be fit into the rectangles. We see from the diagram that six groups of three and two more of a group of three can fit into the five rectangles. Thus, $5 \div \frac{3}{4} = 6 \frac{2}{3}$.

Figure 29. Comparison of fraction division problems

One of the main differences in these two problems is that in the second problem, the divisor did not fit nicely into the original unit. A numerator not equal to one in the divisor led us to need to consider pieces fitting across multiple unit rectangles, not just within the same rectangle. Both problems allowed us to divide the rectangle into fourths, but the second problem made us consider groups of bars within the rectangle instead of one bar at a time. This can be a difficult concept for students of any age to grasp conceptually.

Explaining Misconceptions

According to Lamon (2007), students have difficulty with the multiplication and division of fractions because they are in the habit of multiplication making something *bigger* and division making something *smaller*. Rizvi and Lawson (2007) had the same findings in their study of preservice teachers. Their participants' performance indicated that prospective teachers do not possess a solid understanding of division of fractions. Their existing knowledge about the concepts presented to them was not strong enough for them to complete the division problems for themselves or to help students learn and understand how to solve division problems. Preservice teachers that possess a deep conceptual understanding of fraction multiplication and division will probably be better able to teach their own students fraction multiplication and division.

References

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal   for Research in Mathematics Education, 21*(2), 132-144.

Barnett-Clarke, C., Fisher, W., Marks, R., & Ross, S. (2010). *Developing essential understanding of rational numbers grades 3-5.* R. I. Charles, & R. M. Zbiek (Eds.). Reston, VA: The National Council of Teachers of Mathematics.

Brown, G., & Quinn, R. J. (2007). Fraction proficiency and success in algebra: What does research say? *Australian Mathematics Teacher, 63*(3), 23-30.

Bruner, J. (1966). *Toward a theory of instruction* (Chapter 1, pp. 1-21). New York: W.W. Norton.

Cavey, L. O., & Kinzel, M. T. (2014). From whole numbers to invert and multiply. *Teaching Children Mathematics, 20*(6), 374-383.

D'Ambrosio, B. S., & Mendonga Campos, T. M. (1992). Pre-service teachers' representations of children's understanding of mathematical concepts: Conflicts and conflict resolution. *Educational Studies in Mathematics, 23*(3), 213-230.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W, G, Martin, & D, Schifter (Eds.), *A research companion to Principles and standards for school mathematics* (pp. 275-285). Reston, VA: National Council of   Teachers of Mathematics.

Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning   and   doing mathematics. In L. P. Steffe & P. Nesher (Eds.) *Theories of mathematical learning* (pp. 397-430). Mahwah, NJ: Lawrence Erlbaum Associates.

Gregg, J., & Gregg, D. U. (2007, August). Interpreting the standard division algorithm in a "candy factory" context. *Teaching Children Mathematics, 14*(1), 25-31.

Johanning, D. I., & Mamer, J. D. (2014, February). How did the answer get bigger? *Mathematics Teaching in the Middle School, 19*(6), 344-351.

Kalathil, R. R., & Sherin, M. G. (2000). Role of students' representations in the mathematics classroom. In B. Fishman & S. O'Connor-Divelbiss (Eds.), *Fourth international conference of the learning sciences* (pp. 27-28).

Krach, M. (1998). Teaching fractions using manipulatives. *Ohio Journal of School Mathematics, 37*, 16-23.

Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Publishing.

Lannin, J., Chval, K., & Jones, D. (2013). *Putting essential understanding of multiplication and   division   into practice 3-5.* K. Chval, & B. J. Dougherty (Eds.). Reston, VA: The National Council of Teachers   of Mathematics.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006). *Bits and pieces   I: Understanding fractions, decimals, and percents (Connected Mathematics 2).* Boston, MA: Pearson Prentice Hall.

Lee, S. J., Brown, R. E., & Orrill, C. H. (2011). Mathematics teachers' reasoning about fractions and decimals using drawn representations. *Mathematical Thinking and Learning, 13*(3), 198-220.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. New York, NY: Routledge.

National Council of Teachers of Mathematics. (2007). *Second handbook of research on mathematics teaching and learning.* F. K. Lester, Jr. (Ed.). Charlotte, NC: Information Age Publishing.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Pagni, D. (1999, March). Giving meaning to multiplication and division of fractions. *Australian Mathematics Teacher, 55*(1), 11-13.

Perlwitz, M. D. (2005). Dividing fractions: Reconciling self-generated solutions with algorithmic answers. *Mathematics Teaching in the Middle School, 10*(6), 278-283.

Petit, M. M., Laird, R., & Marsden, E. (2010, August). They "get" fractions as Pies; Now what? *Mathematics Teaching in the Middle School, 16*(1), 5-10.

Prediger, S. (2011, July). Why Johnny can't apply multiplication? Revisiting the choice of operations with fractions. *International Electronic Journal of Mathematics Education, 6*(2), 65-88.

Rizvi, N. F., & Lawson, M. J. (2007). Prospective teachers' knowledge: Concept of division. *International Education Journal, 8*(2), 377-392.

Rule, A. C., & Hallagan, J. E. (Eds.). (2006). Preservice elementary teachers use drawings and make sets of materials to explain multiplication and division of fractions. Proceedings from PMET '06: *The 2nd Annual Preparing Mathematicians to Educate Teachers Conference.* Oswego, NY.

Rule, A. C., & Harrell, M. H. (2006). Symbolic drawings reveal changes in preservice teacher mathematics attitudes after a mathematics methods course. *School Science and Mathematics, 106*(6), 241-258.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Siegler, R. P., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance.

Son, J. W. (2011). A global look at math instruction. *Teaching Children Mathematics, 17*(6), 360-368.

Taber, S. B. (2001). Making connections among different representations: The case of multiplication of fractions. Proceedings from *The Annual Meeting of the American Educational Research Association.* Seattle, WA.

Tirosh, D., & Graeber, A. O. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98-108.

Tsankova, J. K., & Pjanic, K. (2009). The area model of multiplication of fractions. *Mathematics Teaching in the Middle School, 15*(5), 281-285.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2008). *Elementary and middle school mathematics: Teaching developmentally.* Boston, MA: Allyn and Bacon.

Wentworth, N. M., & Monroe, E. E. (1995). What is the whole? *Mathematics Teaching in the Middle School, 1*(5), 356-360.

Wu, H. H. (2001). How to prepare students for algebra. *American Educator, 25*(2), 10-17.

Zazkis, R. & Liljedahl, P. (2004). Understanding primes: The role of representation. *Journal for Research in Mathematics Education, 35*(3), 164-186.

Author Information

Heather K. Ervin
Bloomsburg University
400 East Second Street
Bloomsburg, PA 17815. U.S.A.
hervin@bloomu.edu