Rationale for the Definition of the Particular Solution to an Initial Value Problem: A Unique Solution Is Guaranteed

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ABSTRACT

The purpose of this article is to examine the reasoning behind the wording of the definition of the particular solution to an initial value problem. This article will be of practical importance for students taking a first year calculus course that includes the study of first order linear separable differential equations.

At times, students view definitions (statements of meaning) as being the same as theorems (propositions that can be proved). An accurate nexus between these two types of statements is revealed in the fact that theorems often influence the content of definitions.

This relationship is observed in the common definition of the particular solution to an initial value problem (IVP) and in the Existence-Uniqueness Theorem. An excellent opportunity to help students appreciate this presents itself in AB5 from the 2006 AP Exam, whose solution to part (b) is demonstrated below.

2006-AB5:

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1+y}{x}$, where $x \neq 0$.

b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

Separating the variables and integrating yields:

$$\frac{dy}{1+y} = \frac{dx}{x} \rightarrow \int \tfrac{dy}{1+y} = \int \tfrac{dx}{x} \rightarrow \ln|1+y| =$$

$$\ln|x| + C_1 \rightarrow |1+y| = e^{C_1}|x| \overset{C=e^{C_1}}{\rightarrow} |1+y| = C|x|$$

Applying the initial condition results in:

$$f(-1) = 1 \rightarrow |1+1| = C|-1| \rightarrow 2 = C \rightarrow |1+y| = 2|x|$$

This stage provides an opportunity to emphasize one aspect of the definition of the solution to an IVP; namely, that it must be a function. Our result up to this point does not meet this criterion, it being a one-to-many relation. Two explicit functions implied by this relation are $y = 2x - 1$ and $y = -2x - 1$. It's clear that the second function, $y = -2x - 1$, satisfies the initial condition. We now turn our attention to correctly stating the domain.

To do this, one needs to know the complete definition of a solution to an IVP. As presented in some textbooks, the common definition of the particular solution to an initial value problem "is that of a differentiable function on an open interval that contains the initial x-value." (Lomen, par. 6)

Based on this definition, it might seem that the particular solution is $y = -2x - 1$, with the domain $-\infty < x < \infty$. Upon examination of the corresponding slope field, however, it becomes clear that this open interval is not correct. The reason is that $y = -2x - 1$ has a derivative of -2 at $x = 0$, but our original differential equation is undefined at $x = 0$. It can easily be shown that the general solution to this differential equation is $y = Cx - 1$. As such, if $x = 0$ then $y = -1$, turning dy/dx into the indeterminate form 0/0. (Riddle, pars. 15 & 16)

Indeed, a solution curve must not contradict or go beyond what the original differential equation generates in its corresponding slope field. (See Figure 1)

Over the years, this has led some of my students to assert that the solution to this IVP is $y = -2x - 1,\ x \neq 0$. (In fact, this was a very popular incorrect response back in 2006, perhaps for the simple reason of student confidence in imposing the same domain restriction of the differential equation on the particular solution.)

It's at this juncture that I find students questioning why the definition stipulates an open interval that contains the initial condition. They are certainly able to follow this definition and arrive at the "correct" answer of $y = -2x - 1,\ x < 0$. They generally have no trouble remembering that, when facing a discontinuity, to choose the "side" that contains the x-value of the initial condition. But it is here that students ask *why* this is so and then hopefully realize that a definition provides no help in answering such a question.

The reasons students give for asking "Why?" center around the fact that the function $y = -2x - 1$with its domain restriction $x \neq 0$ harmonizes well with the slope field that the original differential equation generates. Truth be told, this alleged solution y = -2x – 1, $x \neq 0$ does identically satisfy the given differential equation everywhere it is defined. As for the discontinuity at $x = 0$, students recall that if a function is not continuous at a location, then it is not differentiable there. The non-existence of the derivative at $x = 0$ is also implied by the differential equation itself. (See Figure 2)

Nevertheless, the definition of a particular solution to an IVP implies that it must be a continuous function (a consequence of it being a differentiable function) and that "we may not extend a solution across a discontinuity, even if the resulting function formerly satisfies the differential equation on the other side of the discontinuity." (Lomen, par. 16)

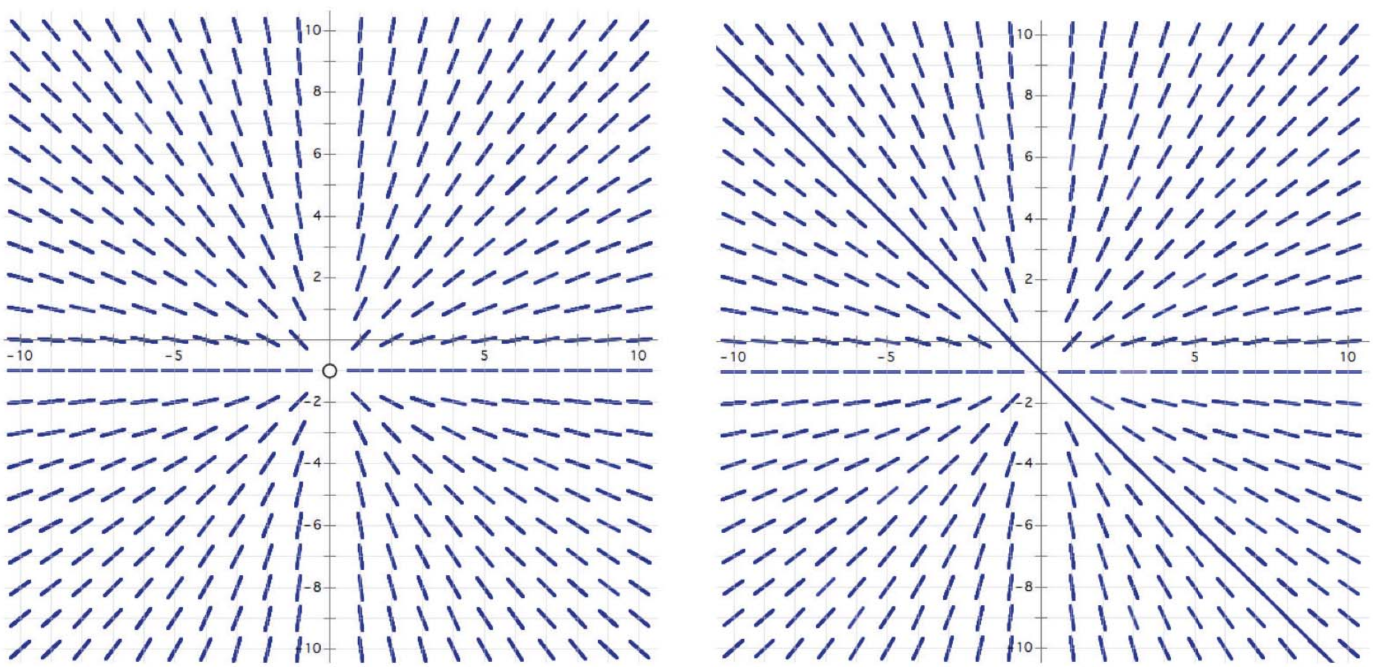

Figure 1. Corresponding slope field of given IVP alongside an incorrect solution.

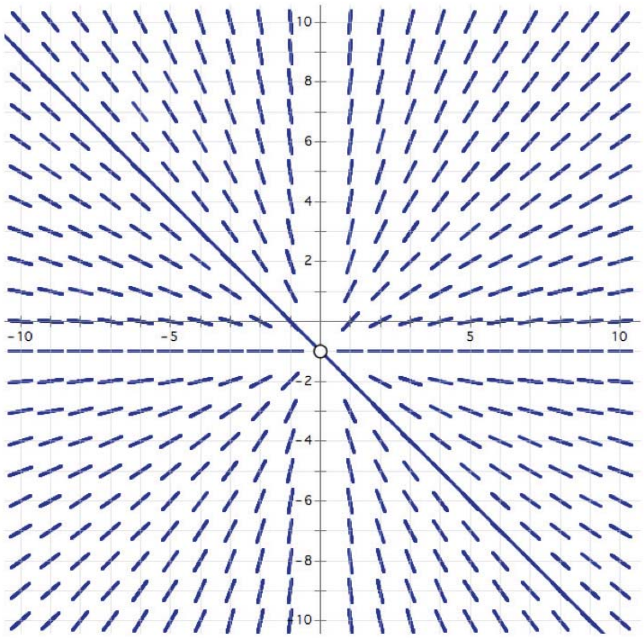

Figure 2. The function y = -2x – 1, matches the slope defined by the given differential equation and has no derivative at x = 0.

The 2006-AB5 problem provides a wonderful opportunity to explain the underlying reasons behind such a definition. These reasons can be found in the Existence-Uniqueness Theorem which is given below.

(Please note that the fact that differential equations are used to model real-life situations is another way to help students appreciate this definition. This is discussed in the article, "The Domains of Solutions to Differential Equations" by Larry Riddle, available on apcental.collegeboard.com)

The Existence-Uniqueness Theorem: If $g(x,y)$ and $\partial g/\partial y$ are defined and continuous in a finite rectangular region containing the point (x_0,y_0) in its interior, then the differential equation $y'=g(x,y)$ has a unique solution passing through the point $y(x_0)=y_0$. This solution is defined for all x for which the solution remains inside the rectangle. (Lomen and Lovelock 56)

This implies that there is an open interval inside the rectangular region and containing x_0 on which there is a unique solution.

This theorem guarantees a unique solution through the initial condition (x_0,y_0) "provided that $g(x,y)$ and $\partial g/\partial y$ are continuous for all points near (x_0,y_0)." (Lomen and Lovelock 56)

For the problem in question $g(x,y)=\dfrac{1+y}{x}$ and $\dfrac{\partial g}{\partial y}=\dfrac{1}{x}$.

We are guaranteed that a unique solution exists on an interval containing $x = -1$, provided that

$g(x,y)=\dfrac{1+y}{x}$ and $\dfrac{\partial g}{\partial y}=\dfrac{1}{x}$ are continuous on a closed finite rectangle in the *xy*-plane that contains the point $(-1,1)$. All points that meet this criteria have $x < 0$, which is the correct domain of the particular solution $y = -2x - 1$ since our analytic work led to no other formula or rule when $x<0$ and $y>-1$.

If we were to include values of x in the domain of the particular solution where either $g(x,y)$ or $\partial g/\partial y$ is not continuous, we run the risk of the possibility of more than one solution existing on this interval. Under these circumstances, "a solution may or may not exist, and if one exists, it may or may not be unique." (Lomen and Lovelock 57)

Splicing intervals together in an attempt to avoid values of x for which $g(x,y)$ and/or $\partial g/\partial y$ are discontinuous cannot guarantee the elimination of this potential ambiguity. If we were to apply this less restrictive definition of the particular solution to 2006-AB5, it quickly becomes clear that more than one function can be defined as we move past the trouble spot of $x = 0$. These functions and their domains are listed below. Please note that all of these functions satisfy the given differential equation and pass through the given initial condition.

Function 1:

$$y=-2x-1, \text{Domain } (-\infty,0)\cup(0,\infty)$$

See **Figure 2** above with a removable discontinuity at the point (0, -1)

Function 2:

$$y=\begin{cases} -2x-1, & x<0 \\ 2x-1, & x>0 \end{cases}$$

Function 3:

$$y=\begin{cases} -2x-1, & x<0 \\ 2x-1, & x\geq 0 \end{cases}$$

Please note the following:

1. All three of these functions harmonize with the slope field generated by the original differential equation.

2. The derivative at $x = 0$ does not exist for any of these

functions. (Although Function 3 is continuous, this function has a corner at $x = 0$ so the derivative does not exist.)

3. The function and domain in common to all three functions is $y = -2x - 1$, $x < 0$.

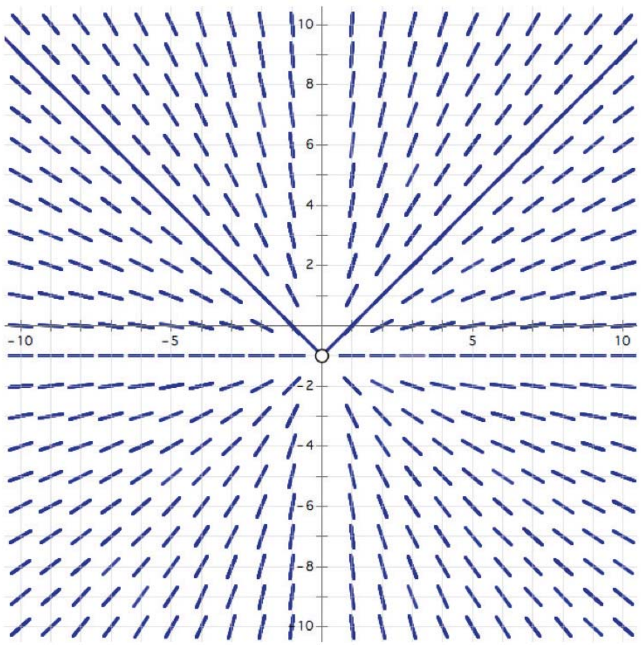

Figure 3. Graph of Function 2 in the slope field with a removable discontinuity at the point (0, -1).

The function $y = -2x - 1$, $x < 0$ is the solution to this IVP because it is the only *unique* function that contains the initial condition on the said domain. Put simply, the wording of the definition of a particular solution to an IVP guarantees the uniqueness of the solution if one exists. Furthermore, it can easily be shown that, if the uniqueness of a solution cannot be established with a given initial condition, then that particular IVP has no solution.

Consider how this is so with the following IVP:

$$\frac{dy}{dx} = \frac{1+y}{x},\ y(0) = -1$$

If we were to plod along and ignore the fact that zero is not in the domain of the natural logarithmic function, and that the separation of variables method is valid only if $x \neq 0$ and $1 + y \neq 0$, we would still be confronted with the fact that this IVP has no unique solution. The following work emphasizes that the ambiguity inherent in this problem cannot be avoided:

$$\frac{dy}{dx} = \frac{1+y}{x},\ y(0) = -1 \rightarrow \frac{dy}{1+y} = \frac{dx}{x} \rightarrow \ln|1+y| =$$

$$\ln|x| + C_1 \rightarrow |1+y| = e^{C_1} \cdot e^{\ln|x|}$$

$$\text{Let } C = e^{C_1} \rightarrow |1+y| = C|x| \rightarrow \text{applying initial}$$

$$\text{condition} \rightarrow |1+(-1)| = C|0| \rightarrow |0| = C|0|$$

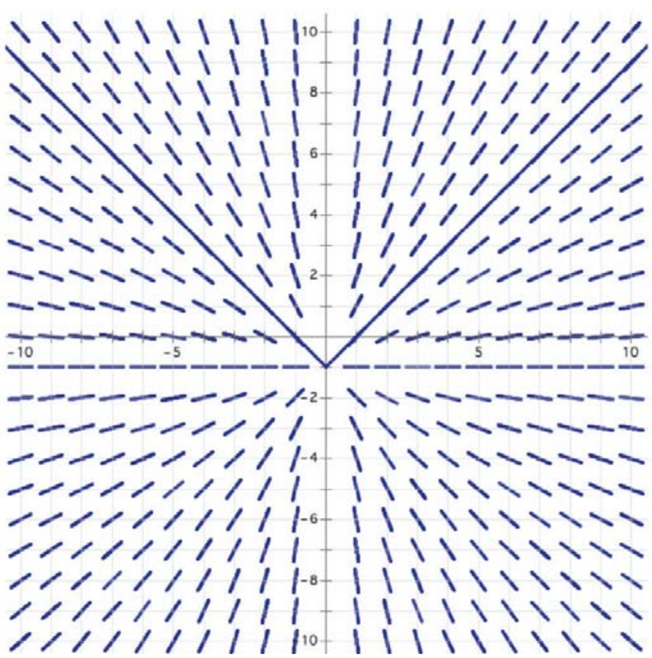

Figure 4. Function 3 in the slope field

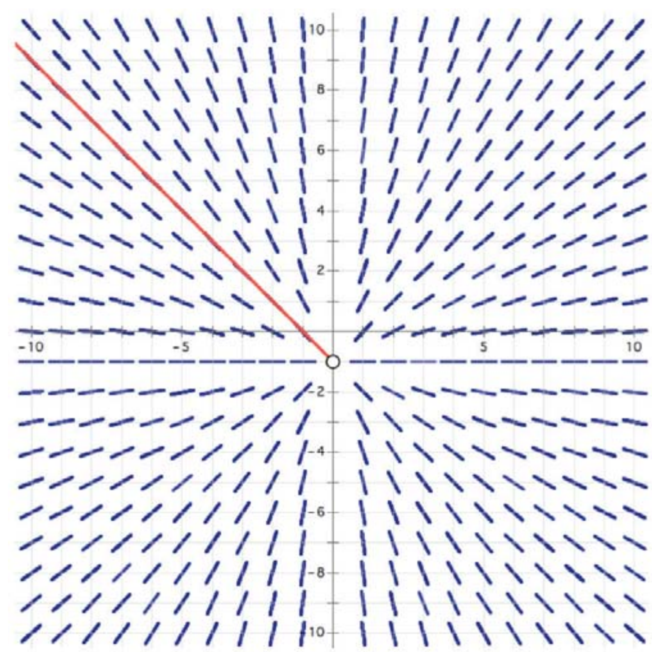

Figure 5. y = -2x – 1, x < 0 is the function held in common with Functions 1-3. It also has the point (-1, 1).

This last statement implies that C can be any real number we wish, creating an endless list of potential solutions on either side of the line $y = -1$.

Another classic example of an IVP that does not have a solution is:

$$g(x,y)=\frac{dy}{dx}=\frac{1}{y}, y\neq 0,\ y(0)=0$$

This differential equation is identically equal to the slope of the tangent line to the integral curves that result when finding the family of solutions (except, of course, when $y = 0$, where this derivative is not defined). It can be verified using the corresponding slope field that, as solution curves approach the x-axis, the tangent lines become vertical, indicating that no numerical slope exists there. When considering the initial condition (0, 0) given in this IVP, we note that both $g(x,y)=\frac{dy}{dx}=\frac{1}{y}$ and $\frac{\partial g}{\partial y}=-\frac{1}{y^2}$ are not continuous in any finite rectangular region containing this point. This alerts us to the possibility that a unique solution may not be guaranteed in this case. Applying the method of separation of variables, along with the initial condition, confirms this suspicion:

$$g(x,y)=\frac{dy}{dx}=\frac{1}{y}\rightarrow ydy=dx\rightarrow\frac{y^2}{2}+C_1=$$

$$x+C_2\overset{C_3=C_2-C_1}{\rightarrow}\frac{y^2}{2}=x+C_3\overset{C=2C_3}{\rightarrow}y^2=2x+C$$

$$\rightarrow y=\pm\sqrt{2x+C}\overset{y(0)=0}{\rightarrow}y=\pm\sqrt{2x}$$

As can be seen from the graph of the two functions (See Figure 6) contained in the last equation, each one of them has an equal claim to the given initial condition. Since we have no basis to choose one of these functions over the other, we are faced with an intractable ambiguity, which means that this IVP has no unique solution.

In summary, students should be helped to appreciate the following:

1. The definition of a solution to an IVP as presented in textbooks is designed, in part, to avoid ambiguous cases.

2. If either $g(x,y)$ or $\partial g/\partial y$ are not continuous for all points near the initial condition, then a unique solution is not guaranteed to exist.

3. Some IVPs may not have any solution due to unavoidable ambiguities.

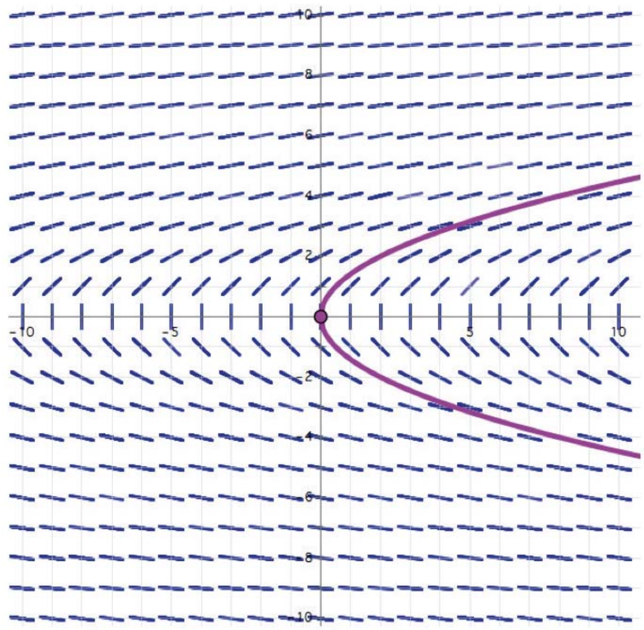

Figure 6. The initial condition (0, 0) creates an ambiguity that cannot be resolved.

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