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Scaffolding the Mathematical "Connections": A New Approach to Preparing Teachers for the Teaching of Lower Secondary Algebra.

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Abstract: This paper discusses the results of a three-year mixed methods study into the effectiveness of a mathematics education unit. This was written for both pre-service primary education students and re-training in-service teachers, to prepare them for the teaching of pre-algebra and early algebra. The unit was taught rom 2013 to 2015 inclusively in a School of Education setting of a university in an Australian capital city. Focusing on the Number and Algebra strand in the Australian Curriculum, its purpose was to better prepare some novice teachers through modelling a more coherent approach to mathematics teaching. The unit's genesis lies in the author's belief that many mathematics teachers conduct their classes in isolated "pockets" of instruction that are not sufficiently informed by a broader, connected understanding of the mathematics. The unit was also prepared as a contribution to the recent call by the Australian Association of Mathematics Teachers for more targeted initiatives to combat the decrease of STEM skills in our schools (AAMT, 2014). Results from the analysis of this study suggest that there might be much to be gained from this new approach.

Introduction

*We need a well-designed and coherent system of support for quality teaching of mathematics that includes quality pre-service, good mentoring into the profession, early career support, ongoing support, **targeted initiatives that address particular issues**, and mature professionals "giving back" to the profession [bolding added].*
Teacher Education Ministerial Advisory Group Consultation Submission (AAMT, 2014).

I think what we learned in the unit is the groundwork, and that ties everything together. I think it's more tied in than what I first thought. Everything links in maths.
Fourth year primary education teacher, respondent in the research project, 2014.

The 2013-2015 reflective project described here employed both quantitative and qualitative research strategies to evaluate the effectiveness of a new mathematics education unit provided at the university at which the author teaches. This was presented both as an elective unit for able primary fourth-year degree students, in all three years, and in 2015 as part of a State-government-funded course for re-training in-service teachers. The unit saw small but consistent enrolments in both on-campus and online versions over this time, and is founded on an original theoretical model for teaching proposed by the author. This model aims to facilitate more effective teaching of all three Australian Curriculum strands in the transition years (Years 6 to 8) of mathematics classrooms. The model will be briefly described in the section headed "The context".

The unit is strongly linked at all times to the *Number and Algebra* strand in the current Australian Curriculum for mathematics, and is aimed at helping pre-service teachers and teachers to more successfully navigate the Curriculum. The workshops and materials present structured conceptual representations of the content in the strand over time, specifically linking the evolution of number concepts with early algebraic thinking. The pre-service and in-service teachers engage in a carefully scaffolded experience that invites them to conceptualise the mathematics strand as a coherent system of thought, moving in a logical way from the Foundation Year to Year 10. The unit also explores and models mathematics curriculum school leadership skills, and incorporates assessment items that ask them to demonstrate these by cohering and presenting their understanding to others in various ways.

Mixed-methods research and reflection, based on the researcher/instructor/writer's anecdotal and recorded observation of the students' experiences as well as the collection of actual data, was carried out over the three years. Figure 1 illustrates the way in which the study was conceptualized. Through interrogating the value for pre-service and in-service teachers of a more organised, *metacognitive* consideration of evolving mathematical concepts, the researcher hopes that this will then ultimately engender deeper and more connected learning experiences for students in primary and secondary mathematics classrooms.

It needs to be stated that the focus of this particular pilot study was not to interrogate closely the minutiae of how any positive changes may have been brought about, but rather to trial some materials and approaches, and to enquire into any overall reported effectiveness of the associated new strategies. The following research questions were addressed, and will be discussed in this paper:

Can a mathematics education unit based on careful mapping materials prepare novice teachers to…

1. *develop some* ***metacognitive understanding*** *of children's evolution of mathematical concepts?*
2. *more clearly relate* ***basic number concepts*** *to* ***early algebraic thinking****?*
3. *develop* ***confidence*** *and* ***self-efficacy*** *as teachers of mathematics?*

Some examples of the unit materials are also provided, so that both the research findings and student commentary may be more clearly contextualised for the reader. This is then followed by further reflection on the pre-service teachers' beliefs about mathematics, and their self-efficacy and confidence concerning mathematics teaching at the conclusion of the unit work.

Literature Review

The Tension between Relational and Instructional Learning in Mathematics Classrooms: Its Impact on Pre-Service and Novice Teachers of Mathematics

Education researchers have in recent years highlighted the tendency of many government bodies to ask for improved teacher "training", with much less attention to the deeper cognitive and relational understanding that is required of the teaching profession (Darling-Hammond & Richardson, 2009; McNally & Blake, 2010; Swierczek, 2010). This is particularly true in the arena of mathematics education in Australia, where there are increasing calls for more strategic or "targeted" approaches to the preparation of classroom-ready and effective mathematics teachers (AAMT, 2014; Office of the Chief Scientist, 2013; Teacher Education Ministerial Advisory Group (TEMAG), 2014). The growing shortage of skilled mathematics teachers across

Australia and in many parts of the western world is well documented and has been at a crisis level for at least fifteen years (Guarino, Santibanez, & Daley, 2006; Ormond, 2011). At the same time, concern about the numeracy skills and basic mathematical knowledge of pre-service teachers' training in primary and lower secondary mathematics continues to be articulated, and early career teachers' understandings and confidence continue to be the subject of current concern (Livy & Herbert, 2014; Hurst, 2014).

Rakes, Valentine, McGatha and Ronau (2010) argue, as many commentators before them, for the importance of *relational* teaching and learning in mathematics. At the same time they lament the intractability of many mathematics teachers in persisting with their preference for *instrumental* mathematics and algorithmic routines, over more *conceptually linked* approaches to the understanding of mathematics.

The persistence of a procedural emphasis in traditional mathematics pedagogy … suggests that although a great deal of evidence supports the importance of teaching mathematics conceptually, the information from that body of research has not yet influenced the teaching profession enough. (p. 391)

Rakes et al. (2010) also remind us that, as long ago as 1976, Richard Skemp

… identified several benefits of building connections among ideas that may explain its effectiveness on improving student achievement: (a) improved ability to adapt to unfamiliar situations, (b) reduced need to memorise rules and heuristics, (c) enhanced student intrinsic motivation to learn mathematics, and (d) increased stimulation of student growth into independent, lifelong learners.

Yet the reality is that "operational" or "relational" learning and teaching (Heibert & Lefevre, 1986; Sfard, 1991, 2008; Skemp, 1986) has always been far more challenging for any teacher to consistently enact in a mathematics classroom than "instructional" teaching. The former type of learning and teaching is far more dependent upon the flexible use of inquiry-based strategies and contextualised conversations with students; and upon a creative attention by teachers to multiple representations that cover both the left-brain verbal/symbolic and right-brain visual/spatial/kinesthetic aspects of a complex mathematical idea (Davis, 2012; O'Halloran, 2005; Skemp, 1986; Sousa, 2008; Ormond, 2012a; Woolner, 2004). Despite the sustained demand by teacher educators for more student-centred approaches in the teaching of classroom mathematics, the majority of mathematics teachers appear to persist much of the time with more teacher-centred ones, and also often with an over-reliance on textbook exercises and algorithmic procedures.

Pre-service teachers are themselves the products of these practices in their own schooling, and as such are very likely to replicate their experiences, without some intervention and thought at the teacher preparation level. Further, this paper argues that the lack of appropriate relational learning in many *mathematics pedagogy* classrooms in teacher education courses is a strong contributing factor to poor teaching in primary schools, and in some cases, secondary schools.

Strategies for Creating Classroom-Ready Mathematics Teachers with Stronger PCK: Visualising the Structure of a Mathematics Curriculum

Current and recent commentators stress the need for students of all school ages to see the *connections and relationships* between mathematical ideas and their representations (Kilpatrick, 2010; Ormond, 2012a; Vogel, 2008); and it is argued here that this is just as true for novice mathematics teachers as it is for students in school classrooms. Further, the researcher believes that there is a great advantage in mapping and revealing mathematical *structures* for pre-service teachers.

Warren (2003) defines knowledge of mathematical structure as “knowledge of mathematical objects and the *relationship* between the objects and the properties of those objects [italics added]”. Being organised about mathematics is not a new idea: in 1991, Briginshaw advocated having a clear conceptual construction in the presentation of any mathematics curriculum or syllabus, claiming that “mathematics is both multi-faceted and multi-layered”. He added that “in order to teach it effectively it would help if those facets and layers were to be differentiated and clearly displayed” (p. 76).

Structure is key in mathematics, whether one considers it from a didactic or a social constructivist perspective. Bauersfeld (1995) looked at mathematical structure and its concept development as a “social practice”, and Vygotsky’s phrase “social constructivism” (1962) relies on “scaffolding” up through a hierarchy of learning. The current Australian Curriculum for mathematics has now provided Briginshaw’s “differentiation and display”, and it does this well: it has offered teachers a logical and carefully mapped structural framework of mathematics content. Yet a truly practical and useful curriculum structure needs also to ensure that its connections and links are visible to the user, thus enabling for teachers something more like Warren’s (2003) “knowledge of structure”. Kilpatrick (2010), who also explains the need for cohesion and organisation in presenting a clear curriculum for teachers, emphasises how important it is to synthesise the theoretical and the practical, and claims that “mathematics educators need to bring research and practice together through an organised system of knowledge that will enable [teachers] to see beyond the specifics of each and explain how they can work together”.

The *specifics* of the mathematics need to *work together*: a huge additional challenge for the writers of a curriculum document. Other analysts have also questioned just how successfully the Curriculum sometimes presents a truly comprehensive and *connected* sense of mathematics. Mulligan, Kavanagh and Keanan-Brown (2012) express this in their final remarks of a paper that critically investigates the *Number and Algebra* strand:

In our discussion we have provided an outline of the Australian Curriculum: Mathematics Number and Algebra content strand, which does to some extent present a connected view of mathematics, by linking number with pattern for algebraic thinking. On the surface this provides a positive sense of expectation about the Number and Algebra strand, with the opportunity for greater conceptual and connected knowledge and the development of teaching practices that focus on relational thinking. The definitions of the Proficiency strands (understanding, fluency, problem solving and reasoning) indicate a curriculum that is concerned about mathematics as patterns, relationships and generalisations, and not just facts, skills and rules. However, the ***structure and depth of the content, as presented through the descriptors, appear to lack the coherence and connectedness that the document promises****, which results in some concerns about the implementation and effectiveness of the strand overall [bolding added]. (p.65)*

In other words, for everyday classroom teachers, the “how” of teaching the mathematics using the Australian Curriculum resource still does not adequately support the “what” of teaching it. The links between the ideas still need to be made by teachers: an essential, but by no means automatic, mathematical process that requires patience and practice.

Experienced mathematics teachers often assume that novice teachers have some solid understanding of an overall framework or paradigm for teaching classroom mathematics, but this paper argues that new teachers often need quite overt assistance in both developing and unearthing their Pedagogical Content Knowledge (PCK). Sound PCK employs a *theoretical* understanding of mathematical ideas together with effective classroom *practice* (Charalambous, Hill & Mitchell, 2012; Davis, 2012; Hill, Rowan & Ball, 2005; Shulman, 1986; Sullivan, 2011). Ball, Thames and Phelps (2008) state that:

Shulman and his colleagues defined a perspective [of knowledge] that highlighted the content-intensive nature of teaching. However they also sought to specify the ways in which the content knowledge for teaching is distinct from disciplinary content knowledge. This had important implications for informing an emerging argument that teaching is professional work with its own unique professional knowledge base. (p.392)

Baumert and his research team (2010) concluded that teachers' content knowledge of mathematics "remains inert in the classroom unless accompanied by a rich repertoire of mathematical knowledge and skills relating directly to the curriculum, instruction, and student learning" (p. 139). Chick, Baker, Pham and Cheng (2006) coin the phrase "clearly PCK" to portray this blending of knowledges for productive teaching, and Roche and Clarke (2011) discuss the confident teacher's ability to "select", "interpret", "demand" and "adapt".

Ball et al. (2008) supports these commentators, saying:

… Shulman argued that knowing a subject for teaching requires more than knowing its facts and concepts. Teachers must also understand the organizing principles and structures and rules for establishing what is legitimate to do and say in the field [italics added]. The teacher need not only understand that something is so; the teacher must further understand why it is so. … Moreover, we expect the teacher to understand why a particular topic is particularly central to a discipline whereas another may be somewhat peripheral. (p. 391)

The phrase *connectionist teaching and learning,* coined by Askew, Brown, Rhodes, Johnson, and Wiliam in their 1997 study, is very relevant to this argument, stressing as it does the importance of an intelligent balance between *transmission* and *discovery* approaches to classroom teaching and learning (Lee, 2014). Teacher instruction and student experimentation each have their parts to play in any dynamic mathematics learning environment, including those designed for novice teachers. It is argued in this paper that a synergy of both quite overtly organised conceptual mapping, with practical inquiry into real classroom contexts and concept-building activities, is essential when preparing pre-service teachers to teach mathematics well.

The research study also included some exploration as to just what the participants in the unit felt about mathematics itself, and whether or not some evidence of a shift of attitude could be detected. Teacher beliefs and attitudes are notoriously difficult to influence, as the research has shown (Aguirre, 2009; Beswick, 2006; Goldin, Rosken & Torner, 2009; Grootenboer, 2008; Lee, Ladele & Ormond, 2013; Rakes et al., 2010). Yet in this paper it is also argued that a clearer charting of a logical curriculum and its connections and relationships may do more than just provide practical support for novice teachers. It may also result in some positive effects in regard to their beliefs about mathematics and mathematics teaching. This will be explored further in a later section of the paper.

Algebraic Thinking in the Classroom: Teachers' Understanding of Students' Needs across the Years

The need for the practice of early algebraic thinking in school classrooms – even as early as Years 2 or 3 – and its importance to all learning of mathematics has attracted a good deal of commentary over the past decade (Kaput, 2008; Mason, 2006; Van Dooren, Vershcaffel & Onghena, 2002). Van Dooren et al. (2002) describe early algebraic thinking skills as "symbolising, generalising, reasoning about relationships, [and] representing unknowns and even operating on them". Radford (2001) said that teachers of mathematics "need to deepen [their] own understanding of the nature of algebraic thinking and the way it relates to generalisation". Warren and Cooper (2009) have more recently claimed "that the power of mathematics lies in the intertwining of algebraic and arithmetic thinking, each enhancing the other as students become numerate." Moreover, intrinsic to these processes is the teacher's informed use of different but *equivalent* mathematical representations when presenting algebraic concepts. This is something that early career teachers need to be encouraged and reminded to do. As Davis (2012) says, enriching further some of Sfard's (2008) perspectives:

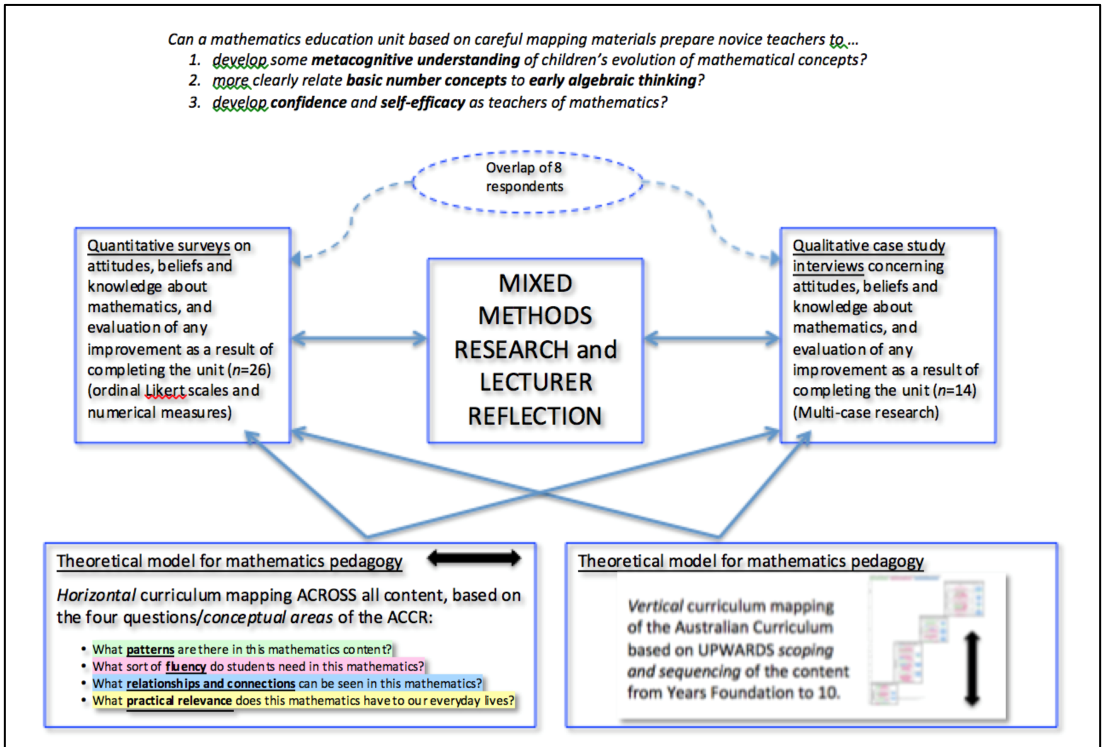
The subtleties of the capacity to move among interpretations tend to be difficult for expert practitioners to appreciate. In particular, teachers and other expert knowers are often unable to differentiate among realizations [representations] which novices are unable to reconcile. (p.4)

Freudenthal (1983) similarly referred to the "automatisms" of practising mathematicians that so often need to be unravelled and reconfigured for students in a mathematics learning setting. This is no doubt true for many pre-service mathematics teachers also.

There is also a large bank of research concerning the natural and often quite understandable confusions that occur for lower secondary students when beginning formal algebra. These are often based on inappropriate mental models for symbols and equations (Booth, 1984; Clement, 1982; Kaput, 2008; Kieran, 1989; Knuth, Stephens, McNeil & Alibali, 2006; MacGregor, 1998; Warren & Cooper, 2009). For example, it is known that young students are very reluctant to move forwards from the implied idea that the equal sign *always and only* means "give the answer" (Ely & Adams, 2012; Kieran, 1981, 1989; Ladele, 2013; MacGregor and Stacey, 1997; Ormond, 2012a). Also, students often misinterpret letter variables as names for objects, and so have difficulty accepting the idea of their variability or their relationship with other quantities (Arzarello, 1998; Herscovics, 1989; MacGregor & Stacey, 1997).

Despite over 30 years of extensive research showing just how essential the successful establishment of these concepts is to students' later success in all areas of mathematics, the early connections between number skills and algebraic thinking are not usually thoroughly interrogated at the primary level of teacher education preparation. The likely algebra misconceptions of the transition years of schooling are also often completely unknown to upper primary mathematics teachers. These two significant problems are addressed in Module 1 and Module 2 of the unit described in this paper. The way in which the author has attempted to redress these shortfalls in the PCK of many novice teachers is also later described.

Method



(Figure 4 best exemplifies the directional perspectives described above.)

Figure 1. The conceptual framework for the study

Both qualitative and quantitative approaches were used to collect, reflect upon and interpret gathered data. A total of 33 respondents were involved in the mixed methods research study over the years 2013, 2014 and 2015 (see Tab.1). These comprised 22 pre-service primary education students who participated in a mathematics specialisation elective unit, and who had maintained an HD average in their mathematics education units in earlier years; and 11 primary and secondary teachers re-training to teach lower secondary mathematics (and sponsored by state government scholarships in a graduate certificate program written by the researcher). Twenty-seven respondents completed the two pre- and post-test surveys, and 14 students agreed to be interviewed after completing the unit. Eight[1] of these 33 responded to both pre- and post-test surveys and also completed case study interviews.

Simons (2009) describes a case study as an "in-depth exploration from multiple perspectives of the complexity and uniqueness of a particular project, policy, institution, programme or system in a 'real life' context", and Yin (2012) claims that it "can be used to document and analyse the outcomes of interventions". Silverman (2011) talks about building a full picture of a situation where people are functioning in a comfortable setting, such as in response to targeted questions in an informal interview. In this study semi-structured interviews and discussions with students were employed, with a set of questions asked (see Appendix Tab. 1) and further commentary invited. The interviews were conducted, audiotaped and transcribed by a disinterested party, by phone or by face-to-face interviews on campus. These transcripts were then later interrogated by the researcher, who looked for *key words and phrases* and

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[1] Case study Students 1 to 7, and Student 14.

evidence of certain dispositions that connected logically with the three research questions. A second disinterested research assistant entered the quantitative data[2] from the two surveys into the statistics program SPSS, also for later analysis by the researcher.

The undergraduate students as part of their degree had completed the unit concerning pre-algebra and algebra up to about Year 8 in school level concepts; the re-training primary and secondary teachers had completed what was essentially the same unit, but with the addition of four more three-hour workshops concerning the learning and teaching of algebra in Years 9 and 10. The pre-test surveys in each case were administered to the students before the unit began, and the post-test surveys were administered at its completion. All participants were assured in writing (in line with the University's ethics policy) that collected data of all kinds would only be examined for the first time after their academic results for the unit had been finalized and submitted.

Instruments and respondents	Fourth year primary maths education students with an HD average, 2013-2015	Re-training primary teachers, 2015	Re-training secondary teachers (non-mathematics), 2015	**Totals**
Pre- and post-test surveys (only)	15	4	-	**19**
Case studies only	-	5	1	**6**
Both surveys and case studies	7	1	-	**8**
Totals	**22**	**10**	**1**	**33**

Table 1. Breakdown of the students' participation: in pre- and post-test surveys *only*, in case study interviews *only*, or in *both* (*n* = 33).

The research methodology was triangulated by using collection and analysis of both qualitative data (the case studies) and quantitative data (the survey questionnaires), so as to ensure the broadest possible perspective (Bergman, 2010; Yin, 2009) in the interrogation of the three research questions described in the introduction. The surveys essentially consisted of requests for percentage breakdowns in students' beliefs about the meaning and teaching of mathematics; and Likert-scaled responses to attitudinal and belief questions concerning mathematical skills and confidence, before and after the teaching of the unit. The former used a statistically validated questionnaire concerning the respondents' beliefs about mathematics and its teaching and learning (Swan, 2005). The latter employed questions designed by the researcher (see Tabs. 2 to 9). These were not statistically validated as their purpose was to provide early indications of possible success in the new teaching approaches, rather than inferentially test for change. In the case of the quantitative data collected from the pre- and post-study surveys, the researcher used descriptive statistics, such as comparative column charts and pie graphs, and numerical comparison of the means of the rank responses between pre- and post-test surveys. Because the sample was very small, inferential testing was largely inappropriate, but it was felt by the researcher that a means-comparison interpretation of ordinal data provided an appropriate method for detecting or observing any meaningful changes from pre- to post-test. A Student Paired t-test was also carried out on the numerical percentage scores obtained from the Swan mathematics questionnaire. The purpose of the research was always intended more as reflection

[2] The number of respondents contributing to this quantitative data decreased on one or two occasions from 27 to 26 as there were some incomplete responses from one participant in one of the surveys.

than as inferential analysis, in preparation for further curriculum offerings of a similar nature should any success appear to have occurred within the three small cohorts.

The Context

The ACCR Theoretical Model

The theoretical model from which the unit materials derive (named by the author as the Australian Curriculum Conceptual Rubric (ACCR)) is only briefly described here. It is important to provide this as a backdrop to the discussion, however, as the model frames all of the thinking behind both the conceptualisation and practical realisation of the unit.

Students are invited from the first workshop to think about *all* mathematics teaching (including the other two strands[3] also) from the perspective of *four big questions*. These overarching and fundamental questions are also the actual origin of the four *conceptual areas* referred to throughout the paper. Moreover, it is made clear to students at the outset that the unit's underlying theoretical model is intended to *clarify* and *enrich* the mandated Curriculum, and in no way to *supplant* it.

The ACCR is designed to direct pre-service and in-service mathematics teachers to appropriate parts of the Australian Curriculum, and to then help them to ask the most effective classroom questions of their pupils, and themselves. These questions, coded by four colours that are used consistently in the unit documentation, are:

- What **patterns** are there in this maths content?
- What sort of **fluency** do students need in this maths?
- What **relationships and connections** can be seen in this maths?
- What **practical relevance** does this maths have to our everyday lives?

The questions involve both big background ideas, and the use of some didactical delivery and practice of skills. Two of the questions are more about the *structure* of mathematics and two are more about how mathematics is *used in an everyday, and classroom, way*.

These notions may again be related back to comments made by Mulligan et al. (2012) concerning the strand of *Number and Algebra*, when they claim that "structure and depth of content" (Sfard's structural learning) are covered satisfactorily in the Curriculum, but that "coherence and connectedness" (operational learning) often are not. Structural teaching and learning is of course also important, and this can be characterized much of the time by a more *transmission* mode of teaching (Askew et al., 1997). It also probably resonates best with the two ACCR conceptual areas[4] (on this occasion using the same colour coding but for *category names* instead of *questions*), *Attain fluency with the facts* and *Use mathematics in the real world*.

Operational learning and teaching is probably more often characterized by the other two ACCR conceptual areas *Look for patterns* and *Use the connecting relationships to reason, and to understand the content.* Operational or relational learning is far more challenging for a teacher to consistently enact or facilitate in a mathematics classroom. It is also, and has always been, a

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[3] Indeed, a second unit focusing on geometry has recently been written using the same theoretical framework.

[4] All of the ACCR *conceptual areas* prompt teachers to ask the important questions that can make the mathematics content come alive in a classroom. Other general ACCR documents (not included in the paper) categorise each Australian Curriculum content description according to its potential for illuminating a particular *area*. This mapping has been carried out in the document across three school-age stages from Years Foundation to 10. It is important to note that because mathematics is so complex and inter-connected, the Australian Curriculum content description codes will often appear under two, three or even four of the conceptual area headings, and does so for all three strands. This is purposeful, as it reflects the need to attend to the different aspects or approaches (the *areas* of the *concepts)* invited by the content description.

dominant theme of discussion in mathematics education. This type of learning and teaching is far more dependent on discovery or inquiry-based strategies and contextualised conversation (Vygotsky, 1962); the use of appropriate analogy and metaphor (Lakoff & Nunez, 2000; Ormond, 2012a); attention to multiple or "multi-modal" representations that cover the left-brain verbal/symbolic and right brain visual/spatial/kinesthetic aspects of a complex mathematical idea (O'Halloran, 2005; Ormond, 2012a; Skemp, 1986; Sousa, 2008); and the "semiotic resources" of language, manipulatives, spontaneous drawings and gesture (Mildenhall, 2013; Radford, 2009; Sfard, 2008).

These ideas, where relevant to the arguments, will be explored further in later sections of the paper.

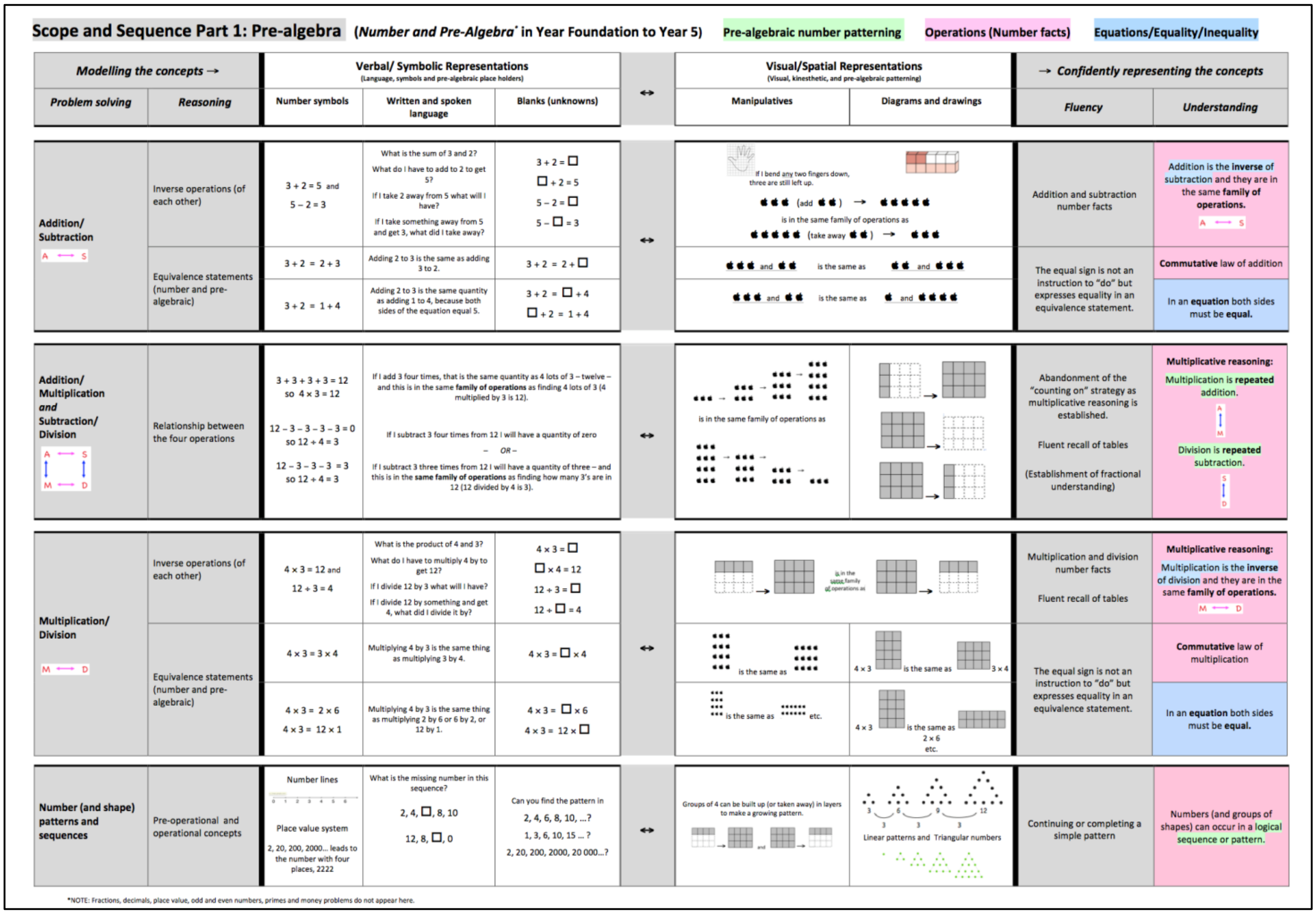
The Documents: Scaffolding the Curriculum Structure

Some examples of unit documentation are now provided for reader so as to better frame an understanding of the content of the paper. Figures 2 to 6 are snapshots[5] of some of the more widely used documents and materials in the unit. The earlier documents illustrate and exemplify through various kinds of mapping the *scope and sequence* of the *Number and Algebra* strand's content, and others summarise the key pedagogical ideas of primary and secondary school that need to be addressed in relation to these (also see Figs. 7, 8 and 9 later in the paper). Consistent reference is also made throughout to the multi-modal representations needed in teaching mathematics, as discussed in the review of the literature. All documents were also provided to students electronically, to accompany PowerPoint lecture/workshops, readings and videoed vignettes of teaching, but the ones seen in Figures 2 and 3 (and in the Appendix) were also distributed as coloured hard copy handouts at the beginning of the unit.

An important objective in the unit is to offer the students two kinds of *directional perspectives* concerning the Curriculum for the strand. Figure 1 attempts to represent these distinctions visually. The first perspective could be described as a *horizontal* one in that it teases out, *across* each Year level, the *connections* in the mathematics, which are seen in terms of the ACCR *conceptual areas* (colour-coded as described above). Figures 2, 3 and 7 all exemplify this horizontal perspective, seen wherever colours are used as coding.

The second *vertical* perspective traces the *sequential evolution* of the concepts (*within* each conceptual area) from Foundation Year to the lower secondary years, and this is seen in Figure 3 in a document that provides a quick and cohesive snapshot of the Australian Curriculum content through the years. This *vertical* scoping mimics the web-based presentation of the Curriculum itself (with the exception that the site moves downwards and not upwards), and this is supported by regular comparative exercises that interrogate the development of ideas in similar activities over the years. These are derived from *Mathletics*© (http://www.mathletics.com.au) worksheets, *First Steps in Number* exemplars, original pre-algebra and algebra tasks, and other resources. The bank of such scope and sequencing exemplars has grown considerably in size since 2013, as students have asked for more and more practical illustrations of the ideas.

[5] Note that the documents presented as figures in the paper are sometimes not as legible as could be desired, as they are very detailed and are in fact often provided in an A3 size to students. However, clearer versions are available to readers, upon request and in some circumstances.



Scope and Sequence Part 1: Pre-algebra (*Number and Pre-Algebra** in Year Foundation to Year 5) | Pre-algebraic number patterning | Operations (Number facts) | Equations/Equality/Inequality

Modelling the concepts →		Verbal/ Symbolic Representations (Language, symbols and pre-algebraic place holders)			↔	Visual/Spatial Representations (Visual, kinesthetic, and pre-algebraic patterning)		*→ Confidently representing the concepts*	
Problem solving	***Reasoning***	**Number symbols**	**Written and spoken language**	**Blanks (unknowns)**	↔	**Manipulatives**	**Diagrams and drawings**	***Fluency***	***Understanding***
Addition/ Subtraction A ⟷ S	Inverse operations (of each other)	3 + 2 = 5 and 5 – 2 = 3	What is the sum of 3 and 2? What do I have to add to 2 to get 5? If I take 2 away from 5 what will I have? If I take something away from 5 and get 3, what did I take away?	3 + 2 = □ □ + 2 = 5 5 – 2 = □ 5 – □ = 3	↔	If I bend any two fingers down, three are still left up. (add) → is in the same family of operations as (take away) →		Addition and subtraction number facts	Addition is the **inverse** of subtraction and they are in the same **family of operations.** A ⟷ S
	Equivalence statements (number and pre-algebraic)	3 + 2 = 2 + 3	Adding 2 to 3 is the same as adding 3 to 2.	3 + 2 = 2 + □		and is the same as and		The equal sign is not an instruction to "do" but expresses equality in an equivalence statement.	**Commutative** law of addition
		3 + 2 = 1 + 4	Adding 2 to 3 is the same quantity as adding 1 to 4, because both sides of the equation equal 5.	3 + 2 = □ + 4 □ + 2 = 1 + 4		and is the same as and			In an **equation** both sides must be **equal.**
Addition/ Multiplication *and* Subtraction/ Division A ⟷ S, M ⟷ D	Relationship between the four operations	3 + 3 + 3 + 3 = 12 so 4 × 3 = 12 12 – 3 – 3 – 3 – 3 = 0 so 12 ÷ 4 = 3 12 – 3 – 3 – 3 = 3 so 12 ÷ 4 = 3	If I add 3 four times, that is the same quantity as 4 lots of 3 – twelve – and this is in the same **family of operations** as finding 4 lots of 3 (4 multiplied by 3 is 12). If I subtract 3 four times from 12 I will have a quantity of zero – OR – If I subtract 3 three times from 12 I will have a quantity of three – and this is in the **same family of operations** as finding how many 3's are in 12 (12 divided by 4 is 3).		↔	is in the same family of operations as		Abandonment of the "counting on" strategy as multiplicative reasoning is established. Fluent recall of tables (Establishment of fractional understanding)	**Multiplicative reasoning:** Multiplication is **repeated** addition. Division is **repeated** subtraction.
Multiplication/ Division M ⟷ D	Inverse operations (of each other)	4 × 3 = 12 and 12 ÷ 3 = 4	What is the product of 4 and 3? What do I have to multiply 4 by to get 12? If I divide 12 by 3 what will I have? If I divide 12 by something and get 4, what did I divide it by?	4 × 3 = □ □ × 4 = 12 12 ÷ 3 = □ 12 ÷ □ = 4		is in the same family of operations as		Multiplication and division number facts Fluent recall of tables	**Multiplicative reasoning:** Multiplication is the **inverse** of division and they are in the same **family of operations.** M ⟷ D
	Equivalence statements (number and pre-algebraic)	4 × 3 = 3 × 4	Multiplying 4 by 3 is the same thing as multiplying 3 by 4.	4 × 3 = □ × 4	↔	is the same as	4 × 3 is the same as 3 × 4	The equal sign is not an instruction to "do" but expresses equality in an equivalence statement.	**Commutative** law of multiplication
		4 × 3 = 2 × 6 4 × 3 = 12 × 1	Multiplying 4 by 3 is the same thing as multiplying 2 by 6 or 6 by 2, or 12 by 1.	4 × 3 = □ × 6 4 × 3 = 12 × □		is the same as etc.	4 × 3 is the same as 2 × 6 etc.		In an **equation** both sides must be **equal.**
Number (and shape) patterns and sequences	Pre-operational and operational concepts	Number lines 0 1 2 3 4 5 6 Place value system 2, 20, 200, 2000... leads to the number with four places, 2222	What is the missing number in this sequence? 2, 4, □, 8, 10 12, 8, □, 0	Can you find the pattern in 2, 4, 6, 8, 10, ...? 1, 3, 6, 10, 15 ... ? 2, 20, 200, 2000, 20 000...?	↔	Groups of 4 can be built up (or taken away) in layers to make a growing pattern. and	3 6 9 12 3 3 3 Linear patterns and Triangular numbers	Continuing or completing a simple pattern	Numbers (and groups of shapes) can occur in a **logical** sequence or pattern.

*NOTE: Fractions, decimals, place value, odd and even numbers, primes and money problems do not appear here.

Figure 2. The first Scope and Sequence document for the Australian Curriculum *Number and Algebra* (Years Foundation to 5). This was handed out in hard copy to all students at the beginning of the unit. Two more such scope and sequence documents were prepared for Years 6 and 7, and Years 7 to 9, and they are formatted in exactly this way also. (This figure provides a slightly clearer image for helping the reader to understand the reference to "Document 2" in Fig. 5.)

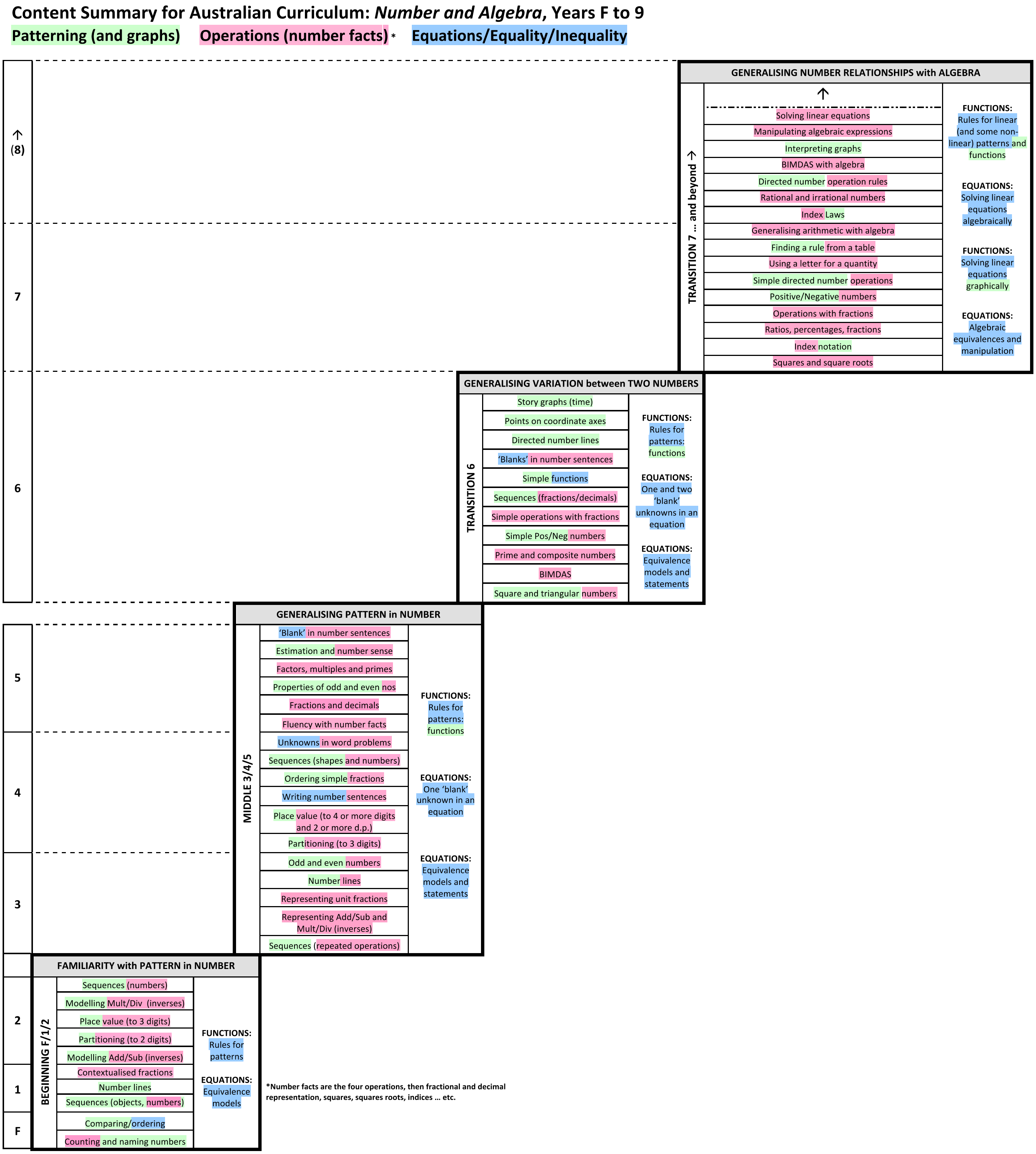


Figure 3. A section of the Australian Curriculum *Number and Algebra* Content Summary document (referred to as the "step-ladder diagram"). This has since been mapped in its entirety from Foundation to Year 10A inclusive.

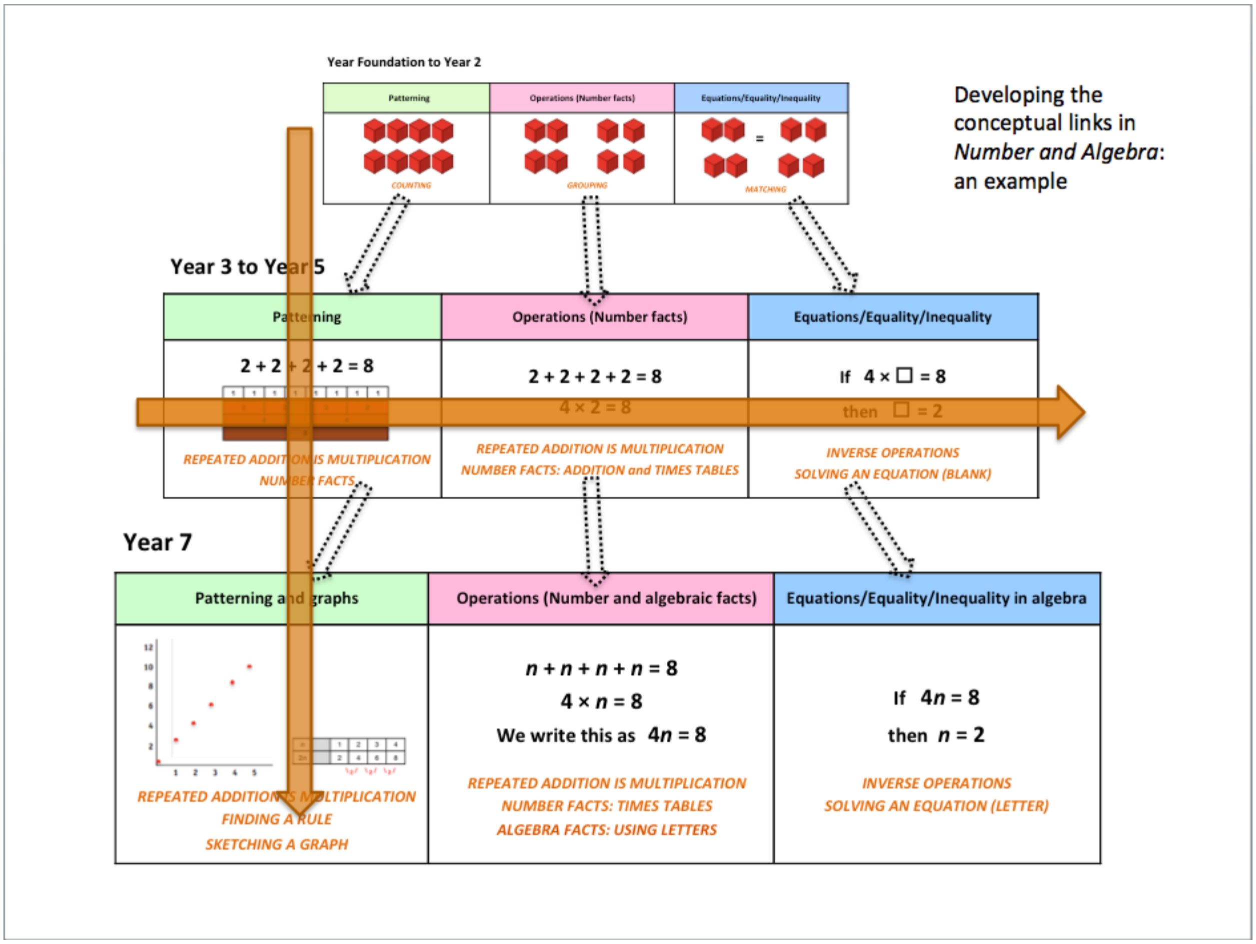
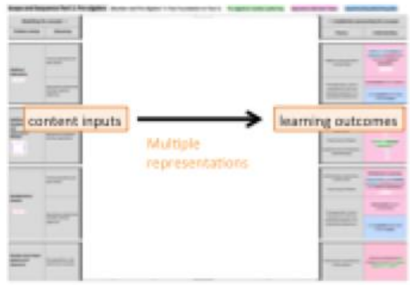
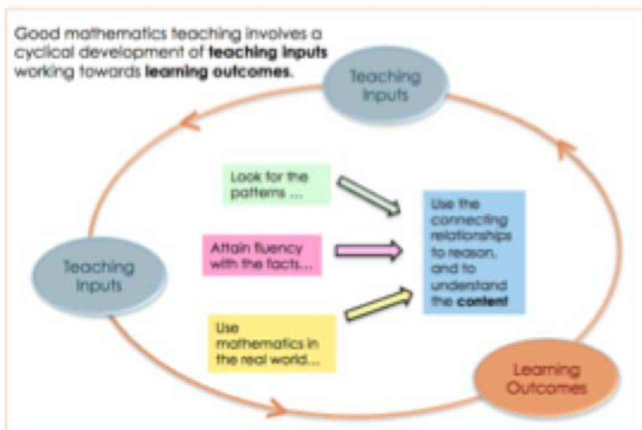


Figure 4. An example of a PowerPoint slide from the introductory materials: showing the *horizontal* and *vertical* perspectives for thinking about the mathematics, to be used as tools for developing stronger pedagogical content knowledge.

Document 2 gives the – perhaps not always very desirable – impression that student learning is *linear*, moving smoothly from left to right.



Of course, the reality is that it is not this simple. Document 2 is useful for mapping a *teaching* path of concepts and looking at the interrelationships of ideas needed in *teaching*, but it is not actually a picture of student *learning*. This image from Workshop 1.1 ... more suitably represents the *cyclical* nature of learning in an effective mathematics classroom, and includes the by now familiar diagram for the **conceptual areas**.



So what *are* **teaching inputs** and **learning outcomes**? I would summarise it thus, something that will continue to be a theme in all of the workshops:

INPUT → **TEACHING CONTENT**, something **taught by a teacher to,** *or* **facilitated by the teacher for,** students.
(Both direct teacher instruction and more indirect teacher prompting (scaffolding),
with students using ***Problem solving*** and ***Reasoning***)

OUTCOME → **STUDENT LEARNING**, a level of **understanding** that is **demonstrated by a student**.
(Individual student knowledge evidenced in continuous assessment, with students
gaining and then showing ***Understanding*** and ***Fluency***)

Figure 5. An early except from the "Weekly Warm-Ups" documentation provided for the four-unit graduate certificate course, of which this unit was a part. (For a clearer image of the second diagram, see the Appendix.)

It was stated earlier that the underlying theoretical model (the ACCR) is not in any way intended to overlay *yet more structure* onto the Curriculum, but to instead provide *a deeper perspective as to its actual teaching in a classroom*. Figure 6 again provides an excerpt from some early unit documentation.

Australian Curriculum Strands

Number and Algebra	Measurement and Geometry	Statistics and Probability
◆ What **patterns** are there in this maths content?		
◆ What sort of **fluency** do students need in this maths?		
◆ What **relationships and connections** can be seen in this maths?		
◆ What **practical relevance** does this maths have to our everyday lives?		

The ACCR has been prepared for your use as part of the unit resource documents, and will be referred to quite often in the unit. **It is important to note at the outset that it is provided as a set of support documents for the Australian Curriculum (AC) for mathematics, and it is not intended in any way to replace it or re-interpret the AC in a different way**. Recent feedback from teachers has strongly indicated that they often struggle with using the AC to plan meaningful and cohesive lessons for their students. The ACCR is the result of careful research, and is an "in-house" resource to be used whenever useful to assist teachers in their use of the AC**.**

Where your unit assessments mention the need to reference the ACCR, they intend this in a relatively simple way: all of the course units are based on the same understanding, that effective teaching of classroom mathematics occurs through four kinds of "lenses", called the four "conceptual areas". These are: looking for patterns, fluency and recall of maths facts, everyday, real life numeracy, and deeper mathematical relationships. Wherever these four colours appear in the documentation, this is all it means. These conceptual aspects of what we do in a classroom with students and mathematics all involve the AC content, and all involve the AC proficiency strands. The conceptual areas are "overarching" aspects of mathematics that are the foundation of the AC, or indeed of **any other mathematics curriculum**.

Figure 6. The four *big questions* for the teaching of all classroom mathematics; and an excerpt from the introduction to the unit.

The ACCR is intended to supplement for teachers some of the "coherence and connectedness" that Mulligan et al. (2012, p. 65) find lacking for the *Number and Algebra* strand. These four distinct but nevertheless linked aspects of the theoretical model, the *conceptual areas*, are used in the unit to evoke or suggest the different possible teaching emphases inherent in an Australian Curriculum *content description*. The students are encouraged to reflect on the fact that the richer content descriptions – that is, those whose potential for interesting mathematical work is more evident – often connect all at once with *patterning*, *fluency*[6]*and recall*, and *everyday numeracy*[7], and they should also say something about

[6] The Fluency conceptual area across all three AC strands in the whole ACCR model (not described in detail in this paper) has much in common, of course, with the Australian Curriculum proficiency strand of *Fluency*. However, it is intended to be a little broader, and also to be intimately linked with the other conceptual areas . While also related to algorithmic skills and the automatic recall of procedures, it is conceived in the model to also involve operational confidence, ready recognition of real contexts and of mathematical structures (patterns), and an *evolving, developmental understanding of the connections* in the mathematics. The acquisition into deep memory of concepts at the level of the conceptual area Equivalence and relationships in mathematics is crucial to the development of the *next* level of Fluency in the following year. (See Appendix Fig. 2.)

mathematical relationships of some kind. An introductory audio-presentation that foregrounds this theoretical model includes a slide illustrating the idea of these two directional perspectives (see Fig. 4).

The Research Questions, the Findings, and Some Early Implications

This section is organised around the three research questions for the study. In each case, both qualitative and quantitative findings will be used as prompts for some discussion and reflection. For the sake of clarity, this discussion is therefore arranged in three associated sub-sections that also briefly consider the implications of the findings.

***Research Question 1*: Can a mathematics education unit based on careful mapping materials prepare novice teachers to <u>develop some metacognitive understanding of children's evolution of mathematical concepts</u>?**

This first question sits at the very heart of the research investigation. The unit's own *theoretical model*[8], discussed in the last section, is based on the premise of the students involved gaining a *newly connected and refreshed understanding* of the structure of the mathematics in the strand. Thus the question involves both *pedagogical content knowledge* (PCK) and *subject content knowledge* of the mathematics (SCK) (Hurrell, 2013; Ma, 1999; Shulman, 1986). The research question also relates to Davis's (2012) concept of teachers' *emergent* understanding.

Both online and hard copy resources, as exemplified by Figures 1 to 9, were consistently successful with the students. Evidence of this can be seen in Table 2, where the mean rank approval ratings derived from the second survey lay between 4 (*agree*) and 5 (*strongly agree*).

Post-test survey questions *(strongly disagree (1), disagree (2), unsure (3), agree (4), strongly agree (5))*	**Mean rank on Likert scale from 1 to 5**	**Standard deviation**
The hard copy materials offered by the unit were clearly laid out and easy to understand.	4.11	0.847
The on-line resources offered by the unit were clearly presented and easy to understand.	4.26	0.712

Table 2. Means of ranks of responses to two questions on the post-test survey (*n* = 27).

The many affirmative comments made by students in the final case studies also support this claim. Just some of these are included here.

> **<u>Student 7 (2014):</u>** [The lecturer] provided us with … [a document: see Fig. 3] … starting from Foundation and all the way up to Year 7 or 8. It had the Curriculum broken down … and that really helped, because with a spread out sheet you could look at and compare each year and look at the evolution or the increasing difficulty of

[7] The "Everyday Numeracy" conceptual area is included where relevant in the unit documentation, but is not featured specifically as part of the theoretical model for this particular strand, as discussed here.
[8] That is, as in the pedagogy, or *andragogy*, of the adult teaching student learners themselves, and not be confused with that for the children they will be teaching.

each Curriculum content description from each year, to really compare them. So that was one document that I refer to a lot.

Student 6 (2014): I loved these documents. I am filing them and keeping them with me. It was great to see how they summarised the content throughout the unit. It provided the information required for you, and the different learning abilities, the progression throughout the year levels - and it was very clear to read and understand. I will definitely be using these as a point of reference for my future teaching career.

Student 8 (2015): They were amazing, the most fabulous documents ever. Just so simple to follow, easy to read, quick to access, comprehensive links to Number and Algebra and the Australian Curriculum. She summarised it so I could go, yeah, I can incorporate this, this and this, in this part of my maths.

Some of the students' own case study interview comments can also be used quite helpfully to illustrate both certain aspects of the unit's rationale, and their connection with the underlying research questions. As was explained earlier, the researcher upon interrogating the transcripts looked for *key words and phrases* that supported the contention of the unit's effectiveness. An example is provided below, with such phrases highlighted.

The students' comments that follow are in response to two case study interview questions. The researcher has also chosen here to roughly categorise the responses, classifying them as to their association with either a *horizontal* or a *vertical perspective*, as defined in the last section of the paper, and seen in the representation of Figure 1. The researcher was looking for evidence of any apparent consciousness in the student respondents of the mapping in two *directions*, with organisation by coloured themes as an added overlay (referred to in the research question as "careful mapping materials").

Research Question 1*: Can a mathematics education unit based on careful mapping materials prepare novice teachers to develop some* ***metacognitive understanding of children's evolution of mathematical concepts****?*

Case study Interviewer*: The unit aimed to develop* ***a more overall, holistic understanding of the way in which pre-algebraic work can support beginning algebra*** *in lower secondary school. Please comment on whether or not this was successful for you. How helpful or otherwise did you find the '****scope and sequencing'*** *documents provided in the unit?*

- **The horizontal perspective: introducing the colour-coded *conceptual areas* across content for connecting the mathematics**

Student 7 (2014): [The lecturer] gave us some big A3 documents. The Scope and Sequence Part 1 Pre-Algebra one was basically a big spreadsheet with divisions [see Appendix Fig. 1]. It had a column: **verbal and symbolic representation of how to represent the maths and also visual and spatial representation**. We talked about the differences between verbal-symbolic and visual-spatial and **how students learn differently** and how it is good to **give students different representations of the maths** for them. And she had it, once again, **highlighted: green for pre-algebra patterning, pink for number operations, and blue for equations, equality and inequality**, **so we could see where these concepts fitted in**. She had **pictures or examples of how to represent**, for example, addition and subtraction in a verbal-symbolic way or a visual way. Then, in division and multiplication, how to represent it symbolically, visually, spatially. Things like that.

- **The vertical perspective: scoping and sequencing the Australian Curriculum up through the years**

Student 2 (2013): I think that after doing the unit, my knowledge of [children's evolution of mathematical concepts] especially has increased. Going back to the **holistic approach** that [the lecturer] had with her documents, it all seemed so clear for the algebra. I know it was only one part of the maths Curriculum but **suddenly it made sense as to why we were teaching them these things in Year 4 and how that was developing them for algebra**.

Student 7 (2014): I hadn't looked at the **curriculum in this depth** and **compared each year in mathematics in such depth** before. So that really did help. You have to **read the curriculum and analyse it to reach that level of understanding**. One thing [the lecturer] talked about was **how important it is to look at the year before and the year after and see where they have come from and where they are going**.

The researcher used the thematic analysis seen above to imitate a type of informal yet organised *coding procedure*, noting any relevant correlations between wording and concepts in the research and case study questions, with the students' own words[9].

It should also be explained at this point that in writing this unit the author was always conscious of a certain tension in the underlying theoretical model. While on the one hand the mathematics was being explained for the pre-service teachers using a very structured conceptualization, on the other hand it was very important that, as classroom teachers, they did *not* therefore assume that their students would also necessarily learn in this very linear and organised manner. Current commentary emphasizes the need for all teachers to understand the sheer complexity and diversity of the manner and timing of student learning (Deans for Impact, 2015). It was hoped that, while a greater sense of pedagogical control and some enhancement of important PCK had been provided for teachers by these holistic frameworks for *Number and Algebra* – and that this would help them in their organisational planning – it would at the same time actually encourage a *better disposition* towards student-centred, inquiry-based teaching and learning. Figure 6 supports this idea: it is an early excerpt from the reflective documentation provided each week to students by way of introduction to the consecutive ideas in the workshops. The importance of clearly distinguishing between *teaching inputs* and *student learning outcomes* is a solid theme in the unit.

Seeing the Connections and the Big Picture: Uncovering the Hiddenness of Understanding

In response to the first research question *("Can the unit … prepare novice teachers to develop some metacognitive understanding of children's evolution of mathematical concepts?")*, the researcher also looked for any evidence of the students starting to *really know what they*

[9] The researcher considered the more formal use of the NVivo qualitative analysis package, or its like, but ultimately decided on this less structured but more intuitive approach to the analysis.

knew. This involved looking for signs of deeper reflection about their actual knowledge, from a personal, yet somehow more distanced (*meta*-), point of view.

It was clear from the quantitative analysis that their general sense of both what and how to teach had improved, and this can be seen in Table 3. (The codes used for the quantitative data entry into the SPSS statistics package are also included here so as to link with the later discussion concerning Fig. 11.)

Below are some important ideas that were addressed in the unit. Please tick the boxes that match best how well that you now feel you understand these **components** or **concepts** in mathematics teaching. *(badly (1), sometimes well (2), soundly (3), excellently (4))*	**Mean rank increase on Likert scale from 1 to 4**	**Standard deviation**
"What" I need to teach in upper primary maths classes, as in *content* (**in**)	0.407	1.0468
How to become an effective teacher of early algebraic ideas (**tea**)	0.852	0.907
How primary mathematics teaching may support early secondary mathematics classrooms (**sec**)	0.519	0.802

Table 3. Mean rank increases from pre-test to post-test responses to three paired sets of questions in the two surveys (*n*=26).

A detectable improvement like this suggests to the researcher that the unit work had indeed elicited in the students some degree of *metacognition* and heightened understanding. Moreover, it could be argued that this had occurred because the students' own understanding of what mathematics *is*, and how it is *structured*, had been encouraged to surface. Recent commentators have argued that teachers usually know more than they realize, and that mathematics teacher educators can help this to "emerge" (Davis, 2012; Ma, 1999). Davis (2012) has claimed that "teachers' knowledge of mathematics might be productively construed as a complex evolving form, a significant dimension of which is tacit knowledge". Mathematics educators often assume that their prepared novice teachers have a solid – or at the very least adequate – understanding of such essential teacher skills as seen in the first column of Table 3. However, it is argued in this paper that many pre-service teachers need much more *consciously deliberate assistance* in developing their understandings. The documents and the theoretical model described in this paper aim to uncover some of that tacit and evolving knowledge, wherever it might exist.

The students in the study – while all competent fourth year degree pre-service students or practising in-service teachers, and all comparatively confident in their basic mathematical skills (see Tab. 7) – nevertheless consistently demonstrated a marked improvement in both their pedagogical and their content understandings. In their comments, all 14 case study interview students attributed much of this to their experience in the unit; and, importantly, each of these respondents made at least one reference to such as experiences as

- a **sudden awareness** or realisation;
- a brand new **big picture overview** of the content and the pedagogy;
- a new sense of the **deep connections** in the mathematics;
- and/or a sense of **continuity** and **scope** in the mathematics.

Again, some highlighting to represent the coding has been added to the actual transcripts here, to help to support this claim. It has been explained that the researcher was looking for pertinent *key words* and indications of *changes in disposition*, and these connections with the original research ideas are exemplified again below. These were categorized by the researcher into groups such as the four dot-pointed above, and links were made with data findings from the surveys. The quotations below represent a few examples of the corroborative evidence being sought.

> **Student 2 (2013)**: I feel that the best thing that I got out of the unit, and the people that I have spoken to that it was the best thing for them too, was the **holistic approach** that we **suddenly could see why** we were teaching kids to count in Year 1 and how that actually made a difference when they were learning algebra in Year 10. Because those links aren't always very clear and [the lecturer]'s documents were fabulous for that … The holistic approach and how **everything seem to just fit**: it **just made sense** to me after doing the unit.

> **Student 4 (2014)**:` It was a **real eye opener** for me. I **didn't realise** how early you were learning algebra as a student. I **didn't realise** it went right down to the younger years … You could see the **progression**. In the first lower primary years you could see the basic activity, and then the middle year activity was similar, so you could see it getting more advanced as you went along … I didn't really understand that was an algebra thing and **didn't think much about it, until wow** - it really is something that they've got to understand when they are young. They have to understand those patterns and how numbers work for them to get the algebra later on.

> **Student 6 (2014):** There are areas in the curriculum that **I wasn't aware that were linked**. **They are all interlinked but sometimes you don't realise it.**

> **Student 10 (2015)**: I would say it was successful for me. It gave me **awareness**.

> **Student 12 (retraining primary teacher, 2015):** I teach Kindergarten this year and we do lots of patterning work and categorising, but I was **not fully aware** before the unit how this would be useful in later years. Now I can also see the **gradual progression across the years** and how each concept prepares the way for the next. … I was not familiar before the unit as to how the pre-algebraic work linked in with algebra in upper years.

> **Student 13 (retraining secondary teacher, 2015):** I found this aspect ["developing a more overall, holistic understanding"] of the unit very interesting: working [also] in a primary school, and looking at my friends' resources and **seeing the pre-algebra in their work, but with no direct focus or highlight of it**, is now really amazing.

This indication of improved understanding about their own – or about their pupils' – learning of mathematics in the strand is also supported by analysis of the quantitative data, seen in Table 4. The overall means of the ranked scores for "mua" and "mui" were 2.96 and 3.00 respectively, and these shifted up to 3.30 and 3.33, respectively. While, when compared with Table 3, the increase of a third of a mean rank score is not large, it still suggests that the students have nevertheless generally moved overall as a group from the "soundly" category towards the "excellently" one. This can also be seen in Figure 11.

Below are some important ideas that were addressed in the unit. Please tick the boxes that match best how well that you now feel you understand these **components** or **concepts** in mathematics teaching. *(badly (1), sometimes well (2), soundly (3), excellently (4))*	**Mean rank increase on Likert scale from 1 to 4**	**Standard deviation**
How to assess or evaluate my *own* understanding of maths concepts (**mua**)	0.333	0.092
How to improve my *own* understanding of maths concepts needed for primary teaching (**mui**)	0.333	0.784

Table 4. Mean rank increases from pre-test to post-test responses to two more paired sets of questions in the two surveys, concerning an understanding of mathematics (*n*=26). (See Table 2.)

***Research Question 2*: Can a mathematics education unit based on careful mapping materials prepare novice teachers to more clearly relate basic number concepts to early algebraic thinking?**

This second research question, and its associated discussion, will be used to further illustrate the nature and purpose of the unit content. An earlier section of the paper concerning the unit's context and rationale focused more upon the first half of the unit (Module 1). The second half of the unit concerns such issues as the transition from Years 5 and 6 pre-algebra to Years 7 and 8 algebra; the likely student misconceptions about pronumerals and equations at this time; and the need for a systematic building of algebraic skills. Here the notion of the colour-coded *conceptual areas* will again be demonstrated with regard to both modules, and again with some student comments and some activity exemplars. Some materials from Module 2 will also be briefly described.

Module 1: Pre-algebraic building blocks in primary school, and practical activities for understanding the scoping and sequencing

As already seen, one case study question asked students to comment on whether or not they felt that the unit had been successful in its professed aim to develop a more overall, holistic understanding of the way in which pre-algebraic work can support beginning algebra in lower secondary school. Some responses to this research question have been categorized according to which of the *conceptual areas* the students were discussing as they replied. In the case of the last student (Student 7), she actually automatically remembered the categories in terms of the colours. The highlighted key phrases are in this instance also colour-coded, imitating once more how the informal coding analysis was carried out by the researcher. The relevant conceptual areas have been added each time for further clarification.

> **Student 8 (2015):** Yes. Children's evolution of maths is really play-based originally; using concrete materials and making and predicting rules ... So they are moving from play-based and concrete materials and this is what I notice in patterns, then moving - taking away I guess some of those concrete materials and focusing then more on what is that pattern, can we name the pattern, does it work any other way, ... and then all the way through to formulating an actual rule using pronumerals and things like that to provide the pattern. (**Patterns and sequences for algebra**)

> **Student 4 (2014):** When they are younger you start by doing your 1 + 1, 2 -1 kind of thing. … The relationship between adding and subtracting, quite young; and then moving onto the relationship between multiplication and division. You understand that base

before going into algebra. Because if you don't understand the difference between them and how they are so similar in what they are, then it is very hard for them to establish it later. (**Inverse operations and number facts**)

Student 5 (2014): [The unit] emphasised the importance of teaching those pre-algebraic ideas in the early years so they could establish that foundation knowledge which they could then build on later years of schooling. For example, teaching children the equals sign. Say 5 rather 2 +3 = 5 can also be 2+3 can also be 4 + 1 and those sorts of things. So that your answers don't just have to be just one set answer [on the right-hand-side] for everything.(**The meaning of equality**)

Student 4 (2014): Having 2 + 2 = and then having the kids figure out that it can be 3 + 1 as well. Like it's the same on both sides. … I didn't really understand that was an algebra thing and didn't think much about it, until wow - it really is something that they've got to understand when they are young. (**The meaning of equality**)

Student 3 (2013): …If you are doing patterning, obviously you start simple and it becomes increasingly more challenging for them, but it's when they understand the purpose of filling that blank spot, that is what they have to understand. That understanding will carry them further into understanding why we put the letters in. (**Using blank unknowns before letters**)

Student 14 (retraining primary teacher, 2015): Once I saw the class demonstration for solving a linear equation and paying attention to inverse operations I could see why the pre-algebraic pattern work was so important and how it transitioned seamlessly from back tracking to balancing. **(Patterns, operations, and equivalence)**

Student 7 (2014): There was a table [see Fig. 7 below] that [the lecturer] had highlighted. She had pre-algebraic number patterning in green, operations in pink, equations in blue. It just helped us to look at it in a different way and compare. For example, number patterning between Year 4 and Year 5 and what's different. This really helped a lot in seeing the evolution. I hadn't looked at the Curriculum in this depth and compared each year in mathematics in such depth before. … You really have to read the Curriculum and analyse it to reach that level of understanding. **(Patterns, operations, and equivalence)**

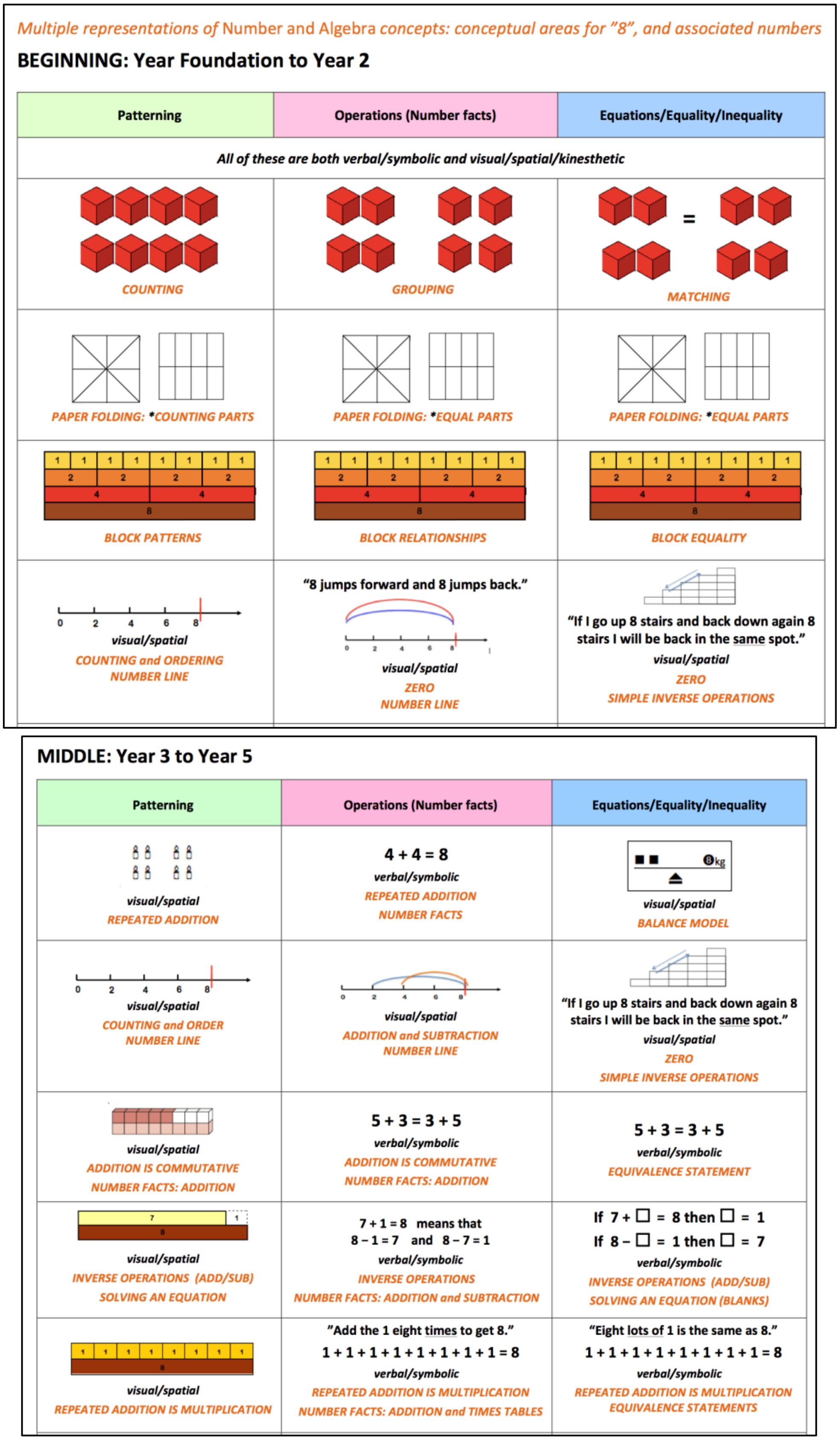
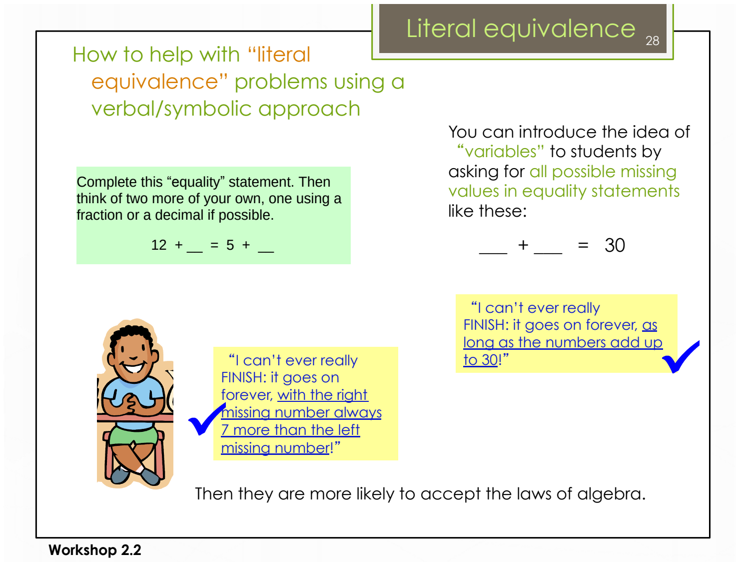


Figure 7. Two sections of a document exemplifying the *conceptual areas* with activity *prompts*, for Year Foundation to Year 5. This is an example of the *horizontal* perspective as discussed.

Module 2: Transition from pre-algebra to secondary algebra: moving from the concrete to the abstract language of algebra

The second module, which focuses on the early secondary years, builds upon the ideas established in the first module. For this purpose, the author has summarized the well-documented common early algebraic misconceptions (Booth, 1984; Clement, 1982; Kieran, 1981; Perso, 1991: see Fig. 8), which were briefly referred to in the review of the literature. In unit work, these misconceptions are closely examined, at the same time refreshing or establishing the pre-service and in-service teachers' own algebraic understandings and skills. The second module engages less in the *scoping and sequencing* of pre-algebraic and algebraic ideas, and focuses more now on *summarising the desirable learning outcomes* that may be achieved, through classroom attention to both the misconceptions and a carefully sequenced teaching of algebraic manipulative skills (see Fig. 8). Wherever possible, this thinking is linked back to the discussions about pre-algebra in the first module, so as to keep students focused upon the primary school number concepts that are so fundamental to early algebra. One such example of this linking of the two modules is seen in Figure 9.

Figure 8. An example of a "summarizing" document used in Module 2: linking the notions of desirable learning outcomes, misconceptions, and some fundamental manipulative skills for transition algebra.



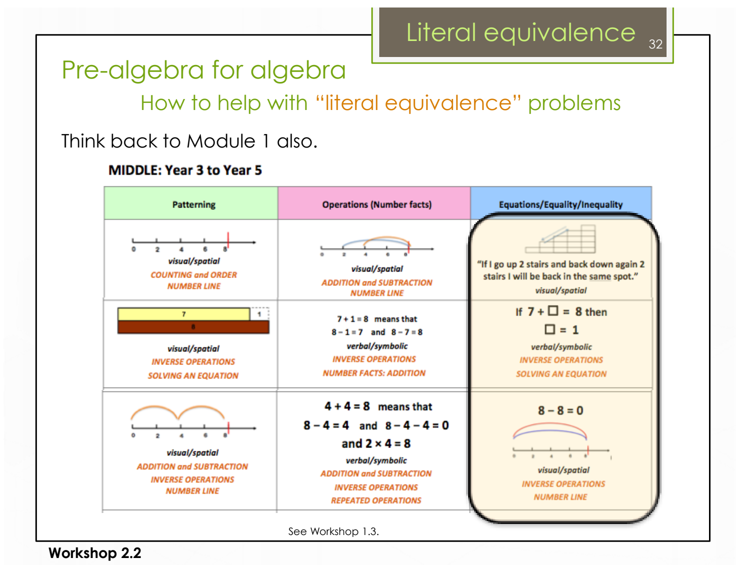


Figure 9. Two PowerPoint slides from a Module 2 workshop: analysing a common algebra misconception, and relating this back to ideas in Module 1.

Some further case study comments concerning the second module are included here, and again they suggest some general overall success of this approach. Greeted with particular enthusiasm by the students was the theme of common student misconceptions in lower secondary school, concerning the introduction of the use of letters and the notion of algebraic equations. (In the following quotations the extra bolding is not added, but the researcher's methods for interrogating the thinking behind the statements was carried out in the same manner as before.)

> **Student 3 (2013)**: Obviously some more defined skills as far as algebra goes are like when we are to introduce the lettering and not to use … like one thing [the lecturer] said that can be confusing, is if we are talking about chairs and tables and using the letter *c* and *t*, which they sort of relate to the object as opposed to relating to the number that it should be.

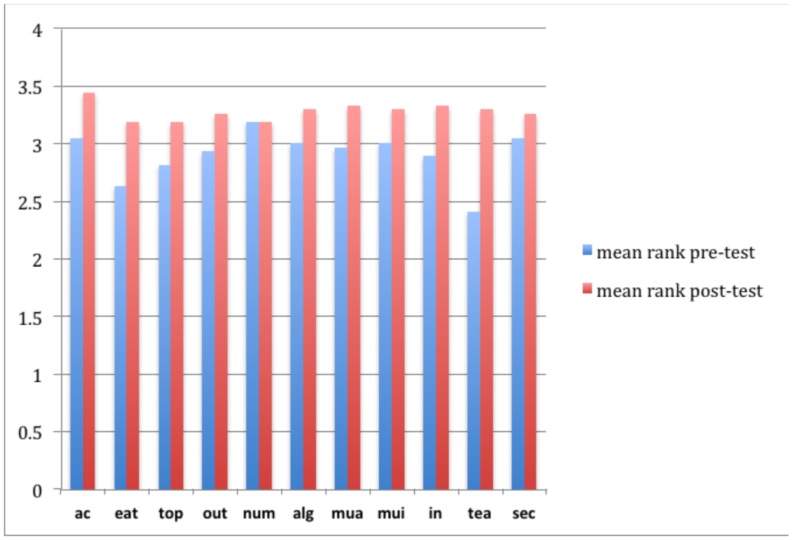
<u>Student 4 (2014):</u> A turning point for me in the unit was actually the scope and sequence and how things join together, and then I think one of the major points was learning about the misconceptions that kids have, and then interpreting how they made that mistake and then how to fix that problem. Because, you can teach them and that is fine, but then you have got to figure out how they have done it wrong. I think that was one of the bigger things that this unit did and a lot of other [mathematics education] units didn't.

<u>Student 13 (retraining secondary teacher, 2015):</u> I truly believe all the misconceptions [the lecturer] talked about, in the sense that I saw [in my colleagues' classrooms] the completed sheets using shapes et cetera in the place of '*x*', but knew that if we transition that to an actual '*x*', the [primary] student would not have been as successful – you could just tell, there was confidence with the shapes but decreased confidence when algebra was mentioned.

The researcher's claim that the students felt generally more enabled in terms of their PCK is also supported once more by the quantitative data (see Tab. 5); this analysis again relied on any evidence of a positive shift in the mean rank between tests. (This also serves as an appropriate introduction to the next section of the paper, in which evidence of improved professional confidence and sense of self-efficacy will be discussed.) The mean rank for "*How to become an effective teacher of early algebraic ideas*" shifted up by almost a whole unit on the scale.

Below are some important ideas that were addressed in the unit. Please tick the boxes that match best how well that you now feel you understand these **components** or **concepts** in mathematics teaching. *(badly (1), sometimes well (2), soundly (3), excellently (4))*	**Mean rank <u>increase</u> on Likert scale from 1 to 4**	**Standard deviation**
How to become an effective teacher of early algebraic ideas (**tea**)	0.852	0.907
How primary mathematics teaching may support early secondary mathematics classrooms (**sec**)	0.519	0.802

Table 5. Mean rank increases from pre-test to post-test responses to two paired sets of questions in the two surveys (*n*=26). (Also see Tabs. 3 and 4.)



Code	Concept/component
ac	The *Number and Algebra* strand in the Australian Curriculum
eat	How to plan and write an early algebra learning task
top	How to plan comprehensively a sequence of lessons in *Number and Algebra*
out	How I can judge what students have learned and can understand in a mathematics classroom
num	Basic concepts (to Year 7) in number skills and patterns
alg	Basic concepts (to Year 7) in pre-algebra and algebra
mua	How to assess or evaluate my *own* understanding of maths concepts
mui	How to improve my *own* understanding of maths concepts needed for primary teaching
in	"What" I need to teach in upper primary maths classes, as in *content*
tea	How to become an effective teacher of early algebraic ideas
sec	How primary mathematics teaching may support early secondary mathematics classrooms

Figure 11. A comparative column chart, with a key explaining codes: illustrating the changes from pre-test to post-test in regard to the paired question: "Please tick the boxes that match best how well that you now feel you understand these components or concepts in mathematics teaching. *(badly (1), sometimes well (2), soundly (3), excellently (4))"* Largest increases are highlighted in grey in the key.

As can be seen in both Table 5 and Figure 11, two of the stronger overall positive shifts in the means of ranked responses occurred for the questions coded "eat" (*How to plan and write an early algebra learning task*) and "tea" (*How to become an effective teacher of early algebraic ideas*). This was an encouraging finding when associated with that second research question (*Can a mathematics education unit based on careful mapping materials prepare novice teachers to <u>more clearly relate basic number concepts to early algebraic thinking</u>?).*

Research Question 3: **Can a mathematics education unit based on careful mapping materials prepare novice teachers to <u>develop confidence and self-efficacy as teachers of mathematics</u>?**

This last part of the paper reports on any self-observed changes in the professional confidence and sense of self-efficacy in the respondents; and finally examines and reflects on some analysis of their attitudes towards classroom mathematics.

Confidence and Self-efficacy

According to the quantitative data from this study, the students' overall professional confidence also generally improved, and their sense of self-efficacy improved slightly also. This is seen in Table 6.

Please select answers to the following questions by placing a tick in the appropriate box: *(strongly disagree (1), disagree (2), unsure (3), agree (4), strongly agree (5)*	**Mean rank increase on Likert scale from 1 to 5**	**Standard deviation**
I feel **confident** when teaching maths in an upper primary (or Year 7) classroom.	0.407	0.572
I can teach upper primary (or Year 7) mathematics **effectively** (with good student learning outcomes).	0.296	0.609

Table 6. Mean rank increases from pre-test to post-test responses to two paired sets of questions in the two surveys (*n*=26). (Also see Tabs. 2, 3 and 4.)

It needs to be noted also that none of the respondents reported low self-confidence at the beginning of the unit, when mostly primary mathematics content was being considered. This is probably to be expected from the demographics of each of the cohorts involved (see Tab. 1). Also evident was their comparatively different responses to the *extra knowledge* about *number* that they felt that they had gained in the unit, as opposed to *pre-algebra and algebra*.

Below are some important ideas that were addressed in the unit. Please tick the boxes that match best how well that you now feel you understand these **components** or **concepts** in mathematics teaching. *(badly (1), sometimes well (2), soundly (3), excellently (4))*	**Mean rank increase on Likert scale from 1 to 4**	**Standard deviation**
Basic concepts (to Year 7) in number skills and patterns [**num**]	0.111	0.751
Basic concepts (to Year 7) in pre-algebra and algebra [**alg**]	0.333	0.877

Table 7. Mean rank increases from pre-test to post-test responses to two paired sets of questions in the two surveys (*n*=26). (See Tabs. 2 to 5.)

Figure 11 shows that the mean rank for the responses to the first question in Table 7, concerning their confidence with number concepts in the primary years, already lay between "soundly" and "excellently" (3.19 pre-test and 3.30 post-test, on a 4-point Likert scale). The mean rank for the algebra item nevertheless also only showed a move from 3.00 pre-test to 3.33 post-test. While anecdotally most case study respondents referred to a general increase in professional effectiveness and confidence, it needs to be acknowledged that all students in the study group were starting already from a fairly firm base in their PCK concerning early number concepts, at the primary teaching level at least.

The researcher also noted with interest that the reported increase in self-confidence was generally more apparent in 2014 and 2015, after she had responded to an obvious fault in the 2013 version of the unit by including many more teaching examples and activities.

> **<u>Student 2 (2013)</u>**: Yeah I think so [that my self-efficacy has improved]. Probably not as much as I was hoping, I don't feel as I got as many ideas and activities. I don't feel as prepared as what I would have liked to after doing a full unit on algebra. The curriculum side of things and the holistic view of where everything fits in the curriculum, why we teach things in year 1, year 3 through, was great, but the specific things of actually how to teach it, I don't feel as prepared as I thought I would after a full unit.

Further case study responses to the questions below indicate one likely source of the renewed confidence: a general sense of a stronger ability to *plan* and to s*equentially link concepts*. Phrases that support this claim are seen in bold in the transcripts below, and some bolding to show connections with the research and case study questions is offered again, as before.

<u>Research Question 1</u>: Can a mathematics education unit based on careful mapping materials prepare novice teachers to develop ***confidence and self-efficacy as teachers of mathematics***?

<u>Case study interviewer</u>*: Has your* ***professional confidence*** *increased due to participation in the unit? Has your* ***professional self-efficacy*** *increased due to participation in the unit?"*

> **<u>Student 1 (2013)</u>**: Okay, so professional confidence … Definitely, I have more of an idea of **how to plan that and what to use and what resources to use**, and **how to stop and plough back if they have those misconceptions there**. Identifying those misconceptions is a big thing that we got from this unit, so **that would definitely affect our professional practice**.

> **<u>Student 6 (2014):</u>** I have felt **I have gained extensive knowledge in pre-algebra concepts** and I am really looking forward to **being able to implement these in the classroom**. For me it is about **knowing about what I need to teach confidently and imparting this onto my students**. I think this unit has helped significantly… The documents provided to us, the lectures, and going through the [Australian] Curriculum … The Curriculum is in front of us, but **being able to unpack it and know exactly what achievement standards are required, and so forth**, it really helps to prepare. My pre-confidence would be a 4 and now I feel up there, maybe being a 9.

> **<u>Student 8 (2015):</u>** Most definitely yes … Many teachers get lost. They don't really have a good idea of algebra themselves, so when they come to try and teach it they like... oh I don't know. Whereas this unit really gave - it **increased your own algebraic thinking**, so then you'd be confident being able **to translate that into teaching**.

> **<u>Student 12 (retraining primary teacher, 2015):</u>** Before the unit I wondered whether I could teach mathematics at this level and **now I know I have the tools to successfully teach it**, although I feel a lot of previous planning would be required in the initial stages.

> **<u>Student 14 (retraining primary teacher, 2015):</u>** Yes it has. I can hold a **conversation with my peers regarding the pedagogy**. I now just need to familiarise myself with Year 10 algebra and vocabulary.

Beliefs about Mathematics

One last interesting aspect of the research will be briefly reflected upon here. The students were also asked to complete a second set of pre- and post-test surveys that attempted to explore their general feelings about classroom mathematics (Swan, 2005). Swan's diagnostic questionnaire interrogates the idea of three preferred "orientations" for teaching – namely *transmission, discovery,* and *connectionist* – first discussed by Askew et al. (1997), as mentioned in the literature review. Teacher beliefs and attitudes are notoriously difficult to influence in any significant way, as the research has shown (Beswick, 2005, 2006; Grootenboer, 2008; Goldin, Rosken, & Torner, 2009; Hurrell, 2013). This was largely seen to be true in this study also.

When the mean distributions of student beliefs were calculated, expressed as a percentage breakdown of three items, the respondents were relatively evenly balanced in their preferences for the three orientations, but with a somewhat stronger preference for the *transmission* mode of teaching and learning. There was very little change to this seen in the post-test, although the connectionist orientation percentages gained on average by 2%, and those for the discovery and transmission orientations lost 1% each. This can be seen in Table 8 and Figure 12 below.

Teaching orientation	*"Mathematics is …"*
Transmission	A body of knowledge and standard procedures; a set of universal truths and rules, which need to be conveyed to students
Discovery	A creative subject in which the teacher should take a facilitating role, allowing students to create their own concepts and methods
Connectionist	An interconnected body of ideas which the teacher and the students create together through discussion

Table 8. Descriptions for the "orientations" associated with students' beliefs about mathematics (Swan, 2005)

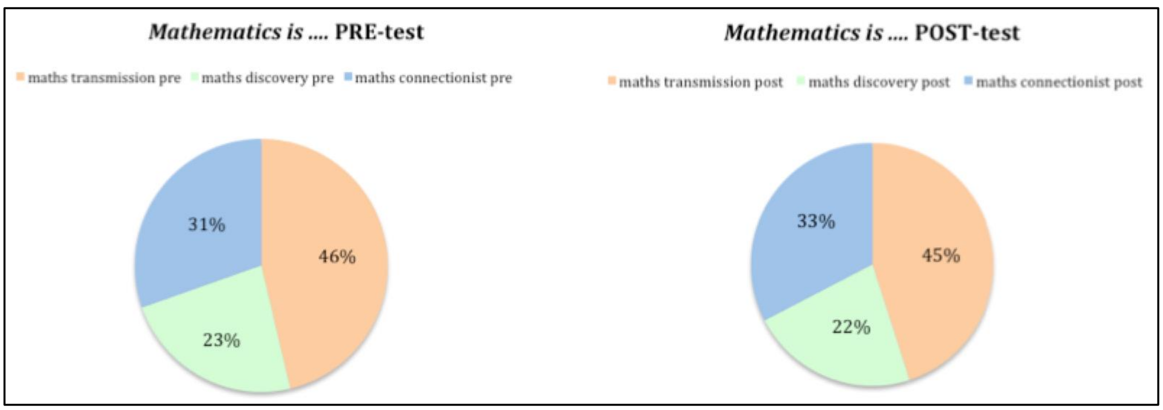


Figure 12. Differences in the overall means of the percentage breakdown scores for *Mathematics*, attributed by students before and after unit participation. ("*Give each of the three statements a percentage (so that the sum of the three percentages adds to 100%.")*

Further analysis, seen in Table 9, supports the finding of a very slight change, with the overall mean of the percentages rising by just 2.17% for the *connectionist* orientation and dropping only very slightly for the *discovery* and *transmission orientations*. In support of this, Paired *t*-tests on these same means are all quite non-significant (p=0.56, p=0.69 and p=0.27).

Teaching orientation	**Pre- and post-test differences** *(Swan, 2005)* *"Mathematics is …"*	**Mean of percentage differences**	**Standard deviation of percentage differences**	***p*-value (Paired Student-t test)**
Transmission	A body of knowledge and standard procedures; a set of universal truths and rules, which need to be conveyed to students [**MT2 – MT1**]	-1.33	11.392	0.556
Discovery	A creative subject in which the teacher should take a facilitating role, allowing students to create their own concepts and methods [**MD2 – MD1**]	-0.95	11.821	0.685
Connectionist	An interconnected body of ideas which the teacher and the students create together through discussion [**MD2 – MD1**]	2.17	9.863	0.274

Table 9. Mean rank differences (decreases or increases, *(% post-test minus % pre-test)*) in pre-test to post-test percentage attribution responses to two paired sets of questions in the two surveys. This is again also analysed using the Paired Student *t*-test (*n*=26).

This result raised an immediate question. Can it be assumed that, in a general sense, this particular small group of pre-service and in-service teachers has firmly maintained their original beliefs? In other words, were they actually relatively unaffected by a unit offering that has consistently argued for the need to recognize the *connectedness* of mathematics and the need for *student inquiry*? It would at first indeed appear so; and also that the perceived usefulness of inquiry-based learning has dropped slightly in the opinions of the group, rather than increasing with the creative approach intended in the unit.

However, it is more revealing to look at this on a case-by-case basis across all 26 respondents to the quantitative surveys. The researcher was prompted to do this by the relatively large standard deviations around these very small mean changes, and Figure 13 provides some interesting extra information about the individuals and their responses. The larger variability of the actual individual movements in the percentage weightings is now seen. For example, Respondents 2[*], 4[*] and 11[*] have each shifted by 10% or even 20% from *transmission* to *connectionism*; and Respondent 18 (Student 7 in the case study) by 20% overall to *discovery* learning and *connectionism*. However, by contrast, Respondents 6, 9 (Students 1 and 3 in the case studies), 4[*] and 15[*] have moved 20% overall away from the more student-centred orientations towards a more traditional *transmission* one. It appears that the group was not so homogenous after all, and only 6 of the 26 individuals (Respondents 3, 5, 10[*], 12[*], 13[*] and 22[*]) have actually maintained exactly the same percentage breakdown in the post-test as in the pre-test, thus appearing to have not changed even slightly in regard to beliefs about classroom mathematics. (Students 2 and 4 in the case study groups are amongst these.)

[*] These are students not involved in the case study interviews, for the reader's interest.

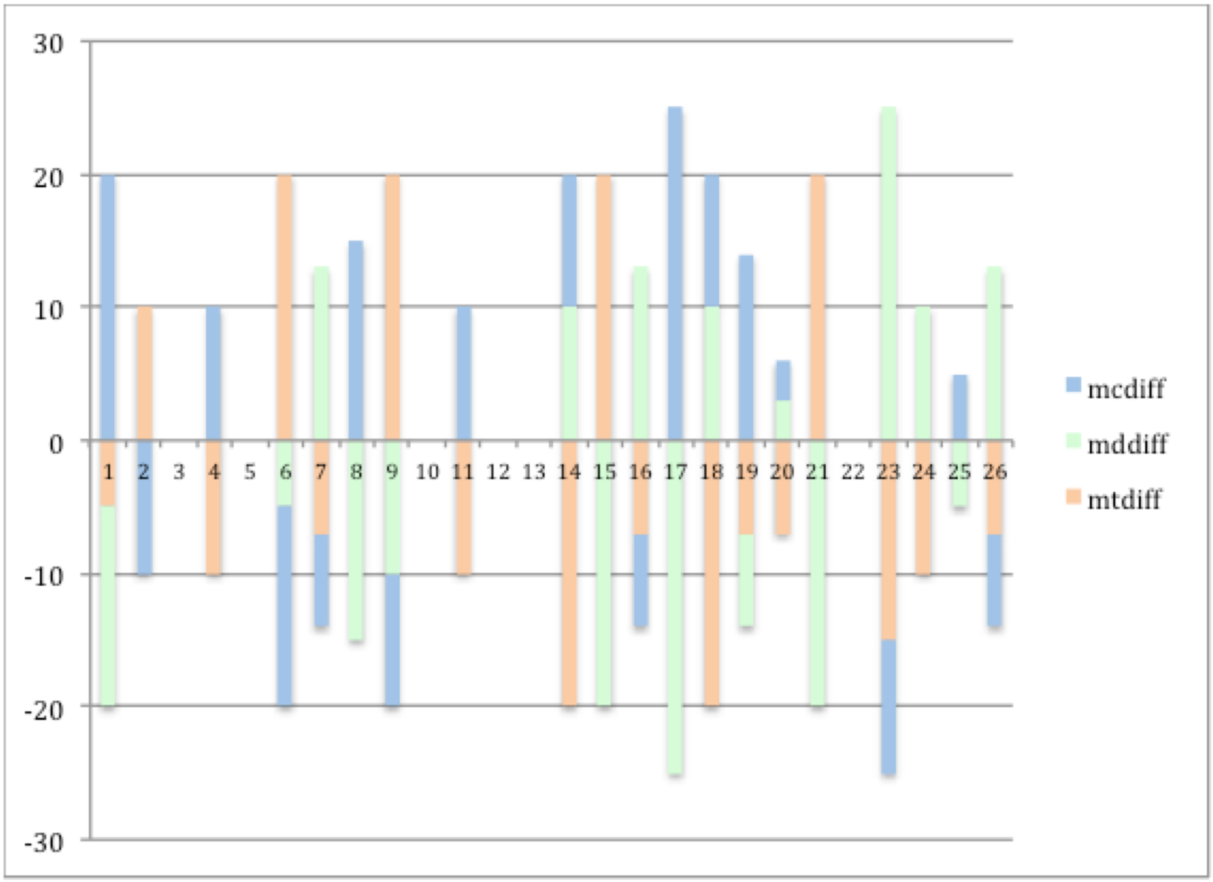


Figure 13. Bar charts showing individual shifts of all student percentage breakdown scores, before and after unit participation. ("*Give each of the three statements a percentage (so that the sum of the three percentages adds to 100%.")*

This further analysis then posed an associated question for the researcher: was the emphasis in the unit upon the importance of inquiry-based, deep learning in these transition years in the strand unheeded by some students, after all? Did, perhaps, the highly structured nature of the documents and of the content – even with the added sense of control and confidence that this appeared to afford students (as revealed anecdotally) – actually move some of them away from more creative teaching, rather than moving them towards it?

The students' future classroom teaching practices can only be guessed at, but one comforting statistic did emerge from the quantitative data that indicated that this might not be quite the case. The students appeared to have moved significantly in their self-confidence and sense of self-efficacy when it came to enacting and understanding *problem solving scenarios* with their own students. Table 10 evinces this claim.

Please tick the boxes that match best how well that you now feel you understand this ***component*** *... in mathematics teaching: (badly (1), sometimes well (2), soundly (3), excellently (4))*	**Mean rank increase on Likert scale from 1 to 4**	**Standard deviation**
How to assess students' mathematical **problem-solving skills** (**ps**)	0.519	1.014

Table 10. Mean rank increases from pre-test to post-test responses to a paired sets of questions in the two surveys (*n*=26). (Also see Tables 2 to 5.)

One case study participant also had this to say in response to the question: *"Has your professional self-efficacy increased due to participation in the unit?"*

> **Student 13 (retraining secondary teacher, 2015):** Yes. I am more confident of what my classes should be able to do and how to get them doing it, I am more adventurous in the classroom and more confident in discussing and following discussion of the direction a course/class/topic should follow.

The researcher hopes that the confidence and knowledge gained by this small cohort of students has, at least to some degree, offset any possible tendencies to equate the structured nature of the unit's *teaching model,* with the much more fluid *nature of mathematics itself,* or of their own students' *learning of mathematics*.

Conclusion

The rationale for the unit described in this paper derives from the researcher's belief that providing teachers with some kind of synthesised and illuminating *interpretative model* for the mandated Australian Curriculum may actually allow them more freedom to develop their lessons in an open and inquiring way, as well as in a better informed one. Without such supports as provided by overviews as the ACCR structure and its summative documents, and without a rich variety of pedagogical exemplars, less experienced teachers are more likely to resort to a fragmented and *purely* linear view of mathematics, or worse still, an un-informed, more textbook-driven one. A deeper and more comprehensive understanding of the Australian Curriculum for mathematics is very much needed by many teachers in both primary and secondary schools.

Just as a sense of balance is needed in the strategic use of those well-researched classroom teaching orientations, the same is no doubt true for mathematics education programs for pre-service teachers. Teaching *how* to teach needs to be modelled at every point. As structured as the discipline of mathematics may be described to be, it has been argued in this paper that even more structure and guidance is needed to support improved teacher PCK. It has been suggested that the guidance needed should be one that attempts to enable or unearth teachers' sense of mathematical *connections* (Ormond, 2012a). Perhaps there is something about a minute analysis of the scoping and sequencing of a mathematics strand that can liberate novice teachers to experiment with inquiry processes – knowing that a sound understanding of some sort of master plan will mean that they can diverge a little without getting lost.

Any research conclusions in this small study are necessarily somewhat limited by the small sample size, and by the more subjective interpretive analysis of a researcher who was more focused upon reflection and interrogation of some pedagogical themes, than upon gathering inferential evidence. Further planned research into these ideas may provide more evidence about the ideas discussed here. It is intended that such future research will look at a much larger sample than seen in this three-year pilot, and interrogate more closely, using a broader and better defined scale, just how any positive changes may be encouraged by similar approaches. The contention of the paper, however, that teaching mathematics in isolated pockets does not work effectively for classroom mathematics students, is unchanged. Experienced teachers have the necessary cohesive vision to avoid this tendency in their teaching, but novice teachers almost inevitably do not.

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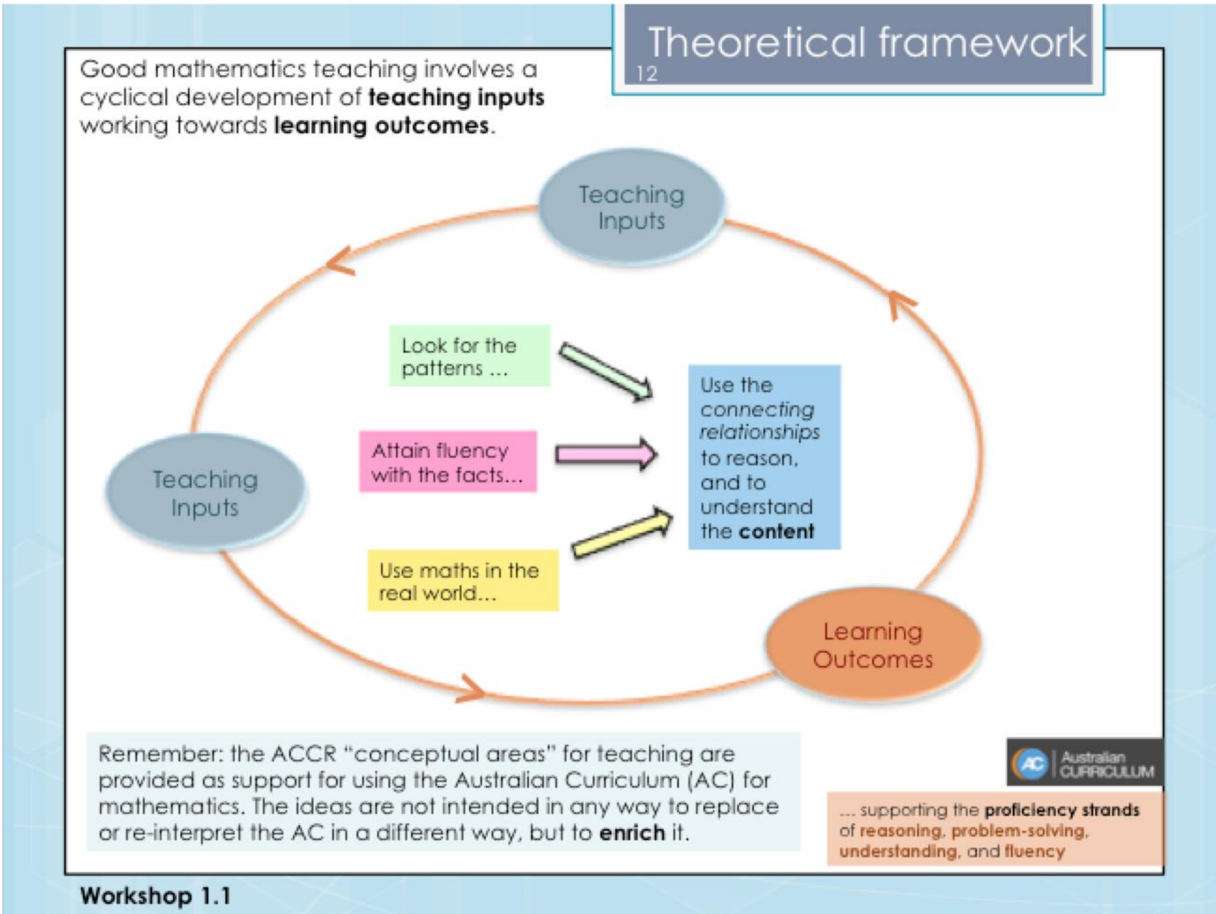
Appendix

1. Please describe your feelings about **mathematics and mathematics teaching**, relating them where relevant to your experience in the unit. Have these feelings changed or remained about the same?	4. Has your **professional confidence** increased due to participation in the unit?
2. The unit aimed to develop a **more overall, holistic understanding of the way in which pre-algebraic work can support beginning algebra** in lower secondary school. Please comment on whether or not this was successful for you.	5. Has your **professional self-efficacy** increased due to participation in the unit?
3. Comment on how you see **children's evolution of mathematical concepts in number and number patterns**, over the latter half of primary school especially, and relate this if relevant to your experience in the unit.	6. Please describe your current understanding of the **cognitive demands** in mathematics as students move from primary to secondary school.

Appendix Table 1. Case study questions for the study.



Appendix Figure 1. A PowerPoint slide illustrating the ACCR conceptual areas and their usefulness in effective teaching practice. This provides a clearer image to help readers better understand Figure 6 in the body of the paper.

AUSTRALIAN CURRICULUM ACHIEVEMENT STANDARDS: Conceptual areas of *Number and Algebra*

Pre-algebraic patterning Operations (number facts) Equations/ Equality/ Inequality

	Foundation	Year 1	Year 2
BEGINNING	By the end of the Foundation year, students make connections between number names, numerals and quantities up to 10. They count to and from 20 and order small collections.	By the end of Year 1, students describe number sequences resulting from skip counting by 2s, 5s and 10s. Students count to and from 100 and locate numbers on a number line. They carry out simple additions and subtractions using counting strategies. They continue simple patterns involving numbers and objects.	By the end of Year 2, students recognise increasing and decreasing number sequences involving 2s, 3s and 5s. They represent multiplication and division by grouping into sets. Students identify the missing element in a number sequence. Students count to and from 1000. They perform simple addition and subtraction calculations using a range of strategies.
	By the end of the Foundation year, students make connections between number names, numerals and quantities up to 10. They count to and from 20 and order small collections.	By the end of Year 1, students describe number sequences resulting from skip counting by 2s, 5s and 10s. Students count to and from 100 and locate numbers on a number line. They carry out simple additions and subtractions using counting strategies. They continue simple patterns involving numbers and objects.	By the end of Year 2, students recognise increasing and decreasing number sequences involving 2s, 3s and 5s. They represent multiplication and division by grouping into sets. Students identify the missing element in a number sequence. Students count to and from 1000. They perform simple addition and subtraction calculations using a range of strategies.
	By the end of the Foundation year, students make connections between number names, numerals and quantities up to 10. They count to and from 20 and order small collections.	By the end of Year 1, students describe number sequences resulting from skip counting by 2s, 5s and 10s. Students count to and from 100 and locate numbers on a number line. They carry out simple additions and subtractions using counting strategies. They continue simple patterns involving numbers and objects.	By the end of Year 2, students recognise increasing and decreasing number sequences involving 2s, 3s and 5s. They represent multiplication and division by grouping into sets. Students identify the missing element in a number sequence. Students count to and from 1000. They perform simple addition and subtraction calculations using a range of strategies.

	Year 3	Year 4	Year 5
MIDDLE	By the end of Year 3, students recognise the connection between addition and subtraction and solve problems using efficient strategies for multiplication. Students count to and from 10 000. They classify numbers as either odd or even. They recall addition and multiplication facts for single digit numbers. They continue number patterns involving addition and subtraction.	By the end of Year 4, students choose appropriate strategies for calculations involving multiplication and division. They identify unknown quantities in number sentences. They describe number patterns resulting from multiplication. Students use the properties of odd and even numbers. They recall multiplication facts to 10 x 10 and related division facts. They continue number sequences involving multiples of single digit numbers and create … [shape] patterns.	By the end of Year 5, students solve simple problems involving the four operations using a range of strategies … Students identify and describe factors and multiples … Students continue patterns by adding and subtracting fractions and decimals. They find unknown quantities in number sentences.
	By the end of Year 3, students recognise the connection between addition and subtraction and solve problems using efficient strategies for multiplication. Students count to and from 10 000. They classify numbers as either odd or even. They recall addition and multiplication facts for single digit numbers. They continue number patterns involving addition and subtraction.	By the end of Year 4, students choose appropriate strategies for calculations involving multiplication and division. They identify unknown quantities in number sentences. They describe number patterns resulting from multiplication. Students use the properties of odd and even numbers. They recall multiplication facts to 10 x 10 and related division facts. They continue number sequences involving multiples of single digit numbers and create … [shape] patterns.	By the end of Year 5, students solve simple problems involving the four operations using a range of strategies … Students identify and describe factors and multiples … Students continue patterns by adding and subtracting fractions and decimals. They find unknown quantities in number sentences.
	By the end of Year 3, students recognise the connection between addition and subtraction and solve problems using efficient strategies for multiplication. Students count to and from 10 000. They classify numbers as either odd or even. They recall addition and multiplication facts for single digit numbers. They continue number patterns involving addition and subtraction.	By the end of Year 4, students choose appropriate strategies for calculations involving multiplication and division. They identify unknown quantities in number sentences. They describe number patterns resulting from multiplication. Students use the properties of odd and even numbers. They recall multiplication facts to 10 x 10 and related division facts. They continue number sequences involving multiples of single digit numbers and create … [shape] patterns.	By the end of Year 5, students solve simple problems involving the four operations using a range of strategies … Students identify and describe factors and multiples … Students continue patterns by adding and subtracting fractions and decimals. They find unknown quantities in number sentences.

	Year 6	Year 7	Year 8
LATER	By the end of Year 6, students recognise the properties of prime, composite, square and triangular numbers … They solve problems involving all four operations with whole numbers. They describe rules used in sequences involving whole numbers, fractions and decimals … They write correct number sentences using brackets and order of operations … Students locate an ordered pair in any one of the four quadrants on the Cartesian plane.	By the end of Year 7, students solve problems involving the comparison, addition and subtraction of integers; make the connections between whole numbers and index notation, and the relationship between perfect squares and square roots; and represent numbers using variables … They connect the laws and properties for numbers to algebra; interpret simple linear representations and model authentic information; and solve simple linear equations and evaluate algebraic expressions after numerical substitution. They assign ordered pairs to given points on the Cartesian plane.	By the end of Year 8, students solve everyday problems involving rates, ratios and percentages. They recognise index laws and apply them to whole numbers … They describe rational and irrational numbers. They make connections between expanding and factorising algebraic expressions, and use efficient mental and written strategies to carry out the four operations with integers. They simplify a variety of algebraic expressions, and solve linear equations and graph linear relationships on the Cartesian plane.
	By the end of Year 6, students recognise the properties of prime, composite, square and triangular numbers … They solve problems involving all four operations with whole numbers. They describe rules used in sequences involving whole numbers, fractions and decimals … They write correct number sentences using brackets and order of operations … Students locate an ordered pair in any one of the four quadrants on the Cartesian plane.	By the end of Year 7, students solve problems involving the comparison, addition and subtraction of integers; make the connections between whole numbers and index notation, and the relationship between perfect squares and square roots; and represent numbers using variables … They connect the laws and properties for numbers to algebra; interpret simple linear representations and model authentic information; and solve simple linear equations and evaluate algebraic expressions after numerical substitution. They assign ordered pairs to given points on the Cartesian plane.	By the end of Year 8, students solve everyday problems involving rates, ratios and percentages. They recognise index laws and apply them to whole numbers … They describe rational and irrational numbers. They make connections between expanding and factorising algebraic expressions, and use efficient mental and written strategies to carry out the four operations with integers. They simplify a variety of algebraic expressions, and solve linear equations and graph linear relationships on the Cartesian plane.
	By the end of Year 6, students recognise the properties of prime, composite, square and triangular numbers … They solve problems involving all four operations with whole numbers. They describe rules used in sequences involving whole numbers, fractions and decimals … They write correct number sentences using brackets and order of operations … Students locate an ordered pair in any one of the four quadrants on the Cartesian plane.	By the end of Year 7, students solve problems involving the comparison, addition and subtraction of integers; make the connections between whole numbers and index notation, and the relationship between perfect squares and square roots; and represent numbers using variables … They connect the laws and properties for numbers to algebra; interpret simple linear representations and model authentic information; and solve simple linear equations and evaluate algebraic expressions after numerical substitution. They assign ordered pairs to given points on the Cartesian plane.	By the end of Year 8, students solve everyday problems involving rates, ratios and percentages. They recognise index laws and apply them to whole numbers … They describe rational and irrational numbers. They make connections between expanding and factorising algebraic expressions, and use efficient mental and written strategies to carry out the four operations with integers. They simplify a variety of algebraic expressions, and solve linear equations and graph linear relationships on the Cartesian plane.

Appendix Figure 2. The Years F to 10 Australian Curriculum *Number and Algebra* Achievement Standards (abridged) document. This allows students to think about the three "conceptual areas" for the strand both as *isolated* ideas in each Standards excerpt, as well as *overlapping* and related ones. An activity required the students to try highlighting these categories first for themselves.

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