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Algebra and Proof in High School: The Case of Algebraic Proof as Discovery

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Abstract

In the United States, students' learning experiences around proof is generally concentrated in the domain of high school geometry with a focus on its verification function. Thus, providing students with a limited conception of what proof entails and the role that it plays in performing mathematics. Moreover, there is a lack of U.S.-based studies addressing how to integrate proof into other mathematical domains within the high school curriculum. In this paper, the author reports results from an interview at the end of a teaching experiment which was designed to integrate algebra and proof into the high school curriculum. Algebraic proof was envisioned as the vehicle that would provide high school students the opportunity to learn about proof in a context other than geometry. Results indicate that most students were able to produce an algebraic proof involving variables and a parameter and its multiples. In doing so, students experienced the discovery function of proof.

Keywords: proof, algebra, discovery, proof

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Álgebra y Prueba en la Secundaria: el Caso de la Demostración Algebráica como Descubrimiento

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Resumen

En USA las experiencias de aprendizaje de los estudiantes sobre demostración se concentran generalmente en el dominio de la geometría de secundaria, con énfasis en su función de verificación. Sin embargo, se les da una concepción limitada de lo que significa. Es más, existe una falta de estudios en USA sobre cómo integrar el uso de las demostraciones en otros dominios de las matemáticas dentro del currículum de secundaria. En este artículo se presentan resultados de una entrevista realizada al final de un experimento de enseñanza que fue diseñado para integrar el álgebra y la demostración en el currículum de secundaria. La demostración algebraica se presenta como un vehículo que puede aportar la oportunidad de aprender sobre la demostración en otros contextos que no sean la geometría. Los resultados indican que la mayoría de los estudiantes son capaces de producir demostraciones algebraicas que involucran variables y parámetros. Haciendo esto, los estudiantes experimentan el descubrimiento de la función de las demostraciones.

Palabras clave: demostración, álgebra, descubrimiento, demostración

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In the U.S., we lack a robust research base focused on students' experiences around proof in high school in domains other than geometry. Hanna (2007) argues that mathematical proof has increasingly played a less prominent role in the secondary mathematics curriculum in the U.S., thus urging, "We need to find ways, through research and classroom experience, to help students master the skills and the understanding they need" (p.15). Indeed, it is surprising the void in the U.S. curriculum regarding proof with the exception of high school geometry, and its emphasis on the two-column format (Herbst, 2002). Despite this past lack of emphasis on proof, however, there currently seems to be a shift in how proof is viewed as part of the secondary mathematics curriculum. The authors argue that proof should be naturally incorporated into all areas of the curriculum. There are currently several other documents advocating for the central role of proof in the teaching and learning of mathematics across all grades (e.g., Stylianou, Blanton, & Knuth, 2009). However, Herbst (2002) argues that a change in how students view proof would require more than minor adjustments or calls for reform. Some research suggests that integrating proof into domains other than geometry holds much promise for students' understanding of proof (e.g., Hanna & Barbeau, 2008; Healy & Hoyles, 2000; Pedemonte, 2008; Stylianou et al., 2009). As we are far from having a sound body of research, it remains to systematically study how to incorporate proof into areas other than geometry in ways that support students' learning of these important concepts. To do this, the author draws from research in countries that do have a tradition of incorporating proof throughout the curriculum both across domains within mathematics and across grades (e.g., Arsac et al., 1992; Balacheff, 1982; Boero, Garuti, & Mariotti, 1996; Pedemonte, 2008). In addition, over the past decade, the learning and teaching of algebra has increasingly become a central component of the mathematics education research agenda (Gutiérrez & Boero, 2006; Stacey, Chick, & Kendal, 2004). In the U.S., algebra is often considered as a gatekeeper to accessing, and ultimately understanding, more advanced mathematics (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2009; National Mathematics Advisory Panel, 2008). Despite its role as gatekeeper, researchers have largely demonstrated the difficulties students have encountered in learning algebra. For example, some studies show that students do not comprehend the use of letters as generalized numbers or as variables (e.g., Booth, 1984; Kuchemann, 1981) and have difficulty operating on unknowns (e.g., Bednarz, 2001; Filloy & Rojano, 1989; Filloy, Rojano, & Puig, 2008; Filloy, Rojano, & Solares, 2010). Additionally, extant research has focused mostly on elementary algebra (the study of patterns, functions, linear equations, etc.); consequently, little is known about students' mathematical experiences around more advanced algebraic concepts and skills, such as algebraic expressions involving multiple variables and parameters, among others, as recommended by the National Mathematics Advisory Panel (2008).

Taken together, I hypothesize that algebraic proof can serve as a vehicle for integrating more advanced algebraic concepts (e.g., problems involving multiple variables and parameters) together with mathematical proof in the U.S. curriculum, thus supporting students' understanding of both important mathematical topics. In fact, some researchers have highlighted that proofs are more than instruments to establish that a mathematical statement is true. Indeed, they embody mathematical knowledge in the form of methods, tools, strategies, and concepts (Hanna & Barbeau, 2008; Rav, 1999). Thus, the field is in need of U.S.-based studies that develop ways of integrating proof in this broader way, across different areas of mathematics and across grade levels, and to systematically study the ways in which students engage with innovative problems that involve algebraic proof.

It is in this context that the author presents findings from a teaching experiment that focused on an integrated approach to the teaching of algebra and proof in high schools in the U.S. Accordingly, the goals of this paper are as follows: (1) to state our working hypothesis that an integrated approach to algebra and proof has great potential to foster students' meaningful learning of both algebra and proof; (2) to make explicit the underlying theoretical principles used to inform the design of problems; and, (3) to describe students' mathematical processes as they worked through a problem that involves multiple variables, and a parameter and its multiples in an individual interview setting at the end of the teaching experiment aimed at assessing student learning. In doing this, I will present an illustration of students using proof as discovery.

Framework

Algebra: A Tool with the Potential to Make Explicit What is Implicit

Regarding algebra, researchers worldwide have proposed varied conceptualizations of school algebra in order to foster students' meaningful learning. Among many, some are the functional approach (e.g., Schwartz & Yerushalmy, 1992) and algebra as generalization (Lee, 1996; Mason, 1996).

Each approach highlights different and centrally important aspects of school algebra. However, from our perspective, what is still missing is an emphasis on one of the most important features of algebra: by manipulating an expression we can read information that was not visible or explicit in the initial expression (Arcavi, 1994). For instance, we can show using algebra that if we add three. In addition, we can also see that the sum will always be the triple of the second number 3(a+1) by factoring 3 from both terms. This aspect of algebra—the use of algebraic notation to make explicit what previously was implicit— has great potential to link algebra with proof due to its capacity to unveil that a certain property holds for all cases (e.g., the sum of any three consecutive integers is (always) a multiple of three). This feature of algebra lies at the centre of the teaching experiment described herein.

Algebra as a Modelling Tool

Given the project's focus on integrating algebra and proof, I employ Chevallard's (1989) framework in which algebra is envisioned as a tool to mathematically model problems. In particular, Chevallard (1989) described the modelling process using algebra in the following way: (1) Define the system to be studied by identifying the pertinent aspects in relation to the study of the systems that are to be carried out … (2) Build a model by establishing a certain number of relations R, R', R'', etc., among the variables chosen in the first stage; the model of the systems to study is the set of these relations. (3) 'Work' the model obtained through stages 1-2 with the goal of producing knowledge of the studied system, knowledge that manifests itself by new relations among the variables of the system. (p. 53)

Recalling the example presented in the previous section, we can use algebra to show (i.e., prove) that the sum of any three consecutive integer numbers is always a multiple of three. First, following Chevallard's (1989) framework, we identified the variables which are the three integers a, b, and c (i.e., first stage). Since they are consecutive we can, as outlined in the second stage, identify the mathematical relations that are at play. Thus, b=a+1, and c=b+1. In addition, since, b=a+1 substituting in c=b+1 gives us c=(a+1)+1=a+2. So far, we have expressed the three consecutive numbers by using the "consecutive" relation obtaining the following: a, a+1, a+2. Now, in the third stage, since we have the same variable, we can work the model as follows: a+(a+1)+(a+2)=(a+a+a)+(1+2)=3a+3=3(a+1). This last expression shows that, in fact, the sum of three consecutive integers is (always) a multiple of three. We can also say with precision that the sum is a specific multiple of three; it is also a multiple of the second consecutive number.

This aspect of algebra, namely the potential to reveal or make explicit new information through the use of properties, has been underplayed in school algebra, yet has great potential for integrating algebra and proof, and builds upon the so-called 'discovery' function of proof as mentioned by De Villiers (1990). By using algebra, we can generalize patterns and represent relations, and thus capture all cases with a general expression. This is necessary when in need of proving a statement with a universal quantifier. In addition, using algebra, we can manipulate and transform an expression into equivalent expressions (e.g., using distributive property); as a consequence, we may make explicit in the later expression what we wanted to show (i.e., prove). These two aspects of algebra are central to constructing algebraic proofs.

Mathematical Proof

The notion of proof in our framework has been conceptualized by bringing together a range of prior research. First, I build on Balacheff's (1982, 1988) notion that proof is an explanation that is accepted by a community at a given time. He distinguishes between an explanation (i.e., the discourse of an individual who aims to establish for somebody else the validity of a statement) and a mathematical proof (i.e., a discourse with a specific structure and follow well-defined rules that have been formalized by logicians).

Second, I draw on the corpus of research that brought to the forefront the multiple functions of proof beyond the verification purpose. Among them, Hanna's (1990) distinction between proofs that (just) prove and proofs that (also) explain. The first type just establishes the validity of a mathematical statement while the second type, in addition to proving, reveals and makes use of the mathematical ideas that motivate it. In a similar vein, Arsac et al. (1992) proposed three roles for proofs: to understand why and/or to know, to decide the truth-value of a conjecture, and to convince oneself or someone else. In a similar vein, De Villiers (1990, 2012) gave the following roles that proofs play in mathematics: verification (i.e., concerned with the truth of the statement); explanation (i.e., providing insight into why it is true); systematization (i.e., the organization of various results into a deductive system of axioms, major concepts and theorems); discovery (i.e., the discovery or invention of new results); and communication (i.e., the transmission of mathematical knowledge). De Villiers (1990) suggests that these other functions of proof can have pedagogical value in the mathematics classroom. Of special interest to us is the function of discovery, namely, when new results are discovered/invented in a deductive way. As it will be shown later, this is the way proof was embodied in the interview.

Third, drawing from Boero's notion of cognity unity (2007, 1996), it is assumed that conjecturing and proving are inter-related and crucial mechanisms in generating mathematical knowledge. Therefore, in the problems analyzed as part of our teaching experiment, students were not provided with the conjecture to prove. Instead, as part of the problem, they had to construct or produce their own conjectures, and then prove them.

In summary, in our teaching experiment, proof was conceived as an explanation accepted within the classroom community (Balacheff, 1982, 1988). Students had the opportunity to engage in problems that required the construction of a proof more from a problem-solving approach. That is, students were not required to prove up front (Boero et al., 2007), rather they had to first investigate a mathematical phenomenon and, as a result, they would produce conjectures. Students were challenged to provide evidence grounded on mathematical relations and logic. Thus, it was intended that students would use examples to explore the problem and, as a result, they would come up with conjectures about aspects of the problem. At this stage, students would engage in a "proving situation" framed as coming up with evidence beyond particular cases that would show that the conjecture is true always, and would also give us insight as to why this was the case. Oftentimes, students would set out to prove a conjecture to end up proving something else, thus discovering through the proving process.

Theoretical Principles and Mathematical Tasks

Drawing from the theoretical framework, I synthesize the principles that guided the design of the Calendar Algebra Problems used in our teaching experiment: (1) to foster students' production of conjectures (Boero et al., 2007) and the entailed interplay between examples and counter-examples; (2) to focus on algebra as a modelling tool (Chevallard, 1989); (3) to promote a broader role of proof (Arsac et al., 1992; De Villiers, 1990, 2012; Hanna, 1990); and (4) to showcase algebraic proof by leveraging the link between algebra and proof that lies in students' production of equivalent algebraic expressions to reveal or make explicit information in the expression (Arcavi, 1994; Martinez, 2011).

The design has also been informed by previous reports such as Barallobres (2004), Bell (1995) and Friedlander and Hershkowitz (2001). The Calendar Algebra Problems share the same context, the calendar, and were presented according to an increasing degree of complexity. For example, in Problem 1 (Figure 1), students worked on a regular calendar with seven days per week and a 2x2-calendar square (see Table 1). In this case, the multiple variables (one independent variable and the rest dependent variables) correspond to the numbers (i.e., day-number) within the 2x2-calendar square. For a more detailed description of the problems, see Martinez (2011). In Problem 9, the length of the week changed from 7 to 9 days (see Table 1). Consequently, the level of difficulty increased given that students not only had to define the multiple variables involved in the problem, but also the mathematical relations among them changed (i.e., a, a+1, a+9, and a+10). This change in the length of the week was intended to lay the groundwork for Problem 18 (Table 1), in which the length of the week was parameterized to *d*-days. For a more detailed description of these problems, see Martinez (2011).

Problem 1

Part 1. Consider squares of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be $\begin{matrix} 1 & 2 \\ 8 & 9 \end{matrix}$.

These squares will be called 2x2-calendar-squares.

Calculate:

1. The product between the number in the upper left corner and the number in the lower right corner.
2. The product between the number in the upper right corner and the number in the lower left corner.
3. To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Find the 2x2-calendar-square that gives the biggest outcome. You may use any month of any year that you want.

Part 2. Show and explain why the outcome is always going to be -7.

Figure 1. Problem 1 from the Calendar Algebra Sequence.

In Problem 12 (Figure 2), students are asked to analyze the nature of the outcome of the described calculation (i.e., subtraction of the cross product). In implementing the first design principle (Boero et al., 2007), rather than providing the conjecture to be proved, the problem was posed in such a way as to offer students an opportunity to come up with their own conjectures. It was intended that the problem would provide an opportunity for students to analyze the dependence/independence of the outcome in regards to the month, year, location of the 4x4-calendar-square, and days in the week. Once they reach a conclusion and produce a conjecture, the challenge becomes to gather evidence to show why this happens. In doing so, students discovered the exact expression for the outcomes, as it will be shown in the Results section.

Table 1
Sequence of problems intended to provide an opportunity to parameterize days in a week.

Lesson Number	Date	Problem Number	Generic square including independent and dependent variables	Dimension of the calendar square	Length of the week (Parameter)	Outcome
1 and 2	Oct. 11 and 18	1	$\begin{matrix} a & a+1 \\ a+7 & a+8 \end{matrix}$	2x2	7 is specific instance of the parameter	-7
13	Feb. 7	9	$\begin{matrix} a & a+1 \\ a+9 & a+10 \end{matrix}$	2x2	9 is a specific instance of the parameter	-9
13	Feb. 7	18	$\begin{matrix} a & a+1 \\ a+d & a+d+1 \end{matrix}$	2x2	*d*	-*d*
Interview	Feb 28-Apr. 4	12	$\begin{matrix} a & a+1 & a+2 & a+3 \\ a+d & a+1+d & a+2+d & a+3+d \\ a+2d & a+1+2d & a+2+2d & a+3+2d \\ a+3d & a+1+3d & a+2+3d & a+3++3d \end{matrix}$	4x4	*d*	-9*d*

Problem 12

In this case you will be working with a month that has *d*-days in each week. The shape to use is a 4x4-calendar-square.

The set of operations to carry out are the following:

1. The product between the number in the upper left corner and the number in the lower right corner.
2. The product between the number in the upper right corner and the number in the lower left corner.
3. To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Your task is to analyze the behavior of the outcome in terms of its dependency on the dimensions of the square, length of the week, and position of the square.

Figure 2. Problem 12 from the Calendar Algebra Sequence.

In line with the second and fourth design principles (Chevallard, 1989; Usiskin, 1988), one of the goals of this teaching experiment was to provide students the opportunity to use algebra as a modelling tool. As part of this process, students would interpret the final expression in terms of their initial conjectures and the context. In doing this, students would have the opportunity to connect the algebraic proof to the reasons that make the outcome a specific value (i.e., -9d), which is the implementation of the third design principle.

Previous Reports on the Calendar Algebra Problems

Students had had ample opportunities during the teaching experiment to analyze problems and generate their own conjectures, and to use algebra as a modelling and proving tool. Specifically, students encountered a variety of problems all with the same context (i.e., the calendar) and had experimented with different shapes (2x2-calendar-squares, 4x4-calendar-squares, 5x2-calendar-rectangles, etc.), different operations (subtraction of the cross product, addition of the product of the middle numbers, etc.) and different outcomes (e.g., constant outcome in Problem 1). During the first lessons of the teaching experiment (Problem 1), students faced challenges related to: what counts as evidence to prove a universal statement, the necessary number of independent variables to model the problem, the deductive nature of the task in comparison to what they were accustomed (i.e., equation-solving and modelling with linear functions), operations involving variables (e.g., a.a=a2 and a+a=2a), and properties (e.g., how to distribute -1, how to multiply binomials and trinomials), among others. These challenges were addressed when they appeared within the context of a problem or, alternatively, sometimes the teacher designed tasks targeting specific problematic issues. The latter was in the case, for instance, of how to distribute -1 within different kinds of polynomials. For a more detailed description of results, see Martinez, Brizuela and Castro Superfine (2011) and Martinez (2011).

By the time students encountered Problems 9 and 18, they had worked on identifying relevant variables, parameter and the relations among them, and representing them using algebraic notation. Students also used properties (e.g., distributive property) to generate a chain of equivalent expressions that allowed them to ultimately produce a final expression that showed explicitly what they wanted to prove. The students continued improving to correctly use mathematical properties (e.g., distributive property) throughout the teaching experiment.

Even though all students ultimately succeeded in completing both problems (i.e., 9 and 18), the process was not straightforward. Students faced various challenges (e.g., correct use of the distributive property, use of parenthesis, etc.) that were overcome through discussion with their peers within the small group and/or by scaffolding provided by the teacher. Students did not face challenges associated with multiple variables per se. This was to be expected given that students had been working with multiple variables for twelve classes during which they worked on eight problems (Table 1). The new challenge at that point in the teaching experiment was the inclusion of a parameter. For a more detailed description, see Martinez and Castro Superfine (2012) and Martinez (2011).

Regarding the role of the teacher, in Martinez and Li (2010), it was reported specifically on the learning environment, and the teacher interventions that aimed at fostering students' mathematical inquiry, and maintaining the cognitive demand of the task (Stein, Smith, Henningsen, & Silver, 2009). Among these, we identified: (1) helping students re-focus their inquiry, (2) helping students select mathematical tools, (3) accepting students' provisory ideas, (4) recognizing the potential in students' ideas and promoting the student to showcase the idea, and (5) reviewing a property using an additional example to preserve the original challenge for students.

Methodology

Participants

Nine out of fifteen high school students (14 and 15-year olds) volunteered to participate in this study after being recruited jointly by their mathematics teacher and the researcher from an Integrated Mathematics and Science class of about twenty students at a public high school in Massachusetts. The students, who varied in terms of their mathematical performance in their regular mathematics class, included four females and five males. Students worked in the same groups throughout the teaching experiment. Abbie, Desiree and Grace were in one group; Chris, Janusz and Audrey in a second group; and, Brian, Cory and Tyler were in the third group. Pseudonyms are not used. Students were familiar with distributive, associative and cancellation properties, and with equation-solving. In their regular mathematics class they had just learned about linear functions with the entailed concepts of rate, slope, y-intercept, etc."

Data Collection

Students participated in a total of fifteen one-hour lessons per week and two individual interviews. Lessons and interviews were video- and audio-taped, and students' written work was collected and scanned for analysis. The author was the teacher and interviewer in this teaching experiment. In this paper, I report on data collected during Interview #2 (Problem 12), which took place at the end of the second part of the teaching experiment.

The goal of this interview was to gather data to assess individual student performance (i.e., confirmatory not exploratory), and mathematical processes after the teaching experiment had concluded. In this interview, each student was asked to solve Problem 12 that was new to the students but similar to the problems discussed in class.

Data Analysis

Data was analyzed qualitatively taking a grounded theory approach (Glaser & Strauss, 1967), which is a bottom-up approach. In other words, starting from the data, theoretical relationships and categories are constructed. As discussed previously, based on students' work on Problem 12, I identified the mathematical processes and the stages (e.g., conjecturing stage) that students went through. In addition, within each stage, I identified obstacles that students faced, ways in which they dealt with these obstacles, and illustrated how they were overcome. In order to do this, I worked with three data sources: video of the interview, students' written work and transcripts of the interview. Transcripts were parsed into stages and sub-stages taking into account what students were doing and saying, and taking into account the overall goal that oriented the activities. Once these were identified, I proceeded to describe students' mathematical processes.

Results

Our analysis suggests that overall students were able to use algebra to prove in an individual interview context at the end of the teaching experiment. In doing so, students worked on identifying relevant variables, parameter and the relations among them, and representing them using algebraic notation. Students also used properties (e.g., distributive property) to generate a chain of equivalent expressions that allowed them to ultimately produce a final expression that revealed the outcome. Only one student (Audrey) arrived at the conjecture, establishing that the outcome is -9d before proving; the rest of the students discovered what the outcome was at the end of the proving process. It is in this way that the algebraic proof functioned as discovery (De Villiers, 2012). In addition, students continued to use mathematical properties (e.g., distributive property) during the individual interview. These are the methods, concepts, and tools (Hanna, 1990; Rav, 1999) involved in the process of proving that students learned and/or with which they became more proficient.

Even though all students, with the exception of one, ultimately succeeded in completing the mathematical task, the process was not straightforward. Students faced various challenges (e.g., how to express some cells using the parameter’s multiple, etc.) that were ultimately overcome through discussion with the interviewer. Examples will be provided within the corresponding stage (or sub-stage) when the challenges occurred.

Our analysis suggested that students engaged in varied mathematical activities when solving Problem 12, thus the description of students' processes is organized around them. Each stage is characterized by the main goal that was orienting the set of activities that students engaged with during that stage. For example, the first stage involved students producing conjectures. Our analysis also suggested that the obstacles that students faced were intrinsic to each of the stages. Therefore, the results of our analysis are organized into the following stages: (1) Conjecturing, and (2) Proving. When students were constructing the proof, I identified distinct sub-stages at the interior of the Proving Stage. Each of these sub-stages corresponds to a sub-goal within the larger process of proving. Consequently, the sub-stages are (2a) Constructing a Generic Square, (2b) Setting up an Initial Expression, (2c) Generating a Chain of Equivalent Expressions, and (2d) Proof as Discovery. Even though all students went through these stages, each student had a unique experience in terms of the way it was approached, difficulties they faced, and how they overcame them.

(1) Conjecturing

Most of the students (six out of nine) started exploring the problem by analyzing numeric examples from a calendar. Some (i.e., Audrey) used them to study the behavior of the outcome; some others (i.e., Abbie, Desiree, and Grace) used them to express the cells in the 4x4-calendar-square; and some others (i.e., Brian and Janusz) used examples for both purposes. The rest of the students (i.e., Chris, Cory and Tyler) proceeded directly to model using algebra.

As a result, students produced two types of conjectures: the outcome conjecture and the existence of dependency conjecture. Only one student, Audrey, was able to conjecture that the outcome is equal to -9d (i.e., the outcome conjecture). The rest conjectured the existence of a relationship between the outcome and the parameter involved in the problem.

Regarding the outcome conjecture, Audrey first explained, “I need to get the outcome via multiplication, and it has d days so I know that d is a big part of the problem... so, because the outcome is -d times, multiply by something else, a number…and with 4x4-calendar-square... that no matter what the outcome is always the same, and is always -d multiply by a specific number. So I have to figure out what that one is.” Thus, Audrey set up in her quest to determine the number that multiplies -d. In order to do so, she used examples to determine the exact formula of the outcome. As a result, she concluded that the expression of the outcome is -9d.

As mentioned before, the rest of the students did not arrive at the conclusion that the outcome is -9d although they concluded that the outcome depends on the number of days in a week (i.e., parameter d). In turn, this less specific conjecture opens up the stage for a potential discovery proof. Only after proving the nature of that relationship is uncovered. The following exchange with Janusz illustrates the existence conjecture:

> Janusz [J]: I'm trying to find whether there is a correlation between the outcome and the days in a week [i.e., parameter d].
> Researcher [R]: Before, you told me it definitely depends on the number of days per week, what is your opinion now? Does the outcome depend on…?
> Janusz: Yes, because if it didn't, I'll still be getting -63 for both. [Referring to the calculation for d=7 and d=50]

Since students were aware that the outcome depends on the parameter involved, what remains to be investigated is how does it exactly depend on the parameter? In Chris' words:

> Chris [C]: For example… uhm, like I need to find some kind of equation that says how dependent it is on what variable or another.

> Researcher [R]: Can you tell me the variables? What variables are you going to consider?
> C: There is the dimension of the square, which is set; there is the length of the week, which is varying; and, there's the position of the square.

In what follows all students, except for Audrey, decided to use algebra to answer the aforementioned question. Desiree's words preface the next sub-stage:

> Researcher [R]: What do you need to figure out?
> Desiree [D]: I guess the outcome and how it relates to the days in the week.
> R: How are you going to proceed?
> D: I think I'm just going to start out with some numbers and, then see if I get a formula going for each corner.
> R: Why?
> D: Because when I make a formula for each corner, I can make one formula, and then, simplify down to get the outcome.

Last, it is worth highlighting the conditions that seem to enable a discovery proof: both, the less specific conjecture, and the confidence that students seem to have developed throughout the teaching experiment with algebraic symbols as it will be shown in the next sections.

(2) Proving

(2a) Constructing a generic 4x4-calendar-square.

At this stage in solving the problem, eight out of nine students created a generic 4x4-calendar-square. Audrey was the only student that did not, and was also the only student that did not produce a proof for the conjecture, even though she conjectured that the outcome equals -9d.

Regarding the rest of the group, several of them (i.e., Abbie, Grace, Janusz, Brian, and Tyler) struggled with the construction of the generic 4x4-calendar-square. These students made mistakes when representing the expressions in some of the cells (Table 2).

Table 2
Students' mistakes when creating a 4x4-calendar-square

	Abbie	Grace	Janusz	Brian	Tyler
Incorrect cells in 4x4-generic-square	- Bottom right corner - Bottom left corner	-Top right corner - Bottom right corner - Bottom left corner	-Top right corner - Bottom right corner - Bottom left corner	-Top right corner - Bottom left corner	- Bottom right corner - Bottom left corner
Type of mistake	- Incorrect parameter multiple	- Incorrect additive term - Incorrect parameter multiple	-Incorrect additive term - Incorrect parameter multiple	- Incorrect additive term - Incorrect parameter multiple	- Incorrect parameter multiple

The other three students (i.e., Desiree, Chris, and Cory) constructed the generic 4x4-calendar-square in the first attempt. Ultimately, all students that created a generic 4x4-calendar-square (8 out of 9) used correctly a parameter and its multiples to represent the pertinent cells.

From our analysis, four types of generic diagrams emerged depending on how it was constructed, namely the mathematical relationships used to generate the cells components (Table 3). The differences among the 4x4-calendar-squares are relevant since they correspond to the different ways in which students conceptualized the mathematical relations in the problem. When expressing relations among cells within the same column, students need to consider the days-in-a-week relation, while among cells within the same row students need to consider the days relation.

Types I and II generalizations differ on how the bottom right cell is generated. In the case of Type I, the bottom right cell is generated using the days relation in contrast to using the days-in-a-week relation as in Type II. In both Types I and II generalizations, students use the days relation to generate the top right cell and the days-in-a-week relation to generate the bottom left cell. In sum, in Type I students use the days relation twice and the days-in-a-week relation once; while, in Type II students use the days relation once and the days-in-a-week relation twice. In other words, Type II generalization is grounded by two underlying conceptualizations that rely on the parameter—this translates in adding the parameter iteratively or its multiple/s—and one on days—translates in adding 1 iteratively or 3 once-; while the opposite happens in Type I. The distinction among I.a., I.b., and I.c. sub-types is based on the presence/absence of unit-composition on the parameter d and on 1.

Lastly, the generic squares constructed and used by students played a fundamental role in laying the groundwork in students' proving process. Indeed, it worked as a placeholder of the multiple variables, parameter and the mathematical relationships among them. Students kept referring to it when writing the expression representing the outcome. It was the first stage within the proving process. What students did in producing this generic 4x4-calendar-square corresponds to the first and second stages in the algebraic modelling process as described by Chevallard (1985, 1989), respectively, which are identifying variables and parameters, and setting up mathematical relationships among them. The generic square worked as a stepping-stone in helping students produce the initial expression to represent the outcome.

(2b) Setting up an Initial Expression.

By the end of this stage, eight out of nine students expressed the initial expression correctly. Audrey was the only student that did not write the initial expression; she was also the only student who did not use a generic 4x4-calendar square. In comparison with the previous stage (i.e., constructing a generic 4x4-calendar-square), students did not struggle. Two students (i.e., Brian and Tyler) went through three trials before arriving at their initial expressions. The multiple attempts are related with prior mistakes in constructing a generic 4x4-calendar-square. This is encouraging in terms of students' learning since it has largely been identified as a problem in the literature (e.g., A Friedlander & Hershkowitz, 1997; Harel & Sowder, 1998; Healy & Hoyles, 2000); thus Calendar Algebra Problems seem to provide a fruitful scenario to address this potential problem.

Table 3
Structure of students' generalizations

Type	Location of the independent variable	Relationship used to generate top right	Relationship used to generate bottom left	Relationship used to generate bottom right	Process
I-a	Top Left	Days relation (Three iterations of +1)	Week relation (Three Iterations of +*d*)	Days relation (+3)	$\begin{pmatrix} x & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \overset{+1}{\xrightarrow{+d}} \begin{pmatrix} x & x+1 & & \\ x+d & & & \\ & & & \\ & & & \end{pmatrix} \overset{+1}{\xrightarrow{+d}} \begin{pmatrix} x & x+1 & x+2 & \\ x+d & & & \\ x+2d & & & \\ & & & \end{pmatrix} \overset{+1}{\xrightarrow{+d}} \begin{pmatrix} x & x+1 & x+2 & x+3 \\ x+d & & & \\ x+2d & & & \\ x+3d & & & \end{pmatrix} \xrightarrow{+3} \begin{pmatrix} x & x+1 & x+2 & x+3 \\ x+d & & & \\ x+2d & & & \\ x+3d & & & x+3d+3 \end{pmatrix}$
I-b	Top left	Days relation (+3)	Week relation (Three Iterations of +*d*)	Days relation (+3)	$\begin{pmatrix} x & \\ & \\ & \\ & \end{pmatrix} \overset{+3}{\xrightarrow{+d}} \begin{pmatrix} x & x+3 \\ x+d & \\ & \\ & \end{pmatrix} \xrightarrow{+d} \begin{pmatrix} x & x+3 \\ x+d & \\ x+2d & \\ & \end{pmatrix} \xrightarrow{+d} \begin{pmatrix} x & x+3 \\ x+d & \\ x+2d & \\ x+3d & \end{pmatrix} \xrightarrow{+3} \begin{pmatrix} x & x+3 \\ x+d & \\ x+2d & \\ x+3d & x+3d+3 \end{pmatrix}$
I-c	Top left	Days relation (+3)	Week relation (addition of parameter multiple, 3*d*)	Days relation (+3)	$\begin{pmatrix} x & \\ & \end{pmatrix} \overset{+3}{\xrightarrow{+3d}} \begin{pmatrix} x & x+3 \\ x+3d & \end{pmatrix} \xrightarrow{+3} \begin{pmatrix} x & x+3 \\ x+3d & x+3d+3 \end{pmatrix}$
II	Top left	Days relation (+3)	Week relation (addition of parameter multiple, 3*d*)	Week relation (addition of parameter multiple, 3*d*)	$\begin{pmatrix} x & \\ & \end{pmatrix} \overset{+3}{\xrightarrow{+3d}} \begin{pmatrix} x & x+3 \\ x+3d & \end{pmatrix} \xrightarrow{+3d} \begin{pmatrix} x & x+3 \\ x+3d & x+3+3d \end{pmatrix}$

(2c) Generating a Chain of Equivalent Expressions.

After setting up the initial expression, students needed to operate on it as well as on its subsequent equivalent expressions in order to produce a proof. Eight students (out of nine) used a derivation strategy when proving. Four students did not make any mistakes when transforming the initial expression into its subsequent equivalent expressions. The rest of the students (4 out of 9) made errors that were corrected while working on the problem. In generating equivalent expressions, students used several properties (e.g., distributive, cancellation, etc.) as tools; in this way properties became furnished with an instrumental utility. Students rarely encounter this aspect of properties in more prevailing school practices and problems.

As mentioned earlier, one student (Audrey) did not use algebra at all to prove her conjecture; she was also the only student that did not use a generic diagram and write an initial expression. This, together with the success of students who used a generic square, points once again towards the importance of the generic square as a tool to structure the relations involved in the problem.

Last, students seemed to have learned, throughout the teaching experiment, of profiting from their own errors. Making mistakes did not stop them from continuing to work on the problem; even though many of them made many attempts and errors, they continued to problem-solve. It seems that students had confidence that they would eventually be able to work their way through the problem, and arrive at an expression for the outcome.

(2d) Proof as discovery.

In this stage, students (all except for Audrey) obtained -9d as the last in a chain of equivalent expressions. Abbie's words illustrate this, “Uhm, so the outcome for all of these operations depends only on days of the number days in your week and not the size of the box. You multiply the number of days in a week times negative 9 and you get your outcome...” Or, in the following exchange with Cory:

> Cory [Co]: So then I can cross out a squared, a squared, 3ad, 3ad, 3a, 3a and so negative 9d that's what I came out with.
> Researcher [R]: And what does it mean?
> Co: Uhm, the answer to this problem is always going to be negative 9 times how many days in a week; I don't know why it's negative 9 though.

By finishing writing the text of the proof, students arrived at the expression for the outcome, thus discovering the nature of the outcome as they proved.

Concluding Remarks

Looking at students' mathematical work as a whole, all students but one produced a correct proof. This was so even when they did not know what exactly needed to be proved, which, in turn, opened the possibility for students to experience proof as discovery. In proving, students used algebra as a modelling tool. In fact, students selected relevant variables and parameters, set-up the relations among them, and worked on obtaining a chain of equivalent expressions that ultimately led to the discovery of the outcome. It seems that algebraic proof as the vehicle to integrate algebra and proof as envisioned by the design principles effectively helped students learn to use algebra to prove. These results are encouraging in that all students but one were able to construct an algebraic proof at the end of the teaching experiment in the context of an individual interview.

However, as previously mentioned, for many students, this was achieved by embarking on different attempts and making and mending errors while working continuously on the problems. This suggests that even though arriving at a viable proof is not easily achieved, it is still within students' reach, and therefore desirable to further students' algebraic learning and production of proofs. This has largely been identified as a problem in the literature (e.g., A Friedlander & Hershkowitz, 1997; Harel & Sowder, 1998; Healy & Hoyles, 2000); the Calendar Algebra Problems thus seems to provide a fruitful scenario to address this potential problem. The U.S.-based study presented here stems from a small teaching experiment, and as such, is a first step into thinking about ways of integrating proof and algebra in the high school curriculum. The author hopes to elicit a dialogue around this issue, and consequently inspire other researchers to design more teaching experiments with the entailed problems, and to conduct studies at a larger scale in countries like the U.S. where proof is not integrated throughout the curriculum.

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