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# The Impact of Early Algebra: Results from a Longitudinal Intervention

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**Abstract**

In this paper, we provide evidence of the impact of early algebra (EA) over time. We document this impact in the following ways: (a) by showing the performance over time of an experimental group of 15 children on an algebra assessment, from 3rd to 5th grade; and (b) by showing how the performance on an algebra assessment of children from an experimental group differs from the performance of a group of comparison students from their same elementary school who did not receive EA instruction from 3rd to 5th grade. We compared students' scores through comparisons of means, correspondence factorial analyses, and hierarchical analyses. Our results highlight the positive impact of an early access to algebra, indicating that this early access is associated, when we compare 3rd graders to 5th graders, with increased scores on items that involve inequalities and graphs. When comparing experimental to comparison-group students we find increased scores on items that involve variables, functional relations, intra-mathematical contexts, tables, and algebraic expressions. The study adds to a body of literature that has been arguing for EA as well as a need to thread algebra throughout the mathematics curriculum, starting in the earliest grades.

**Keywords:** mathematics education, algebra, elementary mathematics.

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# El Impacto del Álgebra Temprana: Resultados de una Intervención Longitudinal[1]

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**Resumen**

En este artículo, presentamos datos sobre el impacto del álgebra temprana (AT) a través del tiempo. Documentamos este impacto de las siguientes maneras: (a) mostrando el rendimiento a través del tiempo de un grupo experimental de 15 niños en una evaluación de álgebra, desde 3º hasta 5º grado de la escuela primaria, y (b) mostrando cómo el rendimiento de los niños de un grupo experimental en una evaluación de álgebra difiere del rendimiento de un grupo de estudiantes de un grupo control de su misma escuela primaria que no recibieron instrucción en AT desde 3º hasta 5º grado. Se compararon las puntuaciones de los estudiantes a través de comparaciones de medias, análisis factorial de correspondencias y análisis jerárquicos. Nuestros resultados ponen de manifiesto el impacto positivo de un acceso temprano al álgebra, lo cual indica que este acceso precoz está asociado, cuando se comparan los estudiantes de 3º grado a los de 5º grado, con puntuaciones mayores en los ítems de la evaluación que involucran desigualdades y gráficos. Al comparar el grupo experimental con el grupo control, encontramos las puntuaciones mayores en los ítems que involucran variables, relaciones funcionales, contextos intra-matemáticos, tablas y expresiones algebraicas. El estudio aporta a un cuerpo de literatura que ha estado discutiendo en favor de AT, así como sobre la necesidad de integrar el álgebra en el currículo de matemáticas a partir de los primeros grados.

**Palabras Clave:** matemática educativa, álgebra, matemáticas en la escuela primaria.

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In this paper, we provide evidence on the specific areas of impact of early algebra (EA) over time. The main argument underlying this paper relates to the benefits of an integration of algebra in early mathematics instruction. Our approach to EA falls within an Arithmetic and Functions approach (Carraher & Schliemann, 2007). In this paper, we adopt Carraher and Schliemann's (2007) definition for EA as, "algebraic reasoning and algebra-related instruction among young learners — from approximately 6 to 12 years of age" (p. 670). We document this impact in the following ways:

a) By showing the performance over time of an experimental group of 15 children on an algebra assessment, from 3rd to 5th grade.

b) By showing how the performance on an algebra assessment of children from an experimental group differs from the performance of a comparison group of students.

As has been pointed out by Carraher and Schliemann (2007), there are different approaches to EA: Arithmetic and Numerical Reasoning; Arithmetic and Quantitative Reasoning; and Arithmetic and Functions. Within the first approach, Arithmetic and Numerical Reasoning, Carpenter, Franke, and Levi (2003) grounded their work on the study of number sentences and their truth-value. Fujii and Stephens (2001) grounded their work on quasi-variables, referring to the implicit variables that students seem to make use of in arithmetical contexts. Within the second approach (i.e., Arithmetic and Quantitative Reasoning), we can distinguish Bruer's (1993) work on young students' reasoning relating number lines and numerical understanding. Smith and Thompson (2008) structured their work on arithmetic and quantities focusing on referent-transforming operations, while Kobayashi (1988) and Goldenberg and Shteingold (2008) among others focus their work on magnitudes and measures. Davydov (1991) and colleagues developed a comprehensive approach to EA where the primary focus was on expressing the basic relationships between explicit and implicit values of quantities. Within the third approach (i.e., Arithmetic and Functions), Kaput's (1998) work was centered on the concept of function to "algebrafy" the school curriculum. Moss, Beatty, McNab, and Eisenband (2006) centered their study on functions as rules for generating collections of figures. Our approach to EA, which will be described in more detail below, has been to take functions as the central object of algebra, expressed through multiple representations.

While the importance of integrating EA instruction into the elementary school curriculum has been increasingly accepted (see, for example, the "Operations and Algebraic Thinking" strand of the Common Core State Standards), to date there has been scarce documentation of the specific areas in which EA would make a difference in children's mathematical learning. The purpose of this paper is to fill this gap by detailing the areas and aspects of algebraic understandings and representations in which EA instruction can have an impact over time.

As has been documented elsewhere (e.g., Carraher & Schliemann, 2007; Schliemann, Carraher, & Brizuela, 2007), until recently much research has highlighted the many difficulties encountered by adolescents when they first learn algebra in middle or high school (e.g., Booth, 1984; Kieran, 1981, 1989; Vergnaud, 1985, 1988; Demana & Leitzel, 1988; MacGregor, 1996; Bednarz, 2001; Bednarz & Janvier, 1996; Wagner, 1981). Much of this literature has explained these difficulties by relying on a developmental framework, arguing that children encounter these difficulties due to the nature of formal thought required by algebra. Thus, from this pessimistic framework, children's difficulties could be explained by their lack of this kind of formal thought. Research of this nature has highlighted the difficulties experienced by children in middle and high school.

This pessimistic perspective has been countered by research in the area of EA. Early attempts to address these negative perspectives were motivated on the one hand, by a mistrust that these difficulties really lie within the children; on the other hand, by an understanding about the nature of early mathematical learning, specifically arithmetic, that does not exclude algebra from its midst. Early exemplars of this approach come from both Russia (see Bodanskii, 1991; Davydov, 1991) and the United States (US; see Davis, 1967, 1971-72, 1985, 1989). EA perspectives hold, in general, that arithmetic has an inherently algebraic character (see Schliemann, Carraher, & Brizuela, 2007) and can be usefully regarded as a part of algebra rather than as a domain distinct from algebra (Carraher & Schliemann, 2007). In our work, we share this perspective and argue that a deep understanding of arithmetic requires mathematical generalizations and understandings of basic algebraic principles.

The research we present here exemplifies a shift in the research community towards understanding how algebraic learning occurs and what resources children utilize in the learning of algebra, as opposed to the previous focus on what children are not able to do. Moreover, our work also exemplifies the centrality of representations in children's algebraic learning, building on their spontaneous ways of representing problems and gradually introducing interactions with conventional representations. Other examples of this shift are Radford, Bardini, and Sabena's (2007) study, in which they described and analyzed the various semiotic resources utilized by 9th grade students in their passage from the particular to the general. Their analysis suggests connections between the syntax of the students' algebraic formulas and the semiotic means of objectification through which the formulas were forged. Another set of studies (e.g., diSessa, Hammer, Sherin, & Kolpakowski, 1991; Hall, 1990; Hall, Kibler, Wenger, & Truxaw, 1989; Izsák, 2004; Meira, 1995) have examined what students can do if they are allowed to build their own representations while solving problems, rather than using external representations prescribed by researchers. This work, as well as ours, highlights not only what students can do but also students' capacities to build meaning for and from representations.

Since 1998, the implementation of EA activities and the documentation of children's learning of algebra in the early grades (e.g., Brizuela & Earnest, 2008; Carraher, Schliemann, & Schwartz, 2008; Schliemann, Carraher, & Brizuela, 2007, 2012) has shown that introducing algebra as part of the early mathematics curriculum is highly feasible and has also clarified how specific representational tools — tables, graphs, numerical and algebraic notations, and certain natural language structures — can be employed to help students express functional relations among numbers and quantities and solve algebra problems. In what follows, we describe four principles of this approach used to analyze data and present results.

*First*, we adopt the position that arithmetical operations can be viewed as functions and we view the introduction of algebraic activities in elementary school as a move from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing relations among variables. This position is similar to a broader perspective that has shown the importance of a functional perspective that stands in contrast to a perspective that focuses on the symbolic manipulation of equations (or equational approach; see Bednarz, Kieran, & Lee, 1996; Chazan & Yerushalmy, 2003; Dubinsky & Harel, 1992; Martinez & Brizuela, 2006; Schliemann, Carraher, & Brizuela, 2007).

*Second*, we also provide students with opportunities to engage in cross-representational activities, looking at how they establish correspondences across various representations (see Brizuela & Earnest, 2008). In a narrow sense, algebraic reasoning concerns only algebraic-symbolic notation. In a broad sense, algebraic reasoning is associated with and embedded in different representational systems. In our research, we adopt a broad view of algebraic reasoning.

*Third*, and related to the second point, we also believe, on the basis of our previous data (Carraher, Schliemann, & Schwartz, 2008), that algebraic notation would facilitate, both for adults and young learners, to give expression to mathematical generalizations (e.g., Mason, 1996). In our work, we provide students with opportunities to use letters to stand both for unknown amounts and for variables. We introduce algebraic notation gradually, allowing it to take on increasing weight over time.

*Fourth*, our approach to EA includes thinking about physical quantities and rich problem contexts as providing essential ways in which to situate and complexify the learning of mathematics. Contexts are crucial to mathematics educators' concerns since students, particularly young students, learn mathematics through reasoning about various types of situations and activities (e.g. Brenner & Moschkovich, 2002; Carraher, Carraher, & Schliemann, 1985; Nunes, Schliemann, & Carraher, 1993; Schwartz, 1996; Smith & Thompson, 2008; Verschaffel, Greer, & De Corte, 2002). Thus, we share Vergnaud's perspective (1982, 1988, 1994) that intellectual complexity can be gained by learning to manage new types of situations, such as different contexts or types of problems.

## Findings from Previous Interventions

Results from two longitudinal studies (e.g., Carraher, Schliemann, & Schwartz, 2008; Schliemann, Carraher, & Brizuela, 2007) point to the conclusion that young students (8-11 years of age) can benefit from EA activities and learn to: (a) think of arithmetical operations as functions rather than as mere computations on particular numbers; (b) learn about negative numbers; (c) grasp the meaning of variables, and not only of instantiated values; (d) shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures; (e) shift from computing numerical answers to describing and representing relations among variables; (f) build and interpret graphs of linear and non-linear functions; (g) solve algebraic problems using multiple representation systems such as tables, graphs, and written equations; and (h) solve equations with variables on both sides of the equal sign. At the end of 4th grade in a second longitudinal study, compared to a comparison group of 5th graders at the same school, more of the experimental group students could: (a) identify that a number sentence of the type 6+9=7+8 is true (85% vs. 65%); (b) handle situations that dealt with unknown quantities (44% vs. 31%); (c) fill out a table by using a functional relation (65% vs. 50%); (d) represent a variable quantity with a letter and a quantity expressed in terms of the variable quantity as [the letter]*3 (70% vs. 29%); (e) identify the correct line in a graph, based on a verbal description of a relationship (78% vs. 46%); (f) express an unknown quantity using a letter, and other quantities in terms of this letter (56% vs. 49%); and (g) write a full equation using letters to stand for unknown quantities (17% vs. 0%).

## Focus of This Paper

In the present paper, we present detailed analyses of the specific areas in which we have been able to identify gains made over time among an experimental group of students that we worked with. Our analyses center on the results of an assessment that was given to both experimental and comparison groups of students at the end of each school year for the duration of our EA intervention. The following are the research questions that guide our analyses:

a) How does the performance on the algebra assessment change among a group of 15 experimental group children, from 3rd to 5th grade?

b) How does the performance on the algebra assessment of experimental group students differ from the performance of comparison group students from their same elementary school who did not receive EA instruction?

## Methodology

### Sample

The sample for this study consists of an experimental group and a comparison group. Both groups used the *Investigations* curriculum in their regular mathematics lessons (Pearson Scott Foresman, 1998). Our EA lessons were implemented by a team of researchers in addition (i.e., not as a partial replacement to *Investigations*) to the standard mathematics curriculum implemented by the teachers. In 3rd and 4th grades, children participated each week in two 60-minute lessons and two 30-minute homework review sessions; in 5th grade they participated in one 90-minute weekly lesson and a 45-minute homework review session. The research team's members implemented lessons for these students and classroom teachers conducted homework reviews. A total of 50 lessons were taught in 3rd grade, 36 in 4th, and 18 in 5th grade. All 104 lessons and corresponding homework reviews were videotaped.

*Experimental group*: The experimental group consisted of the children who participated in our EA lessons from 3rd to 5th grade (2003 to 2006). Specific sample sizes will be provided in the case of each one of the analyses carried out.

*Comparison group*: The comparison group consisted of a group of students from the cohort immediately preceding the experimental group (2002 to 2005) in the same school who, from 3rd to 5th grade, received no EA instruction.

### Assessment

Our source of data for this report is an assessment designed by the research team which was administered once at the end of each school year, from 3rd to 5th grade. The assessment included items that were designed by our research team, as well as items from the Massachusetts Comprehensive Assessment System (MCAS), and the National Assessment of Educational Progress (NAEP). At the end of each grade children in the experimental and in the comparison groups were given this assessment. Because of the paucity of algebra-related questions on the NAEP and the MCAS assessments designed specifically for fourth graders, items were drawn from NAEP and MCAS tests for grades 4, 8 and 10. All fourth grade algebra items were included as well as eighth and tenth grade algebra items that covered themes in our early algebra classes.

For the purposes of this study, we classified the assessment items using a set of categories that are presented below. Each item was scored as correct or incorrect and scores were calculated based on the number of correct assessment items that fell into each category (one point for each correct answer in each category). For instance, if an item was answered correctly 10 times, then each of its related categories (i.e., natural language, algebraic expression, variable, extra-mathematical, functional relation) were scored 10 for that item, and so on for every item.

**Categories considered**

*Representation involved (either in the presentation mode or the answer requested from the students*). Central to EA (see Brizuela & Earnest, 2008) is an emphasis on multiple representations. By representations we mean what Goldin (1998) refers to these as “the shared, somewhat standardized representational systems developed through human social processes” (p. 146). Kaput (1991) referred to these as “materially realizable cultural or linguistic artifacts shared by a cultural or language community” (p. 55). The representations we focus on in our EA research are tables, graphs, natural language, algebraic expressions, and pseudo-algebraic expressions. Thus, for our classification, within representations, we considered:

a) *tables*, by which we mean input-output or function tables;

b) *graphs*, by which we mean graphs of functions in the Cartesian plane or number lines used to represent relations between variables;

c) *natural language*, by which we mean some sort of verbal expression (written or oral) using colloquial language;

d) *algebraic expressions*, sometimes referred to as "symbolic" expressions;

e) *pseudo-algebraic expressions*, an equation or function expressed using pseudo-algebraic notation or non-formal algebraic expression using icons, i.e., 7 x ☐ =21.

*Role of the letter.* In our work, letters in algebraic and pseudo-algebraic expressions can hold one of two roles[2]: as representing an unknown, and as representing a variable. Thus, for our classification, within the roles of the letter, we considered:

a) *unknown*, where the letter represents something that needs to be determined and verifies a set of conditions;

b) *variable*, where the letter represents a set of cases.

*Context*. Our understanding of contexts borrows from Chevallard (1989), who distinguished between *extra-mathematical* and *intra-mathematical* contexts. By *extra-mathematical* we mean contexts where a problematic situation is raised using phenomena external to the mathematical field. In order to be solved, this situation can be modeled using mathematical tools. In other words, the issues to be solved are not about the nature of the mathematical objects themselves, nor about the relations among these objects. The objects on which the relations operate have an intended meaning beyond mathematics. By *intra-mathematical* we mean problems that appear purely from mathematical questions and problems; they involve, "the production of knowledge in one mathematical system through another mathematical system" (Sadovsky, 2005, p. 27).

*Central Object*. In our approach, students solve problems that involve functions or equations/inequalities. Even though we consider our work as a functional approach to algebra, we also include equations as part of our approach by considering equations as only one element of the teaching and learning of algebra. In our work, the notion of a solution to an equation was introduced as the $x$-value of the intersection of two functions. We distinguish three types of central objects:

a) *function*, which can be thought of intuitively as a rule of correspondence between two sets that assigns to each object in the first set exactly one object from the second set;

b) *equation*, which is a condition on a set (i.e., numbers) stated by using the equal sign (or expressed through natural language);

c) *inequality*, which is a condition on a set (i.e., numbers) stated by using >, ≥, ≤, or < signs (or expressed through natural language).

Table 1 shows the classification of all items by category.

Table 1
*Classification of items by presented categories*

| Item | Source | Representation | | | | | Role of letter | | Central object | | | Context | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Tables | Graphs | Nat. Lang. | Algebraic Exp. | Pseudo- Alg. Exp. | Variable | Unknown | Function | Equation | Inequality | Intra-Math | Extra-Math |
| O1 | Our design | | | X | X | | X | | X | | | | X |
| O2 | Our design | | | X | | X | X | | X | | | | X |
| O3A | Our design | X | | | | | X | | X | | | X | |
| O3B | Our design | X | | | X | | X | | X | | | X | |
| O3C | Our design | X | | | | | X | | X | | | X | |
| O4A | Our design | X | | | | | X | | X | | | X | |
| O4B | Our design | X | | | X | | X | | X | | | X | |
| O4C | Our design | X | | | | | X | | X | | | X | |
| O5 | Our design | X | | X | X | | X | | X | | | | X |
| O6A | Our design | | X | X | | | X | | X | | | | X |
| O12A | MCAS2007- 8$^{th}$grade | | | | X | | X | | | | | X | |
| O13 | MCAS2007- 8$^{th}$grade | | | | | X | | X | | X | | X | |
| O15C | NAEP2007- 8$^{th}$grade | | | | | X | | X | X | X | | X | |
| O16 | NAEP2007- 8$^{th}$grade | X | X | X | | | X | | X | | | | X |
| O17A | NAEP2007- 8$^{th}$grade | | | X | | | | X | X | X | | | X |

| Item | Source | Representation | | | | | Role of letter | | Central object | | | Context | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Tables | Graphs | Nat. Lang. | Algebraic Exp. | Pseudo- Alg. Exp. | Variable | Unknown | Function | Equation | Inequality | Intra-Math | Extra-Math |
| O18A | Our design | | X | X | | | | | | | | | X |
| O18B | Our design | | X | X | | | | | | | | | X |
| O18C | Our design | | X | X | | | | | | | | | X |
| O18D | Our design | | X | X | | | | | | | | | X |
| T1A | Our design | | | X | | | X | | X | | | | X |
| T1B | Our design | | | X | | | X | | X | | | | X |
| T2 | Our design | | X | X | X | | X | | X | | | | X |
| T3A | Our design | | | X | | | X | | X | | | | X |
| T3B | Our design | | | X | | | X | | X | | | | X |
| T3C | Our design | | | X | | | X | | X | | | | X |
| T3D | Our design | | | X | X | | X | | X | | | | X |
| T3E | Our design | | | X | | | X | | X | | | | X |
| T4 | MCAS2003-X 8th grade | | | X | | | X | | X | | | | X |
| T5 | MCAS2007- 10th grade | | | X | | X | | X | X | X | | | X |
| T6 | MCAS2007- 10th grade | | | | X | | | X | | | X | X | |
| T7 | Own design | | X | | | | | X | | | X | X | |
| T8 | MCAS2007-X 10th grade | | | | X | | | X | | X | | X | |
| T9 | MCAS2007- 10th grade | | | | X | | | X | X | | X | X | |
| T11A | Own design | | X | X | | | X | | X | | | | X |
| T11B | Own design | | X | X | | | X | | X | | | | X |

| Item | Source | Representation | | | | | Role of letter | | Central object | | | Context | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Tables | Graphs | Nat. Lang. | Algebraic Exp. | Pseudo- Alg. Exp. | Variable | Unknown | Function | Equation | Inequality | Intra-Math | Extra-Math |
| T11C | Own design | | X | X | | | X | | X | | | | X |
| T11D | Own design | | X | X | | | X | | X | | | | X |
| T11E | Own design | | X | X | | | X | | X | | | | X |
| T12 | MCAS2007-10thgrade | | | X | | | X | | X | | | | X |
| T13A | NAEP2003-8thgrade | | X | X | | | X | | X | | | | X |
| T13B | NAEP2003-8thgrade | | X | X | | | X | | X | | | | X |
| T13C | NAEP2003-8thgrade | | X | X | | | X | | X | | | | X |
| T14 | MCAS2003-4thgrade | | X | X | | | X | | X | | | | X |
| Totals | | 10 | 17 | 30 | 10 | 4 | 31 | 8 | 34 | 5 | 3 | 13 | 30 |

**Statistical Analyses**

Three types of statistical analyses were carried out. The first helped us determine the differences in students' performance on the algebra assessment across different groups (i.e., across grades and across experimental and comparison groups). The second helped us establish the categories of items, as previously shown in Table 1, that were associated with particular groups. The third helped us to establish the clusters of groups of students that were formed when we considered their performance on the assessment items, allowing us to establish which groups were most similar as well as dissimilar to each other. The set of three different kinds of analyses helped us answer our research questions, related to the change in performance on the algebra assessment from 3rd to 5th grade among the experimental group children as well as among the 3rd to 5th grade experimental and comparison group children. The last two analyses also allowed us to identify the kinds of questions that were more and less challenging among these groups of students.

**Comparisons of Means**

Four nonparametric statistical tests, using the software SPSS, were used to compare means due to the small size of the sample: A Kruskal-Wallis Analysis of Variance (KWANOVA) was carried out to determine the effects of the classroom intervention on the students with the end-of-year assessments on those students (experimental group) who were present for 3rd, 4th, and 5th grades of the EA intervention. Additionally, three Mann-Whitney U-tests were performed comparing the scores of assessments of the students who took part in the intervention with a comparison group of students from the same school, also in 3rd, 4th, and 5th grades. Scores in each grade level were the number of correct responses.

**Correspondence Factorial Analysis**

Correspondence factorial analyses (CFA) were carried out in order to study relations among categories of assessment items, grade levels, and experimental or comparison group membership using the software SPAD Recherche 5.6. Correspondence factorial analysis is a technique of multivariate analysis that relates active variables (in our case, categories of assessment items and a combined grade level/experimental or comparison group membership variable), modalities of such variables, and individuals by projecting such relations on a factorial plane (see Scheuer, de la Cruz, Pozo, Huarte, & Sola, 2006). According to customary criteria, a variable modality is taken into account when its contribution to one or both factorial axes is higher than the average of all variable modality contributions. Groups are formed by associating modalities with a contribution that are higher than average and that lie relatively close on the factorial plane.

From the original assessments, we only included those 43 items that were the same across every grade. For instance, at the end of 5th grade we included an additional set of items. Since we cannot compare the students' performance across grades for these additional items, they were not included in our analyses.

Categories of assessment items and grade level/experimental or comparison group membership (3rd grade experimental; 4th grade experimental; 5th grade experimental; 3rd grade comparison; 4th grade comparison; 5th grade comparison) were considered as active variables. The following were the CFA carried out:

1) *3rd – 5th grade experimental group students*: active variables were: the 12 sub-categories of assessment items (see Table 1); 3rd grade experimental group students; 4th grade experimental group students; and 5th grade experimental group students. We tested the association between group membership and each one of the 12 categories of assessment items. The question underlying this analysis was: did the association to particular kinds of items change over time, from 3rd to 5th grade (that is, with increased exposure to our EA interventions)?

2) *3rd – 5th grade experimental vs. 3rd – 5th comparison group students*: active variables were the 12 sub-categories of assessment items; 3rd grade experimental group students; 4th grade experimental group students; 5th grade experimental group students; 3rd grade comparison group students; 4th grade comparison group students; and 5th grade comparison group students. We tested the association between group and grade membership and each one of the 12 categories of assessment items. The question underlying this analysis was: did the association to particular kinds of items change over time, from 3rd to 5th grade differently for students who participated in our EA interventions and those who did not?

**Hierarchical Cluster Analysis**

To verify the grouping of individuals obtained from the CFA, a Hierarchical Cluster Analysis (HCA) was performed, using the most important components obtained from the CFA as variables. For this analysis the order in Ward's method was used to identify data partitioning (Lebart, Morineau, & Piron, 1995; Saporta, 1990), using the software SPAD Recherche 5.6.

## Results

### Third – Fifth Grade Experimental Group Students

For these analyses, we considered as our sample a sub-group of 15 students from a larger group of 26 students[3] with whom we had worked from Fall 2003 to Spring 2006 while they were in 3rd – 5th grade of elementary school, at a public school in the Boston area, who participated in our EA activities. A Kruskal-Wallis Analysis of Variance (KWANOVA) and post hoc analyses were conducted to compare students' end-of-year assessments across all three grades. The analyses revealed that scores rose significantly from grade 3 (mean=23.6) to grade 4 (mean=33.3), but there was no significant difference between the assessment scores in grade 4 and grade 5 (mean=29.7), $\chi^2(2, N=15)=19.90$, $p<0.001$[4]. In contrast, when looking at the 22 comparison students that were present for grades 3 (M=22.1), 4 (M=21.9), and 5 (M=23.2), the scores did not significantly change across any grade levels, $\chi^2(2, N=22)=0.076$, $p=0.963$.

In general, what we found through interpretation of the groups formed in the CFA planes, which are not shown here due to space limitations, is that, relative to their scores in 4th and 5th grade, 3rd graders found most challenging assessment items that involve inequalities and graphs.

Table 2
*Contributions to the variation in students' performance on the twelve sub-categories of assessment items of the combined active variable related to grade/group membership. In bold, we indicate contributions to the variation that are greater than the mean.*

| Grade / Group Membership | Axis 1 | Axis 2 |
|---|---|---|
| 3rd Grade Experimental | **70.59** | 0.12 |
| 5th Grade Experimental | 15.94 | **51.15** |
| 4th Grade Experimental | 13.46 | **48.73** |

Thus, we could put forth the hypotheses, at this point, that our EA intervention was helpful and successful in these areas, helping students as they progressed in their EA experience. In addition, an interesting result of this analysis is how well 3rd graders perform on items that involve variables and algebraic expressions, which are at the center of the content typically associated with algebra. See Tables 2 and 3 for value contributions.

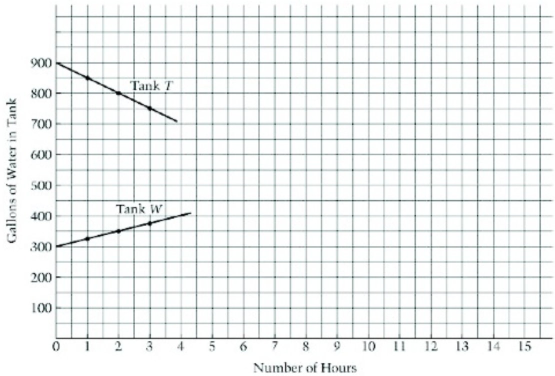
Table 3
*Contributions to the variation in students' performance on the twelve sub-categories of assessment items of the active variable related to sub-categories of assessment items. In bold, we indicate contributions to the variation that are greater than the mean.*

| Sub-categories of Assessment Items | Axis 1 | Axis 2 |
|---|---|---|
| Representation | | |
| Table | **22.29** | 8.08 |
| Graph | **29.84** | 4.85 |
| Natural Language | 4.94 | 8.23 |
| Algebraic Expression | **14.57** | 3.90 |
| Pseudo-Algebraic Expression | 1.82 | **19.13** |

Items T13A and T13B (see Figure 1) are the items in which students had the most difficulties when they were in 3rd grade (the only items in which 3rd graders scored zero points). Item 13 involves two functions that are given by their graphs in the Cartesian plane. In the first question, the students are asked to interpret the graph that shows volume as a function of time. In the second question, students are asked to determine the *x*-value of the intersection point. The questions involve a graph, natural language, variables, an extra-mathematical context, and a functional relation. This question was the most difficult for 3rd graders in our study.

To verify the grouping of individuals obtained from the CFA, a HCA was carried out after the CFA, using the software SPAD Recherche 5.6. The HCA confirmed the grouping of 4th and 5th grade on one hand and 3rd grade on the other. Students' performance is clearly grouped together when they are in 4th and 5th grade, and distinguished from their performance in 3rd grade. The cluster analysis also confirmed that the group formed by 3rd grade is characterized by items that involve *tables*

13.
Two large storage tanks, T and W, contain water. T starts losing water at the same time additional water starts flowing into W. the graph below shows the amount of water in each tank over a period of hours.




Assume that the rates of water loss and water gain continue as shown. At what number of hours will the amount of water in T be equal to the amount of water in W?

*Figure 1*. Items T13A (i.e., interpreting the graph and extending both lines until they meet) and T13B (i.e., “assume that the rates…”).

(test value = 1.98, p = 0.0236) and under-characterized by items that involve *graphs* (test value = -2.30, p = 0.0083). The results of the cluster analysis, along with the CFA, shows that students’ performance is highly associated with the kinds of representations involved in the assessment items.

Given our results some may argue that these are simply a matter of children growing older, of development, and that our EA intervention had nothing to do with these gains. Thus, for these reasons, we carried out a second analysis, including not only the experimental group students from 3rd-5th grades, but also the comparison group students for whom we have parallel data.

**Third – Fifth Grade Experimental vs. Comparison Group Students**

Using a Mann-Whitney U-test with the software SPSS, we compared the end-of-year assessment scores of the 23 experimental students who participated in the study in 3rd grade (M=24.09) with the scores of 30 students from the same school who took the same assessment in their 3rd grade year, one year prior to the beginning of our EA intervention (M=21.08). The experimental students did perform better than the comparison students, but the difference was not significant, U(51) = 307.00, p = 0.495. With the 4th grade assessments, we compared the scores of the 27 experimental students who participated in the study in 4th grade with the scores of the same 30 comparison students described above in 4th grade[5]. A Mann-Whitney U-test indicated that in the 4th grade, the experimental students (M=31.00) performed significantly better than the comparison students (M=21.41), U(55) = 194.00, p = 0.001. Finally, we compared the scores of the 22 experimental students who participated in the study in 5th grade with the scores of 24 students from the same school who took the same assessment in their 5th grade year, one year prior to the year in which our experimental students were in 5th grade. A Mann-Whitney U-test indicated that the experimental students (M=29.36) once again performed significantly better than the comparison students (M=23.21), U(44) = 155.50, p = 0.017.

For the CFA, we considered as the experimental group the sub-group of 15 students who remained in the study for its duration, as detailed above. As the comparison group, we considered 22 students from a larger group 30 students[6] in the cohort immediately preceding the experimental group (2002 to 2005) who, from 3rd to 5th grade, had been taught by the school's regular elementary school teachers and received no algebra instruction.

The results of the CFA suggested the sub-categories of assessment items on which each of the different groups did better than what they typically did (See Tables 4 and 5). Scrutiny of the factorial plane reveals a dichotomy in the kinds of items that are associated with experimental group students on one hand and comparison group students on the other hand. The three comparison group grades are associated with items that involve natural language, equalities, inequalities, graphs, unknowns, and pseudo-algebraic expressions. The experimental group grades are associated with items that involve variables, functional relations, intra-mathematical contexts, tables, and algebraic expressions. The experimental and comparison groups can be distinguished by these kinds of items. Another way to see this is to say that our intervention made the most difference on students' performance on items that involve *variables, functional relations, intra-mathematical contexts, tables, and algebraic expressions*.

Table 4

*Contributions to the variation in students' performance on the twelve sub-categories of assessment items of the combined active variable related to grade/group membership. In bold, we indicate contributions to the variation that are greater than the mean.*

| Grade / Group Memberhip | Axis 1 | Axis 2 | Axis 3 |
|---|---|---|---|
| 3rd grade control | 6.01 | 11.01 | 13.94 |
| 4th grade control | 6.01 | 11.01 | 13.94 |
| 5th grade control | **20.61** | 0.11 | **57.69** |
| 3rd grade experimental | **61.89** | **17.52** | 6.98 |
| 4th grade experimental | 2.91 | **35.90** | 7.46 |
| 5th grade experimental | 2.57 | **24.47** | 0.00 |

A HCA carried out after the CFA confirmed the grouping of experimental grades on the one hand and comparison groups on the other. The first distinction is between $3^{rd}$ grade experimental students and the rest of the students. The second split that occurs in the cluster analysis is between $4^{th}$ and $5^{th}$ grade experimental students on one hand and all the comparison groups on the other. Our interpretation is that the intervention helped to distinguish, among the experimental group students, between $3^{rd}$ grade and $4^{th}$ and $5^{th}$ grade. In the comparison groups, the lack of an EA intervention led all students, from $3^{rd}$ to $5^{th}$ grades, to perform quite similarly. The cluster analysis also confirmed that the group formed by the comparison group students is under-characterized by items that involve algebraic expressions (test value = -2.60, $p = 0.0047$). In contrast, the group formed by $3^{rd}$ grade experimental students is characterized by items that involve algebraic expressions (test value = 1.99, p = 0.0232) and under-characterized by items that involve graphs (test value = -2.12 p = 0.0169). Once again, similarly to what we found in the comparison among 3rd-4th grade experimental students, students' performance depends on the type of representation involved in the assessment item given that students' performance is highly associated to the kinds of representations involved in the assessment items.

Table 5

*Contributions to the variation in students' performance on the twelve sub-categories of assessment items of the active variable related to sub-categories of assessment items. In bold, we indicate contributions to the variation that are greater than the mean.*

| Sub-categories of assessment items | Axis 1 | Axis 2 | Axis 3 |
|---|---|---|---|
| Representation | | | |
| Table | **17.09** | 8.00 | 0.16 |
| Graph | **16.31** | 2.88 | **14.32** |
| Natural Language | 2.27 | 1.20 | 2.17 |
| Algebraic Expression | **33.80** | 4.93 | 0.89 |
| Pseudo-Algebraic Expression | 7.02 | **39.20** | 0.00 |
| Role of the letter | | | |
| Variable | 7.51 | 1.90 | 0.24 |
| Unknown | 5.84 | 0.97 | 3.39 |
| Central object | | | |
| Equality | 2.18 | 4.98 | **13.17** |
| Inequality | 4.52 | **25.34** | 4.24 |
| Functional relation | 1.06 | 0.36 | 7.29 |
| Context | | | |
| Intra-mathematical context | 0.12 | **9.03** | **51.96** |
| Extra-mathematical context | 2.27 | 1.20 | 2.17 |

Items O5 (see Figure 2), O6A (see Figure 3), and T7 (see Figure 4) are interesting in that they are the items in which comparison group students performed the least well and scored the lowest.

5.

Mary earns three times as much money as John.

Let's use the letter N to show how much John earns per hour. If John earns N dollars, how much money would Mary earn? (Show your answer below).

| John | Mary |
|---|---|
| N | |

*Figure 2*. Item O5 from the assessment.

Item O5 involves a functional relation described using colloquial language where students are asked to write the formula. The item involves a table, natural language, an algebraic expression, the use of a variable, an extra-mathematical context, and a functional relation. Item O6A involves a functional relation presented in colloquial language

6. Which of the graphs shows that Mary has three times as much money as John? Explain.

How do you know you chose the correct graph?

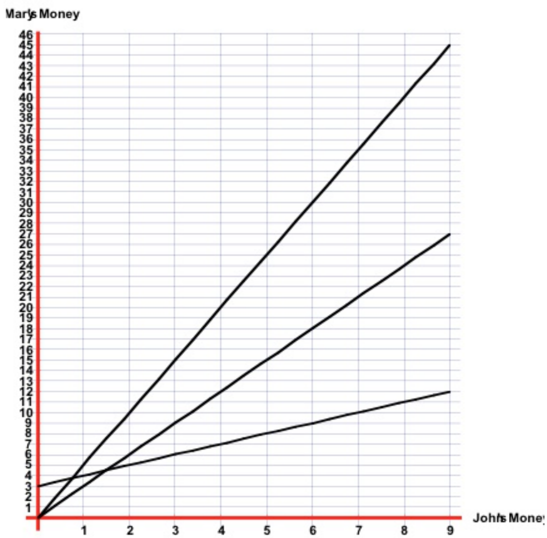



*Figure 3*. Item O6A (i.e., "which of the graphs…") from the assessment.

where students are asked to choose a graph among three different graphs represented in the same Cartesian plane. The item involves a graph, natural language, the use of a variable, an extra-mathematical context, and a functional relation. In item T7 students are presented with a set described by a simple inequality (greater than or equal to -1 and less than or equal to 3) and are asked to represent it in the number line. The item involves a graph, an unknown, an intra-mathematical context, and an inequality.

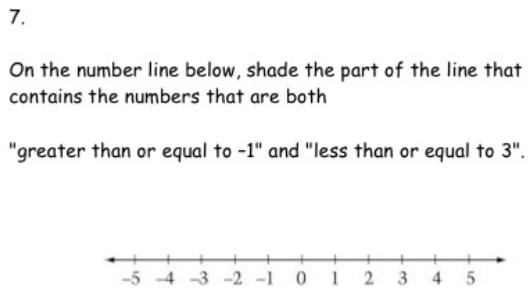



*Figure 4*. Item T7 from the assessment.

In summary, our intervention had an impact on students' performance on items that involve *variables, functional relations, intra-mathematical contexts, tables,* and *algebraic expressions*. In these specific areas, children who have been exposed to an EA intervention seem to do particularly well.

Item O5, described above, and item T2, in which students must work on a "K-number line" (see Figures 7 and 8) are examples of items that comparison students found challenging, while our experimental students did not. Item O5 asked students to generate an algebraic rule from a story expressed through natural language. Experimental students tended to express the rule algebraically (see Figure 5) while comparison students tended to express this as a single numeric answer, or left the question blank (see Figure 6).

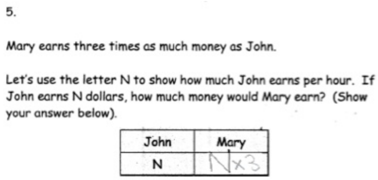



*Figure 5*. I.P. (3rd grade experimental) displays work on item O5.

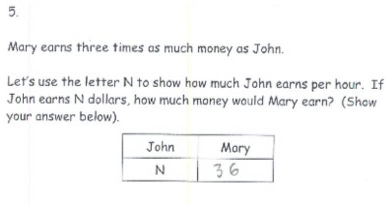



*Figure 6*. J.T. (5th grade comparison) displays work on item O5.

In item T2, students are told of a number of transformations on a K-number line. Students must express the transformations on the number line and provide a final result. Many comparison students failed to express that this was a story that began at K and instead treated the number line as static, expressing the final answer as -4 when the correct

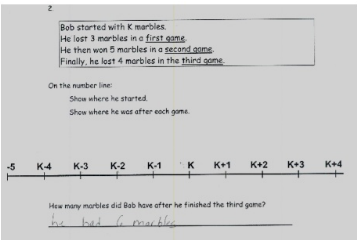



*Figure 7*. D.H. (5th grade comparison) displays work on item T2.

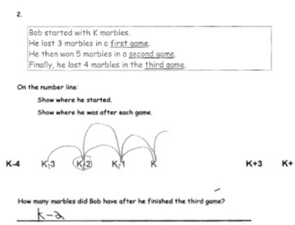



*Figure 8*. A.V. (3rd grade experimental) displays work on item T2.

answer was K-2, or providing some other numerical answer (see Figure 7). Experimental group students, even in the 3rd grade, were able to express the relationships in terms of a variable quantity (see Figure 8).

## Discussion

In this paper, we have been able to provide evidence, through different sources, of the specific areas of impact of EA instruction on students' performance in algebra. This paper adds to the growing body of evidence regarding the benefits of EA by providing specific information regarding the representations and concepts in which students exposed to

EA most distinguish themselves. Furthermore, it also adds to earlier research, such as that carried out by Bodanskii (1991) and Davydov (1991) regarding the positive impact of an early access to algebra.

As described in the results, the statistical comparisons presented show significant differences in performance between children who have and have not had access to an EA intervention. In terms of the CFA and cluster analyses, when comparing experimental and comparison group students, we found that our EA intervention had the most impact on children's understanding, acquisition, and performance on items that dealt with variables, functional relations, intra-mathematical contexts, tables, and algebraic expressions. As stated in our introduction, the focus of our EA approach includes four main components: (1) focus on a functional approach to algebra; (2) focus on multiple representations; (3) focus on the use of letters to represent variables; and (4) focus on exploring problems in a diversity of extra-mathematical contexts. Therefore, we find a high degree of overlap between the focus of our instruction and the areas in which our experimental group students did better than their comparison group peers. This evidence shows, in turn, that young students can learn algebra meaningfully, and make sense of and use algebraic tools; this stands in contrast to previous research that has highlighted what students cannot do after receiving instruction on algebra.

As stated in the introduction, the literature has tended to document the difficulties that young students face when learning algebra. We argue that the data we have here presented helps to frame the above shortcomings regarding young students' abilities and approaches in algebra in terms of a difficulty that lies mostly if not entirely in adults' hands: we have not had the sense to teach algebra earlier. We believe our data help to confirm that the above shortcomings are not usual once children have engaged in the study of algebraic reasoning in the context of an EA experience. Furthermore, the results of our CFA indicate that 3rd graders in the experimental group perform relatively well compared to when they are in 4$^{th}$ and 5$^{th}$ grade and on items that involve variables and algebraic expressions. Our results would look quite different if children exposed to an EA experience had simply done well on problems involving verbal statements, or pseudo-algebraic expressions.

However, they tended to do well on what many consider to be the defining points of algebra: variables and algebraic expressions.

In the different analyses carried out we found time and again that students' performance was associated with the type of representation involved in the assessment item. For instance, comparison group students find items that involve algebraic expressions particularly difficult compared to the experimental group students and other kinds of items, whereas students in the 3rd grade experimental group find them relatively less difficult compared to other items and to themselves in later grades. Graphs, on the other hand, are comparatively more difficult for 3$^{rd}$ grade experimental group students. Furthermore, when compared to their frequencies of correct responses in 4$^{th}$ and 5$^{th}$ grade, 3$^{rd}$ graders do relatively better on items that involve tables and relatively less well than expected on items that involve graphs. These data connect to a larger line of research that indicate the relevance of exploring understandings through different representations (see Brizuela & Earnest, 2008) as well as on the difference that it makes to explore seemingly similar problems through different representations (see Zhang, 1997; Zhang & Norman, 1994, 1995). Doing well on an algebra problem does not mean doing well in all kinds of representations, for instance. Further, we need to explore students' understandings through a variety of different representations in order to gain a full picture of their conceptualizations.

We also evaluated the experimental group students' results on the Massachusetts standardized test in mathematics (the Massachusetts Comprehensive Assessment System or MCAS) for 4$^{th}$ graders[7] before (2002 and 2003) and after (2004 and 2005) the project's intervention. The percentage of children at Advanced or Proficient levels increased to 44% from 36%, while the percentage of those in the Needs Improvement and Fail categories decreased to 56% from 63%. The overall improvement of the experimental group was also better than the overall improvement in the school district (8.5% vs. 4.5% points), but these differences were not statistically significant.

We do note, however, the limitations of this study. First, the small sample size (one school) only allowed for a comparison group as opposed to a matched control group. Thus, we do not know if the two groups began at the same level of ability (though, anecdotally from the teachers, we are led to believe that if any mismatch existed at the start, it would be that our experimental group was thought to be the weaker of the two cohorts). Additionally, the lack of an established assessment in algebra for elementary school students and thus use of many items of our own design does not allow for comparison of the results to a larger group of students who were administered the same items. Finally, as this was supplementary instruction, the comparison group may have received less overall time engaged in mathematics instruction.

### Future Directions

Much work remains needed regarding the short and medium term benefits of EA. For instance, it is our hypotheses that access to algebra needs to happen early, needs to be sustained over time, and needs to occur frequently. That is, algebra cannot be considered as an add-on unit for the end of the curriculum, but needs to be threaded throughout the mathematics curriculum, as has been frequently advocated by Kaput (1991, 1998). The impacts over time, we believe, are significant. Further evidence in support of this hypothesis is still needed.

In this paper, we have presented data showing that young students can learn algebra, detailing the specific areas of impact. Results from a recent study (see Schliemann, Carraher, & Brizuela, 2012) have also shown that EA, besides promoting the learning of algebra in primary school, promotes improvement in later algebra learning and later mathematical learning. This kind of data could have great impacts on teacher preparation. If we are to seriously advocate for the early introduction of algebra in elementary school, then we need to also engage in an exploration of adequate teacher preparation. Finally, positive results in the long term could eventually lead us to inspect and eventual modify the contents of algebra courses at the middle school and high school levels.

## Notes

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lizing the paper's analysis. The authors also thank the other members of the research team, particularly Mary Caddle, who assisted in the data processing.

[2] We are not claiming that these are the only two uses of letters (Usiskin, 1988). These are just the most prominent in our work.

[3] These 15 students were those who remained with us for the duration of the study, from 3rd through 5th grade.

[4] Our hypothesis for this lack of increase in the algebra assessment from 4th to 5th grade is that the focus of most of our teaching in 5th grade was the manipulation of equations. This content was not part of the assessment items given from 3rd-5th grade. The 5th grade students did receive an additional assessment on equations which is not part of this analysis. Furthermore, the intensity of our teaching decreased significantly in 5th grade, when we were in the classes for only 18 90-minute lessons. We would like to hypothesize, on the basis of this data, that algebra instruction needs to happen not only early, but also needs to be sustained and frequent.

[5] The numbers of students in each grade level varied because of new arrivals as well as departures from the school from year to year.

[6] For the purposes of this CFA, we will only focus on the 22 students who remained in the school from 3rd through 5th grade, and for whom we have longitudinal data.

[7] At the time of this intervention, MCAS mathematics assessments were not administered in the 3rd or 5th grades.

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