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Mathematical Teaching Strategies: Pathways to Critical Thinking and Metacognition

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Abstract

A teacher that emphasizes reasoning, logic and validity gives their students access to mathematics as an effective way of practicing critical thinking. All students have the ability to enhance and expand their critical thinking when learning mathematics. Students can develop this ability when confronting mathematical problems, identifying possible solutions and evaluating and justifying their reasons for the results, thereby allowing students to become confident critical thinkers. Critical thinking and reasoning allows students to think about how they utilize their discipline of mathematical skills (i.e., they think about their method of thinking). Metacognition helps students to recognize that math is logical reasoning on solutions to problems. Students are taught how to: identify scenarios; evaluate; select problem-solving strategies; identify possible conclusions; select logical conclusions; describe how a solution was summarized; and indicate how those solutions can be applied to more advanced math problems. This paper indicates the necessity of applying critical thinking and provides an example of how critical thinking; creativity and flexibility in finding such ways help students to better understand the concepts of number sense. This discipline of reasoning results with students who develop the ability with focused thinking, planning and strategizing, which have been identified as key aspects of organizational success, decision making, and life choices.

Key words: Critical thinking; Metacognition; Intellectual skills; Mathematical reasoning; Number sense; Multiplication

Introduction

Teachers have the ability to bring together the most fundamental and widely applicable intellectual skills that have been identified by educators, executives and organizational leaders. Mathematical knowledge and the ability to solve quantifiable problems and utilize critical thinking skills enhance the abilities of students to think and make decisions. Analyzing, evaluating, reasoning and communicating knowledge and skills provides a pathway to new discoveries. Historically, there have been strong links between mathematics and critical thinking, since many great mathematicians were also great critical thinkers. In the ancient Greek origins Pythagoras, Plato, Aristotle, Euclid and Archimedes were both mathematicians and critical thinkers, while the same can be said of many present-day French, American, British and Russian mathematicians..

Subdivide text into unnumbered sections, using short, meaningful sub-headings. Please do not use numbered headings. Please limit heading use to three levels. Please use 12-point bold for first-level headings, 10-point bold for second-level headings, and 10-point italics for third -level headings with an initial capital letter for any proper nouns. Leave one blank line after each heading and two blank lines before each heading. (Exception: leave one line between consecutive headings.)

Critical Thinking and Higher Level

When teaching mathematics, critical thinking skills can be used, practiced and enhanced by effective cognitive methods. Critical thinking can enhance creative problem solving options by encouraging students to seek new strategies when solving mathematical problems. Mathematics teachers know the importance of mathematical reasoning, for it builds the skills required for higher-level mathematics. Van Gelder (2005) argued improving critical thinking abilities requires practice and to be actively engaged in the skill of thinking critically. Van Gelder’s (2001) recommendations for improving critical thinking included practice of: active engagement,

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transfer of learning, understanding theories, thinking map skills, the ability to identify biases and being open to what should be considered 'truth.' The results of various studies support the fact that, with practice, students can improve their critical thinking skill levels in this area (Pascarella & Terenzini, 1991). The work of Reichenbach (2003) and other studies indicate that students can expand their thinking skills, including their clarity, accuracy, precision, relevance, depth, breadth and logic (stills identified in the Universal Intellectual Standards).

Math Reasoning and Metacognition

The sharper their critical thinking skills, the better mathematical student are able to solve problems and to formulate arguments by drawing on a wide base of knowledge. When teaching mathematics options for solving problems or during computations, teachers can assist students by expanding those math reasoning skills associated with advanced mathematics, which require a higher level of thinking, critical thinking or thinking about thinking (often referred to as metacognition). Thus, instructors should provide more options and activities that would allow students to challenge present concepts and allow them to continue expanding their mathematical abilities. Cooperative learning and metacognitive training enhance mathematical reasoning. Although metacognition has been defined simply as thinking about thinking, a better understanding of the definition of metacognition is as follows: higher order thinking that enables understanding, analysis and control of one's cognitive process, especially when engaged in learning. Essentially, when using metacognition, a student becomes aware of their own style of learning and is able to recognize and implement strategies, which is often most effective when solving problems within groups or during cooperative learning. Kramarski and Mevarecch (2001) studied the effects of four instructional methods on the mathematical reasoning and metacognitive knowledge of 384 eighth-grade students. They found that that the cooperative learning and metacognition groups significantly outperformed the individual and metacognition groups, which in turn significantly outperformed the cooperative learning and individual groups on graph interpretation and various aspects of mathematical explanations. Furthermore, the metacognitive groups (the cooperative and metacognition and independent and metacognition groups) outperformed their counterparts (the cooperative and individual groups) on graph construction (transfer tasks) and metacognitive knowledge. These findings demonstrate the importance of metacognitive training or thinking about the cognitive process in results of mathematical reasoning.

Emphasis on Teaching for Critical Thinking

Association of American Colleges and Universities (AACU) survey of business and non-profit leaders found that 93% believe "a demonstrated capacity to think critically, communicate clearly, and solve complex problems is more important than undergraduate major." More than 75% of those surveyed stated that they want more emphasis on critical thinking, complex problem solving, written and oral communication, and applied knowledge in real-world settings for all colleges and universities. The procon.org website, which lists the pros and cons of controversial issues, identifies the following concerning critical thinking (with research backing each statement)

- 93% of higher education faculties believe critical thinking is an essential learning outcome.
- Critical thinking is considered the second most important life skill after interpersonal skill.
- Learning and discussing controversial issues in school helps students become more informed and more active citizens.
- Learning about controversial topics in school increases students' political participation.
- Students who debate controversial issues in school are more likely to be engaged and active citizens.
- Discussing current events and debating controversial issues are associated with higher scores on the National Assessment of Educational Progress (NAEP), the largest national standardized test in the United States.
- Teaching controversial topics helps students develop non-violent strategies for dealing with conflict.
- Controversial issue assignments increase critical thinking skills and appreciation of cultural diversity.
- Studying and debating controversial topics in school helps increase student attention, motivation, achievement, creativity and self-esteem.

  Employers' value demonstrated critical thinking over job candidates' undergraduate Majors.

Bloom's Taxonomy and Reasoning on the Cognitive Domain

Mathematics teachers indicate that using Bloom's Taxonomy discussion on the cognitive domain can accelerate and guide the process from the original domains (knowledge, comprehension application, analysis, synthesis evaluation) toward higher levels of thinking (remembering, understanding, applying analyzing, evaluating and creating). When questioning students during math classes, it is useful to assist them with problem solving by explaining with words, drawings, diagrams and numbers. Also, having students question helps them become better problem solvers. Asking students to explain their Math answers (using words, drawings or diagrams and numbers), is an excellent method for assessing their understanding of the materials discussed. One approach to assessing whether they have truly understood the taught concept is to have the students explain a number pattern, thus engaging them with critical thinking skills. Having students ask questions provides an opportunity for them to obtain reasoning and responses to the questions. By using inductive and deductive reasoning and questioning, while adhering to the Blooms Taxonomy, students learn mathematical concepts and solve mathematical problems and also recognize the extent to which reasoning applies to mathematics and how their critical thinking pathway can facilitate decision making and strategic planning. Bloom's Taxonomy provides a framework for critical thinking. These taxonomies provide a method for preparing instructional units, assessing progress by providing a step-by-step learning process. Bloom's Revised Taxonomy has assisted instructors with a method of usability in critical thinking with this cognitive knowledge matrix. The following website provides utilization of Bloom's Taxonomy for mathematics: http://www4.uwm.edu/Org/mmp/ACM201213-files/ACM-March15-BloomRevisedMath.pdf

There are various terms used to refer to 'reasoning' (e.g., critical thinking, higher-order thinking and logical reasoning) and the preferred terms change according to the subject area. Resnick (Resnick, 1987, 2-3) reported that although we cannot define higher order of thinking in an exact term, we do recognize it when it happens. Resnick listed the features of higher order of thinking involves the following:

- higher order thinking is nonalgorithmic….a path of action not fully specified in advance
- tends to be complex. The total path is not "visible" (mentally speaking) from any single vantage point
- often yields multiple solutions, each with costs and benefits, rather than unique solutions
- nuanced judgment and interpretation
- the application of multiple criteria, which sometimes conflict with one another
- uncertainty. Not everything that bears on the task at hand is known
- self-regulation of the thinking process. Involves imposing meaning, finding structure in apparent order
- And is effortful…Considerable mental work involved in the kinds of elaborations and judgments required. (p. 2-3)

Those areas identified can be observed when mathematic teachers and students can develop and enhance their critical thinking skills by indicating optional methods and perhaps simplifying the process. Below is an example of how critical thinking can be used with simple mathematics. Students can develop and enhance their critical thinking skills as a result of instructors providing optional methods for simplifying the mathematical process.

For example, there are many ways to multiplying two numbers, some of which are easier than the standard approaches. Critical thinking, creativity and flexibility in finding such ways helps students to better understand the concepts of numbers and multiplication. Is there a faster way to compute that is fun and requires a deeper thinking of understanding of the concepts involved? Here, we emphasize a novel balancing method that is useful and enjoyable to experience taking into consideration Bloom's Taxonomy and Resnick's observation of critical thinkers. Utilizing this higher order of thinking can be "viewed" in the following multiplication example answering the question. Can multiplication be simplified?

Can Multiplication Be Simplified?

Multiplying Numbers Close to 100: Let's start with surprisingly simple examples that utilize a complementary approach.

Example 1: 103×106. Instead of multiplying them, we choose their complements to 100, which are 3 and 6, shown below. We multiply the complements: $3 \times 6 = 18$.

103 3
106 6

Interestingly, the cross sums are equal: 103 + 6 = 106 + 3 = 109. Write this three-digit number, 109 in front of the previous 2-digit product 18 to get 10,918. This is the exact product of our two original numbers 103 and 106.

We present the traditional, or expanded multiplication strategy here for comparison purposes:
(100+3) (100+6) = 10,000 + 3 × 100 + 6 × 100 + 3 × 6 = 10,918.
Example 2: 97 × 94. Their complements to 100 are (-3) and (-6). Their product is (-3) × (-6) = 18.

97   -3
94   -6

The cross-sums are again equal: 97 + (-6) = 94 + (-3) = 91. Write this number 91 in front of the previous product 18 to obtain 9118. This is the product of our two original numbers, 97 and 94.
This strategy works also when one number is larger and the other one is smaller than 100.

Traditional Format: (100 – 3) × (100 – 6) = 10,000 – 3 × 100 – 6 × 100 + 3 × 18 = 9100 + 18 = 9,118.
Example 3: 103 × 94. Their complementary numbers are 103 – 100 = 3 and 94 - 100 = -6. Multiply them to get 3 × (-6) = 18.

103   3
94   -6

The equal cross-sums are 103 + (-6) = 94 +3 = 97. We want to write this "in front" of (-18). This means, we have to subtract 18 from 9700. Thus, the original product is: 9682.

Traditional Format: 103 × 93 = (100 + 3) (90 + 3) = 100 × 90 + 100 × 3 + 3 × 90 + 3 × 3 = 9682.
If we ignore a trivial multiplying by 1 in the traditional method, the complementary strategy does not look simpler, but it allows the students to have fun doing it. Of course, both strategies work very well simply because the two numbers chosen were very close to 100. As the numbers get farther from 100, their complements become larger and multiplying the complements becomes less advantageous. Say, 83 × 76 have complements (-17) and (-24), which are too large to easily multiply.

However, as numbers are farther away from 100, some simplification in the complementary strategy compared to the traditional can be achieved by choosing complements with respect to not 100, but some other number, as we demonstrate below. The balancing method always works as shown by the proof in the appendix section. The question here is whether or not it can be more efficient than the traditional method.

Introducing Balance Number

In many cases the multiplication procedure can be simplified by choosing a round number close to both multipliers. We call this number a Balance number and denote by B. Now, we consider the complement of a number N relative to B and call it *Disbalance,* D = N – B.

Example 4: Consider 63 × 47. Let B = 50. The *Disbalance* numbers are 13 and (-3), which are smaller and easier to multiply (-39). We find the cross-sums as before, which we denote as S.

B = 50	6 3	+13	S = 60	S = 63 – 3 = 47 +13 = 60
	4 7	- 3		
		-3 9		

Now we multiply S by B, 60 × 50 = 3,000 and add that number to the previous product. The answer to our original multiplication is 3000 – 39 = 2961.

(60 + 3) (50 – 3) = 60 × 50 + 3 × 50 – 3 × 60 – 3 × 3 = 3,000 + 150 – 180 – 9.

The Balance number 50 works here not because it is half of 100, but because it is close to both of our initial numbers. If our numbers are 78×68, we can choose 80 as the B number (B=80). Then the *Disbalance* are $80 - 78 = 2$ and $80 - 68 = 12$. We have their product simply $2 \times 12 = 24$. The cross-sum $S = 78 - 12 = 68 - 2 = 66$. Now, we have to multiply B by S, 80×66, and add to the previous smaller product (2×12).

Traditional Format: $(78 \times 68) = (70 + 8) \times (60 + 8) = 70 \times 60 + 70 \times 8 + 8 \times 60 + 8 \times 8 = 5{,}304$
The question is: Can we find a better Balance such that the corresponding S and the *Disbalance* are easy to multiply?

Example 5 (Balance B=70):

B = 70   7 9   +9   S = 73
6 4   -6
-54

Thus: $79 \times 64 = 70 \times 73 - 6 \times 9 = 5056$

Traditional Format: $(70 + 9)(60 + 4) = 70 \times 60 + 9 \times 60 + 4 \times 70 + 4 \times 9$

Disbalance are + 6, -2. $S = 79 - (-6) = 64 + 9 = 73$. Now, $B \times S = 74 \times 70$ is easier.
The choice of the Balance is vital. The best choice of B comes with practice.

What should we do when one number is small and the other large, say 34×73? Choosing Balance 100 or 50 will result in two-digit *Disbalance*. Let's try balance 40.

Example 6

B=40   7 3   +33   S = 67
3 4   -6
-198

In this case, we have to calculate $67 \times 40 - 6 \times 33$.
To be able to compare the two strategies, next, we introduce a criterion.

Comparison of Efficiency of the Standard and the Balancing methods

We will find how many one-digit by one-digit multiplications are involved in both of these strategies. The less that number is the simpler or more efficient the multiplication process.

In all three examples 1 – 3, we have 1 versus 1. In example 4, we have 3 versus. 4. In example 5, it is 3 versus 4. But, in example 6, we have 4 versus 4. We see some simplification of the complementary method versus the traditional.

B = 30 will result in single digit complement multipliers. If the balance chosen is 20, we will have (+53) and (+18) (both multipliers are two-digits). Similarly, if B =50, this will also result in two-digit multipliers.

Let's choose a Balance number not a multiple of 10:

B = 35   7 3   +38   S = 74 $74 \cdot 35 + 38 = 2482$
3 4   +1
38

In this example, we have 6 versus 4. The complementary method is a lot more tedious, because the B number does not end in zero.

As you can see, although it is easy in the first part of the solution, when it comes to multiplying the Sum and the Balance, it becomes harder than choosing multiple of tens as a Balance; because it is still one digit in a manner of speaking.

General Rule: We choose as a Balance either of the two multipliers rounded to the ***nearest*** multiples of 10. In either way, we need to make one of the numbers B or S ending in zero to simplify the multiplication process.
Note: Trying to choose B such that S will end in zero doesn't simplify the multiplication.
If both multipliers fall within (B – 10, B + 10) where B is a multiple of 10, then we have only two multiplications, which means two digit by digit multiplication, versus 4. The multiplication is essential simplified in this method.
If both multipliers fall within 10k ± 20, then in the worst case, we will have two 2-digit numbers staring with the digit "1". As we noted, multiplying by 1 is a trivial operation and we ignored it before in considering in the traditional operation. Then we are "saving" or "decreasing the number of steps, if we multiply them using the traditional method.
For example, let's choose two multipliers between 70 – 20 and 70 + 20:

B= 70	8 7	+17	S = 63
	5 6	-14	

Thus, saving one multiplication procedure compared to the traditional method.
Utilizing the B-method, students have the opportunities to use various number combinations while having fun. The B-method also allows the students to practice the execution of addition, subtraction, and positive and negative numbers mentally through investigation while solving multiplication problems presented to them versus the repeated process in the traditional method.

Multiplying one-digit numbers

To get a better idea of how the B-method works, try these examples and decide if it is easier or not?

B = 10	7	-3	S = 5	$10 \cdot 5 + 2 \cdot 3 = 56$
	8	-2		$50 + 6 = 56$
		6		
B = 5	7	+2	S = 5	$5 \cdot 5 - 2 \cdot 2 = 21$
	3	-2		$25 - 4 = 21$
		-4		
	25			
		21		
B = 5	8	+3	S = 5	$5 \cdot 5 - 3 \cdot 3 = 16$
	2	-3		$25 - 9 = 16$
		-9		
	25			
		16		

In the last two cases, it's easier to complement only the largest number to 10 versus choosing B = 5 as shown below:

7 . 3 = (10 – 3) . 3 = 10 . 3 – 3 . 3 = 21
8 . 2 = (10 – 2) . 2 = 10 . 2 – 2 . 2 = 16

Multiplying multi-digit numbers

Can we extend this simplifying approach to multi-digit numbers? Yes, but with some more elaboration. We can always reduce; simplify the number of digits of one of the numbers say, the smaller one by one and choose the closest multiple of 10. Then we have a new process, where we can again reduce by one digit the smaller number and so on until we get to a target number less than or equal to 5.

Example 7 Step 1 (Balance B=100):

8 7 5 +7 7 5 S = 898 × 100 = 89800

1 2 3 +2 3

Step 2 (Balance B=20)

7 7 5 +7 5 5 S = 778 × 20 = 15560 + 2265 = 17825

2 3 + 3 17825 + 89800 = 107625

2265

Example 8 Step 1 (Balance B=700):

8 7 3 +1 7 3 S = 901 × 700 = 630700

7 2 8 +2 8

Step 2 (Balance B=30)

1 7 3 +1 4 3 S = 171 × 30 = 5130 - 286 = 4844

2 8 -2 630700 + 4844 = 635544 Yes! It works!

-286

This understandably works for any number of digits. The question remains, is this method effective for multi-digit multiplication?

Why it works?

How does this new approach work? Instead of proving directly, algebraically, we will work on the same examples and use classical digit – place value concepts.

Example 1 (Balance B=100): Start with the above example 1, and use the place representation
97 · 94 = (100 – 3) (100 – 6) = 100 · 100 – 3 · 100 – 6 · 100 + 3 · 6

We have here four simple multiplication tasks which then must be added together. But, let's see if we can compare this with the steps from the new algorithm. Note that the part (3 × 6) = 18 is common for both methods. We need to see where the 91 (to be added in front of 18) comes from. It's easy to see where 9100 comes from.

97 · 94
= (100 – 3) (100 – 6)
= 100 · 100 – 3 ·100 – 6 · 100 + 3 · 6
= 100 (100 – 3 – 6) – 3 · 6
= 100 · **91** + 3 · 6

Yes! We distributed and grouped the terms partially. Not only have we showed that the new method works in this case, but we saw a different way of coming to this new procedure from the standard representation of decimal numbers. We also see S. In the following examples, S will be highlighted in BOLD.

Example 2 (Balance B=100):
103 · 106
= (100 + 3) (100 + 6)
= 100 · 100 + 3 · 100 + 6 · 100 + 3 · 6
= 100 (100 + 3 + 6) + 3 · 6
= 100 · **109** + 3 · 6
This shows where 109 come from to the front of 18.

Example 3 (Balance B=100):

$$
\begin{aligned}
&103 \cdot 94\\
&= (100 + 3)\ (100 - 6)\\
&= 100 \cdot 100 + 3 \cdot 100 - 6 \cdot 100 - 3 \cdot 6\\
&= 100\ (100 + 3 - 6) - 3 \cdot 6\\
&= 100 \cdot \mathbf{97} - 3 \cdot 6
\end{aligned}
$$

So far we have regrouped the steps of the standard multiplication to see the steps of the new approach.

Example 4 (Balance B=50):

We have chosen an "innovation" a new balance 50 instead of 100 in this case. Let's see if we can justify this example with standards.

$$
\begin{aligned}
&63 \cdot 47\\
&= (50 + 13) \cdot (50 - 3)\\
&= 50 \cdot 50 + 13 \cdot 50 - 3 \cdot 50 - 3 \cdot 13\\
&= 50 \cdot (50 + 13 - 3) - 3 \cdot 13\\
&= 50 \cdot \mathbf{60} - 3 \cdot 13
\end{aligned}
$$

<u>Three-digit Multiplication</u>
Example 7 Step 1 (Balance B=100):

$$
\begin{aligned}
&875 \cdot 123\\
&= (100 + 775)\ (100 +23)\\
&= 100 \cdot 100 + 23 \cdot 100 + 775 \cdot 100 + 775 \cdot 23\\
&= 100\ (100 + 23 + 775) + 775 \cdot 23\\
&= 100 \cdot \mathbf{898} + 775 \cdot 23
\end{aligned}
$$

Step 2 (Balance B=20)

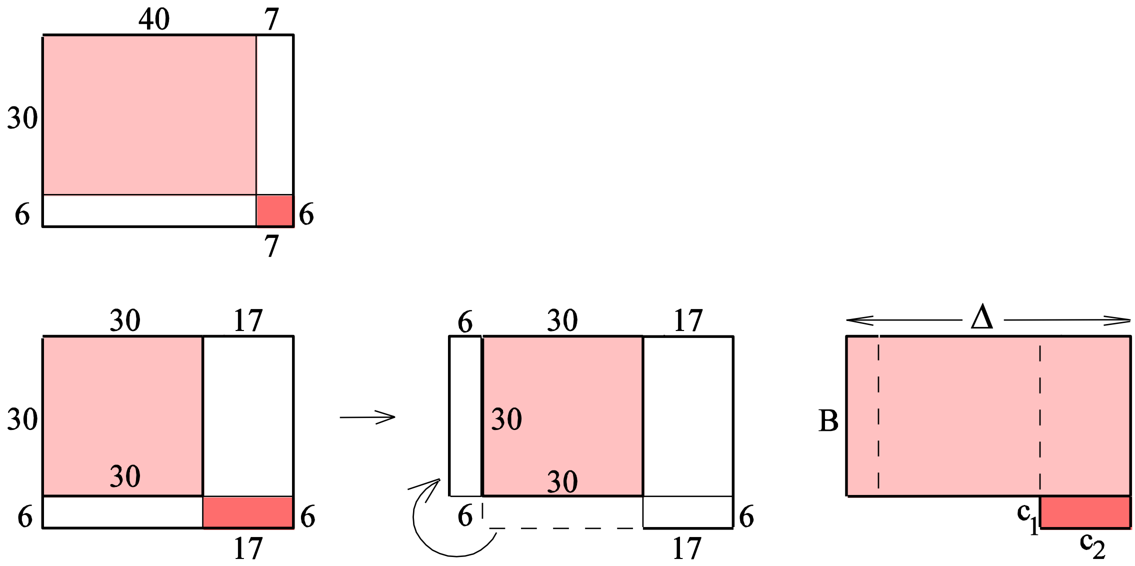
$$
\begin{aligned}
&775 \cdot 23\\
&= (755 + 20)\ (20 + 3)\\
&= 755 \cdot 20 + 3 \cdot 20 + 20 \cdot 20 + 755 \cdot 3\\
&= 20(755 + 20 + 3) + 755 \cdot 3\\
&= 20 \cdot \mathbf{778} + 755 \cdot 3
\end{aligned}
$$

We leave the demonstrations of other examples to the readers. In all examples, we can deduce the new procedure from standard operation using the chosen balance number. This procedure works when the two numbers being multiplied are relatively closed to each other if their difference $N_{2\,-}N_1$ is less or above 1/10 of each. The question is: Is standard procedure with a different balance better than with decimal balance (10, 100, etc.)?

Geometric Interpretation of both methods

<u>Standard Method</u>

Let's take a two-digit number, 47 by 34. We represent 47 as 40 and 7. 47 × 34 is a rectangle and we need to find the area in unit squares – the area of this rectangle. We take 40 separated by 10 units; 40 + 7 and 30 + 4 and draw two sub-division lines. As a result, we get four rectangles; 30 by 40; 30 by 7; 4 by 40 and the smallest one is 4 by 7. We need to find the area of each of the four rectangles and then add them together. Each of these four rectangles is reduced area which multiply two one-digit numbers. So, we could multiply two one-digit numbers as one unit of basic multiplication operation. So, we are third units of operation to be one here, one digit by one digit; two by third; two by seven; third by third; by seven.



New Method

Notice that the standard interpretation of the standard method is very unique, because 37 is represented by 30 + 7. Let's take the same rectangle 47 by 34 and choose the balancing number to be for example 30, is up to us which number to choose. Here is 30 and here is 47. So what we do here, we cut the rectangle by line at 30 and then we take the vertical line also at 30 here. So, that is the same B = 30. What we do with the rectangle is, we are separating a square, which is 30 by 30. Then 17, 30, 4, 17, 30, 4. Now, you have rectangles throughout, but one of them is a square, which indicates both sides are equal. Why is this interesting? Because the first bottom rectangle has the same length as the height of this rectangle. I am going to take the bottom of the rectangle, whose horizontal length and rotate it here vertically and attached it to the side of square. That's the geometric interpretation of this new method. Now we have two rectangles, one big and one little. The big one has one side which is B (balancing number) and the other side is exactly Delta, which is $N_1 + N_2 – B$. So, what we did here in the rectangle 34 by 47, which is the initial problem to multiply. We separated the squares with side is our balancing number B, then the rectangle immediately under that, we have length which is equal to the side, so taken advantage of that, we rotate the rectangle here to get the longer side is on the top and the smaller left over rectangle similar to what we have before in the standard case, now we have one elongated rectangle whose height is the balancing number B, and whose total left is the other side is what we called Delta. The smaller leftover rectangle is $D_{1\times} D_2$. Now we have an elongated rectangle - $S \times B + D_{1\times} D_2$.

Four units of operations to be done here . . .30 × 40; 30 × 7; 4 × 40 and 4 × 7. Choose the balancing number, for example 30 – it is up to us which balance number to choose.

The square allows us to rotate the bottom rectangle to the top to form the square into the new elongated rectangle.

$N_1 + N_2 = 0$

General Proof

The general proof is as follows. To multiply any two numbers, N_1 and N_2, we choose Balance B.

B N_1 D_1 S
N_2 D_2

Then, Disbalance are $D_1 = N_1 – B$
$D_2 = N_2 – B$

And we find the common cross-sum S:

$S = N_1 + D_2 = N_2 + D_1$

Taken separately, we have

$$\left.\begin{array}{l} S = N_1 + D_2 = N_1 + N_2 - B \\ S = N_2 + D_1 = N_2 + N_1 - B \end{array}\right\} \text{The same (again)}$$

Now, we claim that

$B \cdot S + D_1 \cdot D_2 = N_1 \cdot N_2$

Really,

$BS = B\,(N_1 + N_2 - B) = BN_1 + BN_2 - B^2$
$D_1D_2 = (N_1 - B)\,(N_2 - B) = N_1\,N_2 - BN_1 - BN_2 + B^2$

Adding we see everything cancelled except for N_1N_2

$BS + D_1D_2 = N_1N_2$

This is the proof that the new procedure works for any two numbers and with any number of digits!

Conclusion

When instructors teach critical thinking skills, students will be able to find the necessary information needed, evaluate the merits and consequences of the information and solve problems. Critical thinking allows students to process information in a logical manner and to prepare themselves for self-directed learning. Students with critical thinking skills can determine what information is important and what is irrelevant or not useful. Such students can identify logical errors but can be open to other points-of-view and reappraise their core values, opinions and knowledge, as well as determine what information is important and eliminate data that is non-useful, irrelevant and biased information. Students with critical thinking skills can also weigh various facts and points-of-view and identify logical errors, thus helping to solve problems. Critical thinking brings about clarity of perception, vision and a logical communication method of explanation. If a student can think critically and solve problems independently and in a systematic and logical manner, the student will be able to succeed in making wise decisions across all areas where decisions need to be made. The critical thinker is able to make educational decisions, understand options and expand knowledge though creative problem solving. A critically thinking student can realize that one can select the correct response or respond to any problem or decision that might arise. The teacher’s role is to focus on those characteristics of active mathematical strategies, promoting critical thinking and metacognition for life.

Recommendations

Educators indicate the need to replace our current math classes with meaningful mathematical experiences, which teach “how to think through Math” rather than memorizing formulas. These recommendations were indicated while reviewing several mathematic educators’ blogs and online postings. t. The process of ‘thinking through math’ helps students to think critically and discover new relationships and patterns of mathematical equations, leading to innovation and creation of new ideas and implementations, for which there is high demand within our this information age. The use of multiplication to enhance critical thinking with the question: Can multiplication be simplified with the example indicated provides an example of how mathematical knowledge and the ability to solve quantifiable problems can enhance the abilities of students to think and make decisions. Teachers of mathematics encouraging students with analyzing, evaluating, reasoning and communicating knowledge and skills will provides a pathway to new discoveries for the new generation.

References

Bloom, B. S., & Krathwohl, D. R. (1966). Taxonomy of educational objectives. Handbook I: Cognitive domain. Bloom's Taxonomy for mathematics: Retrieved from: http://www4.uwm.edu/Org/mmp/ACM201213-files/ACM-March15-BloomRevisedMath.pdf

Critical Thinking Skills in Education and Life Retrieved from http://www.asa3.org/ASA/education/think/critical.htm#critical-thinker

Hart, R. (2013). It takes more than a major: employer priorities for college learning and student success. *Washington, DC: The Association of American Colleges and Universities*.

Heick, T. (10). *10 Characteristics of a Highly Effective Learning Environment*. Washington, DC: American Psychological Association.

Kramarski, B., & Mevarech, Z. R. (2003). *Enhancing mathematical reasoning in the classroom: The effects of cooperative learning and metacognitive training*. American Educational Research Journal, 40, 281-310.

Martin, L., Thompson, S. D., & Richards, L. (2008*). Enhancing Critical Thinking Skills*. Washington, DC: American Psychological Association.

Pascarella, E. T., & Terenzini, P. T. (1991). *How college affects students*. San Francisco: Jossey-Bass.

Reichenbach, B. R. (2001*). Introduction to critical thinking*. Washington, DC: American Psychological Association.

Resnick, L. (1987). *Education and Learning to Think.* Washington, DC: National Academy Press.

Reichenbach, B. R. (2001). *Introduction to critical thinking*. Retrieved from http://mhhe.com/socscience/philosophy/reichenbach/m1_chap02studyguide.html

Schoenfeld, A & Paul, Richard, *Critical Thinking and Mathematical Problem Solving*. Washington, DC: PBS Adult Learning Satellite Service. https://www.criticalthinking.org/store/products/critical-thinking-and-mathematical-problem-solving/233

Scruton, R. (1996). The eclipse of listening. *The New Criterion, 15*(30), 5-13.

Su, H. F., Marinas, Furner, J. (2011). *Number sense made simple using number patterns.* The International Journal for Technology in Mathematics Education (*IJTME*). Vol. 18, No. 1. pp. 39 – 50. March, 2011.

Su, H. F., Marinas, Furner, J. (2010). *Investigating numeric relationships via an interactive tool: covering number sense concepts for the Middle Grades.* Creative Education Journal. Vol. 1 No. 2, September, 2010. pp.121-127.

Su, H.F., Sico, J., Vicaro, E., Su, T.C., Zeleke, M. (2010). <u>A Simple approach to Algebra and Higher Order Thinking Skills.</u> An Amazon.com publication. August 4, 2010. ISBN-10: 1453735410.

The reflective educator, Retrieved from: http://davidwees.com/content/mathematics-education-blogs /

Teaching Controversial Issues and Critical Thinking: Evidence of Importance and Outcomes, http://www.procon.org/view.resource.php?resourceID=5860

Van Gelder, T. (2005). *Teaching critical thinking*. College Teaching, 45, 1-6.

Van Gelder, T. (2001). How to improve critical thinking using educational technology. Meeting at the crossroads: Proceedings of the 18th Annual Conference of the Australasian Society for Computers in Learning in Tertiary Education. Retrieved from http://www4.uwm.edu/Org/mmp/ACM201213-files/ACM-March15-BloomRevisedMath.pdf