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Prospective Elementary Teachers' Conceptual Understanding of Integers

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Abstract

This investigation examined the degree to which prospective elementary teachers had developed a meaningful and conceptual understanding of what integers are and explored their development of models for multiplication with integers that are related to everyday activities. Additionally, this study explored how these understandings informed prospective elementary teachers envisioning of their teaching of integers concepts and integers with operations to their future students.

Introduction

The success of the current efforts for reform in mathematics education depends largely on teachers' knowledge and understanding of the mathematics they will teach. In light of this, the importance of teachers' content knowledge has been a major focus in the literature for more than three decades (Ball, 1993; Ma, 1999; Shulman, 1986; Ball, Hill, & Bass, 2005; Utley & Reeder, 2012) and increasing teachers' mathematical knowledge remains a concern for both education policy and research (Greenberg & Walsh, 2008; National Mathematics Advisory Panel, 2008). Despite this focus, a significant number of elementary teachers continue to be underprepared and uncomfortable with the mathematics content they are expected to teach (Greenberg & Walsh, 2008).

Research has further provided ample evidence that teacher content knowledge and teacher pedagogical content knowledge influence student understanding (Fennema & Franke, 1992; Hill, Rowan, & Ball, 2005; Ball et al., 2005). If we assume that teacher understanding affects practice and that teacher practice affects student understanding (Hurry, Nunes, Bryant, & Pretzlik, 2005; Smith & Baker, 2001), then it is important that we investigate both teacher understanding and their pedagogic practice.

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply (Ball, 1990, 1993; Ball et al., 2005). However, practicing and prospective elementary teachers often know the procedures and algorithms for elementary mathematics they teach, but do not understand the mathematics conceptually (Ball, 1990, 1993; Ma, 1999). Further confounding the problem is the fact that by the time prospective elementary teachers begin their education or mathematics methods coursework they have spent years learning mathematics from teachers whose pedagogic practices are predominantly teacher centered and driven by the aim for algorithmic and procedural proficiency. Studies of prospective elementary teachers have found that this population tends to exhibit weak number sense, even after having completed their required college mathematics courses (Utley & Reeder, 2012; Tsao, 2005; Yang, Reys, & Reys, 2009; Young & Reichwein Zientek, 2011). Additionally, prospective elementary teachers' experiences with mathematics at the university level tend to focus on college level mathematics and as such do not do much to improve their conceptual understanding of the mathematics they will teach. It is the culmination of these experiences with school and university level mathematics that often develops limited understandings and a sense of low self-confidence with mathematics. The culmination of these experiences creates not only content knowledge for teaching challenges but also difficulties for prospective elementary teachers' envisioning mathematics teaching for understanding from a perspective that is more aligned with constructivist ideas about learning.

This study aimed to understand the degree to which prospective elementary teachers had developed a meaningful and conceptual understanding of what integers are and to explore their development of models for multiplication with integers that are related to everyday activities. Additionally, this study explored how these understandings informed prospective elementary teachers envisioning of their teaching these concepts to their future students.

Background

Research has shown that many prospective elementary teachers have developed misunderstandings and/or limited understandings of various basic mathematics concepts (Ball, 1988; Ma, 1999; Utley & Reeder, 2012; Young &Reichwein Zientek,, 2011). This limited understanding is often and unfortunately passed on to prospective elementary teachers' future students thus creating a cycle of limited knowledge in mathematics and an inability to critically question the mathematics one is being taught. The National Council of Teachers of Mathematics (NCTM) stated that:

> [C]entral to the preparation for teaching mathematics is the development of a deep understanding of the mathematics of the school curriculum and how it fits within the discipline of mathematics. Too often, it is taken for granted that teachers' knowledge of the content of school mathematics is in place by the time they complete their own K-12 learning experiences. Teachers need opportunities to revisit school mathematics topics in ways that will allow them to develop deeper understandings …. (1991, p. 134)

Despite this admonition from NCTM more than two decades ago, it remains the case today that most prospective elementary teachers are engaged in mathematics at the university level that includes topics well beyond what they will eventually teach. This common practice may be based on an underlying belief that if prospective elementary teachers are exposed to "higher level" mathematics they will naturally make sense of and deepen their understanding of the mathematics content they will teach to their students.

Conceptual Understanding of Integers

Traditionally, students have been required to memorize rules for operations with integers thus leaving them without the fluency or flexibility to use the mathematics learned in situations different than those in which they first learned them. Students often get confused about which rule to follow and are left to rely on their instincts to solve problems dealing with integers (Bolyard & Moyer-Packenham, 2006; Ferguson, 1993).

When students first encounter negative numbers they are unable to relate them to the models they have previously made sense of with counting numbers because they cannot "see" negative numbers. According to Hiebert and Carpenter (1992), the models used to teach counting numbers and fractions should make sense to students so they can remember rules that are generalized for performing the operations. However, working with models that will make sense to students is more challenging with integers because the models can only be employed in abstract ways in an effort to relate integers to what students have previously learned. Students struggle with the signs used to indicate positive or negative integers since they are the same signs used for addition and subtraction operations. Battista (1983) recognized the challenge of helping students learn integers and operations with integers stating "teaching students the four basic operations on the set of integers in a meaningful way is a difficult task" (p. 26) and suggested that it might be easier if a single physical model were used. Models most frequently used to teach integers include the neutralization models and the number line which are often represented by two-color counters or positive and negative charges scenarios (Reeves & Webb, 2004; Nurnberger-Haag, 2007). Studies unfortunately indicate that even when students are introduced to integers in ways that they can relate to, they often do not develop an understanding of the "negativeness" or "positiveness" of the situation in which the integers are embedded (Hackbarth, 2000). In light of these challenges, this study explored the following research questions related to prospective elementary teachers:

1. What are these prospective elementary teachers' conceptual understandings of integers?
2. What are these prospective elementary teachers' understandings of multiplication of integers?
3. How do their understandings inform their vision of their own teaching of integers and multiplication of integers to their future students?

Methodology

Data Collection

Journal entries, participants' short-answer responses, and video-taped classroom discussions constituted the data sources for this study. Three different groups of prospective elementary teachers comprised the participants for the study and data were collected over two semesters in two sections of a senior level mathematics methods course in the fall (N = 32) and one section of the same course (N= 29) during the spring. In both the fall and the spring, the instructor for the course collected the data as an ongoing and normal part of the coursework and assignments.

Data were collected after the prospective teachers had participated in a variety of activities focusing on what integers are and experiences aimed at developing their understanding of operations with integers in everyday contexts. Participants were asked to respond to a set of questions and/or prompts (See Figure 1):

> What are integers?
>
> Where do we encounter them or in what ways are integers used in the "real world"?
>
> How would you teach each of the following scenarios with integers?
>
> a. (+)x(+), b. (+)x(-), c. (-)x(+), and d. (-)x(-)
>
> Write a story or scenario for each of a through d above.
>
> Reflecting on your own learning experiences answer the following:
>
> Were you taught to multiply integers by using the "rules"?
>
> Did you question these rules when you were taught them? Why or why not?
>
> How did this classroom experience make you feel?
>
> Do you feel like you can now teach these concepts better through an inquiry approach?

Figure 1. Data collection questions/prompts.

Setting

The participants involved in this study were prospective elementary teachers at a university in the Southwestern region of the United States in a teacher preparation program wherein prospective elementary teachers took four mathematics content courses and three additional courses focused on elementary mathematics methods. Social constructivism was the underlying philosophical and theoretical belief for all instructors and developers of the methods courses. This philosophical and theoretical underpinning drove the decision to engage students in activities and experiences that supported problem-centered learning approaches to teaching and learning mathematics (Wheatley, 1991; Wheatley & Abshire, 2002; Van de Walle, 2004).

The course in which this study was conducted served as the final mathematics methods course prospective elementary teachers took in their final semester prior to their student teaching internship. While the course focused on teaching mathematics in grades four through eight, an overarching goal of the course was to further develop the *mathematical power* (NCTM, 1989) of the prospective elementary teachers. This goal was met through engaging the prospective elementary teachers in continual problem solving, critical questioning of the ways in which they had been taught mathematics, and integration of mathematics with issues of social justice and other content areas. The activities and experiences designed for this course were purposely selected in order to improve the mathematics content knowledge for teaching of prospective elementary teachers, to help them make connections in and among mathematics and other content areas, and to engage them in mathematics in every day contexts (specifically issues of social justice) (Gutstein & Peterson, 2013). For example, throughout the course prospective teachers were engaged in course readings that included articles from *Teaching Children Mathematics* and *Mathematics Teaching in the Middle School* published by NCTM, worked to solve non-routine problems using manipulatives, and were asked frequently to write their own problems using rational number concepts and present multiple ways their future students might approach solving the problem. Prospective teachers in this course were also engaged in learning about grouping methods and how to involve students in productive mathematical discussion, methods for questioning, and were involved in activities that integrated mathematics with other content areas.

The overall purpose of this course was to improve the prospective teachers' content knowledge for teaching while improving pedagogical knowledge, and their *mathematical power* (NCTM, 1989). The course was developed with the idea in mind that teachers tend to teach mathematics they are comfortable with, and if their mathematical understanding is limited or based on misconceptions, they are likely to avoid teaching mathematics as much as possible or simply focus their teaching efforts on procedural proficiency. The combined mathematics methods courses developed for these prospective teachers aimed to build their content knowledge, improve their pedagogic practices, and provide them with options for teaching mathematics in meaningful ways that were not limited to algorithmic approaches.

Data Analysis

Data were analyzed using an interpretive framework. The initial phase of data analysis was an exploratory examination of prospective elementary teachers' journals and video recordings. We hoped to develop an overall idea of the prospective elementary teacher perceptions of integers. Both researchers examined and analyzed the data independent of one another in order to identify overall themes and categories. Following our independent analysis of the data, we met to discuss our initial findings. We discussed our sense of the themes that seemed to be emerging and possible categories that might flesh out in further analysis. At this point, we were confident that our findings and understandings of the data were in alignment. This was followed by several subsequent independent readings and viewings of the videos in an effort to form common trends and categories. During these cycles of analysis we kept our initial themes and categories in mind and also remained open for other categories to emerge. After several cycles of analysis through the data and the formation of themes, we began to identify answers to the following questions. (1) How did these prospective elementary teachers describe integers? (2) What methods did they develop for teaching multiplication of integers? (3) How did their understandings inform their ideas about teaching these concepts to their future students? Data were then summarized and one final read through of all data was conducted by each researcher independently to verify the questions of the study had been answered. During the final analysis cycle, examples were extracted by each researcher independent of the other for use in illustrating the results. A final examination of the illustrative examples that were extracted helped to confirm the validity of the themes and categories identified through data analysis. In almost all cases, the researchers either pulled the same or similar examples to illustrate each category, verifying that each understood the categories in similar ways.

Research Results

The results of this study indicated that after having had multiple experiences with integers through discussion about where integers are encountered in the everyday activities of life, solving non-routine problems with integers, working with a variety of models typically used to teach integers and operations with integers (two-color counters, positive and negative charge models, and number line models), and creating problems (stories and scenarios) with integers for each operation, the prospective elementary teachers in this study could discuss what integers are and where they are "encountered in the real world." They seemed comfortable with multiplication of two positive integers and multiplication of one positive integer with one negative integer; however, they remained hesitant in their abilities to demonstrate an understanding of multiplication of two negative integers (particularly in writing a "real world" story problem). A minority of prospective elementary teachers seemed to be cautious about using approaches and models aimed at conceptual understanding of operations with integers and instead believed that the use of "gimmicks" or "sayings" will help their future students memorize the "rules" for multiplication of integers.

Integers in "Everyday" Encounters

Analysis of the data collected revealed that 100% of the participants could accurately discuss what integers are and where they are likely encountered in everyday activities. Likewise, 100% of participants indicated that integers can be encountered in the game of football (yards lost or gained), borrowing money and/or debt with credit cards, temperature, above and below sea level, and a variety of board games. All seemed to be ways that made sense to the prospective elementary teachers and in contexts in which they felt comfortable.

Multiplication with Integers

The second focus in data collection asked the participants to discuss specifically how they would teach multiplication of integers to future students. They were asked to address all possible combinations of positive and negative integers in multiplication (i.e., (+)x(+), (+)x(-), (-)x(+), and (-)x(-)). Most responses to this question included participants citing one or more of the models that had been presented in class (typically two-color counters or a number line model). Analysis of these data revealed that in all cases wherein a participant indicated a model for teaching multiplication of integers, she/he did so correctly. Additionally, many participants also indicated that they would use repeated addition or multiplication by grouping relying on the context of everyday activities with integers. Approximately 48% of the participants who cited everyday activities with integers and combined it with an array, two-color counters, or number line model did so correctly across all four combinations of multiplication with positive and negative numbers. The following is an example of one participant's response that included both models and everyday contexts (See Figure 2).

Although many prospective elementary teachers such as Maggie were able to create story problems for multiplication of two negative integers (i.e. the casino problem above), during discussions that proceeded writing these scenarios, many of them expressed concern about the mathematical validity of the (-)x(-) word problems. Debates ensued about whether or not saving or maintaining money is the same as gaining money. The most common concern was some form of the idea that if you begin with $0 and do not go to the casino for three days, you are still at $0, not at $25. Is a problem such as a.) + x +

Arrays, two-color counters, lattice multiplication, groups of items, drawings.

Sally had 3 baskets with 5 apples in each basket. How many apples does she have in total?

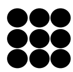
b.) + x –

Story problems. Teach – x + first and then commutative property.

4 friends have $-20 in each of their bank accounts. What is the total of their bank accounts put together?

c.) – x +

Two-color counters, pictures.
-3 x 3



(Maggie drew open circles with a capital letter "R" in each one to indicate red for negative)

You scored -1 (1 birdie below par) on every hole on an 18-hole golf course. How far below par are you?

d.) – x –

Every time I go to the casino I lose $25. I don't go for 3 days. How much $ did I save?

Figure 2. Maggie's response in her journal.

Maggie's accurately portraying a negative integer times a negative integer? Many prospective elementary teachers emerged from the class believing that it does; however, for some, concerns pushed them to ponder the matter further. Although this issue seemed to cause significant dissonance for most prospective elementary teachers in class, after discussing the matter for a few minutes, all but one class seemed temporarily satisfied with the idea of "saving money" as a positive integer answer and so discussion of the matter was terminated. However, in one class, the following example was proposed:

> "Your friend paid for your lunch three times this week. Lunch costs $5 each time. In other words, at the end of the week, when you pay your friend back your bank account will decrease by $15, or 3 x -5 = -15. Your friend decides to take away two of those lunch charges, so she relieves you of two $5 charges, or -2 x -5 = 10. Your account will decrease by $5 instead of $15, so you have gained $10."

Prospective elementary teachers in this particular class then began asking whether it was necessary to find a "real world" example of a negative integer times a negative integer, because it seemed to be the case that in the "real world" it would make more sense to simply use the terminology of a positive integer times a positive integer. Ultimately, one prospective elementary teacher proposed that although it may not be common to phrase "real world" multiplication problems in the way this example illustrates, the point of the example would be to portray the logic supporting a negative times a negative in the "real world" (i.e. there is a "real world" scenario for negative integer multiplication). Although only one class seemed to debate the matter for a lengthy period of time, prospective elementary teachers from all the classes continued to be uncomfortable with the idea of finding a contextual example for (-)x(-), as many of them revisited and questioned this concept periodically throughout the remainder of the semester.

Unfortunately, about 20% of the prospective elementary teachers indicated they would rely solely on methods of memorization, pneumonic devises, songs, or "sayings" to teach multiplication with integers. For example, Shelly discussed the fact that she believed that teaching "rules" is still a necessary part of what she will need to do as an effective mathematics teacher (See Figure 3).

> I know that we should be allowing the students to explore math and come up with their own rules and definitions without actually teaching them the rules; however, sometimes you have to teach the rules and then create opportunities to prove they are correct. I found this poem online while trying to research why a – x – = +. I think this is a great tool to use when teaching negative integers.
>
> When good things happen to bad people, that's bad. (+ x – = –)
>
> When bad things happen to bad people, that's good. (– x – = +)
>
> When bad things happen to good people, that's bad. (– x + = –)
>
> When good things happen to good people, that's good. (+ x + = +)

Figure 3. Shelly's discussion in her journal.

Jennifer likewise indicated that she would not necessarily teach the "rules" but instead found an analogy or saying she thinks is better for teaching multiplication with integers. She believed she will use the "Dating Game" (See Figure 4).

The final focus of this research study was to examine the perceived impact of experiences in a methods course aimed at improving prospective elementary teachers overall mathematical empowerment related to their vision of how they would teach mathematics as it pertains to multiplication of integers. Asking the prospective elementary teachers to write about whether they feel like they can teach integers and multiplication of integers through inquiry approaches revealed in some cases the resilience of their misunderstandings

The Dating Game! a.) I like you, you like me – it's positive. b.) I like you, you don't like me – it's negative. c.) I don't like you, you like me – it's negative. d.) I don't like you, you don't like me – it's positive.

Figure 4. Jennifer's entry in her journal.

and challenges with mathematics. For example, Julie discussed her own learning of the "rules" for the multiplication of integers revealing misconceptions and limitations in her content understanding:

> As I mentioned in class, I don't really remember being taught those rules. The only thing that really sticks out of my head is when I was taught that you take the bigger number's sign when multiplying. I remember having a hard time learning these rules. I couldn't understand why you would need to know this equation in real life. I believe that I was my biggest downfall in mathematics, not being able to relate math to real life experiences. (Prospective elementary teacher journal)

Brittany echoed common sentiments about frustrations with learning mathematics in her journal:

> Thinking more critically about how to teach (-)x(+) and (-)x(-) and story problems that have students doing this mathematics was at first mindboggling to me. For so long I just knew the rule and it was hard for me to think about why I thought this. Doing the (+)x(+), (-)x(+), and (+)x(-) were much easier than (-)x(-). When it came to creating a problem, for (-)x(-) I was frustrated because every "everyday" situation I thought of wouldn't work for this problem. (Prospective elementary teacher journal)

The experience of learning with the goal of conceptual understanding in mind, and being asked to reflect on their own mathematics learning, seemed to have a positive impact on most of the prospective elementary teachers. As they envisioned how they would eventually teach their future students, most prospective elementary teachers seemed to believe a "deeper understanding" should be a necessary component of their future classrooms. Allison wrote:

> Growing up, I was always very frustrated by math and got through it purely by memorization and repetition. However, even though I was constantly confused by the many varying rules, I never once questioned them or thought to ask why they were applicable. I think this is mainly because my math was taught to me in fairly traditional manner with little social justice or authentic applications or connections looking back. I wish that I had learned differently because I feel that it would help me teach, but I'm also thankful that I learned the way in which I

> did because it will help me understand the many different perspectives of my students. I do think it is better to teach integers through an inquiry approach because it allows the students to develop their own individual understanding and methods prior to being subjected to the "right" way. (Prospective elementary teacher journal)

Anne continued with this sentiment by stating:

> I feel like I can teach these concepts better now that I have analyzed the meaning behind them. Having this deeper understanding will allow me to ask probing questions that will cause the students to analyze and consider these concepts. Using story problems will help the students put these concepts into a context that makes sense to them. (Prospective elementary teacher journal)

Although some prospective elementary teachers indicated that they were unsure whether they could teach in ways that would foster more conceptual understanding due to their own frustrations with mathematics, all prospective elementary teachers revealed a belief that mathematics is better understood, and better taught, in the context of everyday scenarios. Kim stated "relating math to real life scenarios helps everyone understand it." The participants expressed overwhelmingly that with integers in particular, the use of everyday contexts helped them examine their own misconceptions and limited understandings of multiplication with integers.

Discussion

Teachers tend to teach mathematics in a way that is comfortable to them (Sowder, Phillip, Armstrong, & Schappelle, 1998). If they are only exposed to algorithmic and procedural methods and do not develop a deep understanding of mathematics and pedagogical practices that support the development of meaningful mathematics understanding, they may be left with very few options but to teach in traditional ways that will result in their own students' shallow understanding. When the idea of developing the *mathematical power* of prospective elementary teachers is infused in a mathematics methods course it may be possible to not only improve the content knowledge prospective elementary teachers need for teaching, but also provide an opportunity for them to envision teaching mathematics for conceptual understanding. The results of this study indicated that providing prospective elementary teachers the opportunity to examine their own mathematics understanding through challenging them regularly to make sense of mathematics in everyday contexts can be powerful. While some of the prospective elementary teachers who participated in this study still indicated they would rely on memorization based methods for teaching their own students multiplication with integers, the overwhelming majority indicated the power of using everyday contexts or scenarios for teaching mathematics. They also indicated that through the use of everyday contexts and a variety of models for understanding integers, they feel more empowered not only in their own mathematics knowledge but also in the ways they now envision teaching their future students.

Research has shown that beliefs about mathematics teaching and learning are resilient despite the efforts and work that takes place in mathematics courses (Reeder, Utley, & Cassel, 2009; Sowder, Phillip, Armstrong, & Schappelle, 1998). While this study demonstrates efforts in a mathematics methods course can positively impact prospective teachers' understandings of mathematics concepts and planned pedagogic practices, it is possible these teachers will return to the more traditional approaches that are more comfortable and more familiar to them (Sowder, Phillip, Armstrong, & Schappelle, 1998). It is difficult to determine how much effort, how many courses, how many years, it takes for prospective teachers to unlearn mathematical misunderstandings and develop conceptual understanding of the mathematics they will teach, but the findings of this study offer hope. However, most methods courses are focused on the development of pedagogic knowledge for teaching rather than content knowledge needed for teaching.

The prospective elementary teachers who participated in this study have experienced more than 12 years of school mathematics and an additional four university level mathematics courses. The courses at the university level focused on mathematics beyond the scope of the content of the school mathematics these prospective teachers will eventually teach, thus doing little to help clear up misunderstandings and/or the limited understandings of many basic mathematics concepts (Ball, 1988; Ma, 1999; Utley & Reeder, 2012; Young et al., 2011). NCTM's stance that "teachers need opportunities to revisit school mathematics topics in ways that will allow them to develop deeper understandings …." (1991, p. 134) is supported by this study. Courses with a concentrated focus on topics prospective elementary teachers will eventually teach rather than "higher level" mathematics may be what is necessary in order to develop their deep conceptual understanding of the content knowledge they will need for teaching.

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