ELPSA AS A LESSON DESIGN FRAMEWORK

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Abstract

This paper offers a framework for mathematics lesson design that is consistent with the way we learn about, and discover, most things in life. In addition, the framework provides a structure for identifying how mathematical concepts and understanding are acquired and developed. This framework is called ELPSA and represents five learning components, namely: Experience, Language, Pictorial, Symbolic and Applications. This framework has been used in developing lessons and teacher professional programs in Indonesia since 2012 in cooperation with the World Bank. This paper describes the theory that underlines the framework in general and in relation to each inter-connected component. Two explicit learning sequences for classroom practice are described, associated with Pythagoras theorem and probability. This paper then concludes with recommendations for using ELPSA in various institutional contexts.

Keywords: ELPSA, lesson design framework, Pythagoras theorem, probability

Abstrak

Tulisan ini menawarkan suatu kerangka rancangan pembelajaran matematika yang sifatnya konsisten dengan cara kita mempelajari sesuatu atau menemukan hal-hal dalam kehidupan ini. Kerangka ini menyiapkan suatu struktur untuk mengidentifikasi bagaimana konsep dan pemahaman matematika diperoleh dan dikembangkan. Kerangka rancangan ini disebut ELPSA dan memuat lima komponen pembelajaran: Experience (pengalaman), Language (bahasa), Pictorial (gambar), Symbolic (symbol) and Applications (aplikasi). Kerangka ini telah digunakan dalam mengembangkan pembelajaran matematika dan program pengembangan guru-guru di Indonesia sejak tahun 2012 atas kerjasama penulis dengan Bank Dunia. Tulisan ini mendeskripsikan teori yang mendasari kerangka ELPSA baik secara umum maupun dalam hubungannya dengan setiap komponen-komponen ELPSA yang sifatnya saling berkaitan. Tulisan ini juga menyajikan dua urutan pembelajaran untuk pengajaran matematika di kelas, berkaitan dengan teorema Pythagoras dan peluang. Tulisan ini kemudian diakhiri dengan rekomendasi penggunaan ELPSA dalam konteks institusi yang beragam.

Kata Kunci: ELPSA, kerangka rancangan pembelajaran, Teorema Pythagoras, peluang

This paper describes a learning framework that has been developed and modified over a 20-year period. From a foundational perspective, our conceptualisation of the framework was informed by a book written by Liebeck (1984) titled *How Children Learn Mathematics*. During the mid 1990s, the initial work was modified by the first author as a way of introducing mathematics pedagogy to undergraduate primary and secondary students. Various iterations of the framework were undertaken over a 10-year period by colleagues at Charles Sturt University in Australia. The model explained in this paper is the one initially proposed by the first author.

The initial model outlined by Liebeck (1984) was focused on how concepts are developed in terms of mathematics understanding. Our insights and development are more closely aligned to mathematics lesson design and, in particular, how mathematics ideas should be sequenced in order for students to make sense of these ideas. In addition, our learning framework affords opportunities for teachers to better understand the way in which pedagogical practices and learning experiences can be effectively presented to students in ways that reflect students' concept development. The specific activities and mathematics lessons presented in this paper illustrate the practical use of our learning framework, which have been developed as both authors worked closely with mathematics education academic and classroom teachers in an Indonesian context. To this point, the learning framework provides opportunities for teachers to enhance the content pedagogical knowledge as well as provides learning experiences for children that could be classified as culturally appropriate and personalised.

It is our view that the learning design described in this paper follows an authentic sequence that stimulates concept development. That is, the learning design reflects the manner in which individuals make sense of mathematics ideas. In classroom situations, teachers often concentrate on presenting mathematical ideas in symbolic ways. Although it is critical for students to develop flexible thinking that utilise symbols, full understanding can only be attained when ideas are presented in ways that lead to the necessary (or efficient) employment of symbolic representations. Typically, this can most effectively be achieved if multiple forms of representation precede symbolic understanding. From our experience, junior high school teachers overly focus on symbolic reasoning to the detriment of other concrete and pictorial representation of mathematics—and this is certainly the case in Indonesia (World Bank, 2010).

Our learning framework is underpinned by theories about learning that are considered constructivist and social in nature. The ELPSA framework views learning as an active process where students construct their own ways of knowing (developing understanding) through both individual thinking and social interactions with others. Several variations of how students develop concepts or establish abstract thought have been mentioned in the literature. Liebeck (1984) proposed that mathematics was an abstraction from reality in that a specific sequence of events took place as concept formation led to understanding. In her model, her sequence was framed around E (Experience); L (spoken Language that describes the experience); P (Pictures that represent the experience); and S (written Symbols that generalise the experience). We have expanded these stages of progress in describing mathematical understanding to include an application phase, which describes how the acquisition of knowledge can be applied to different situations.

Our framework assumes that *experiences*, both personal and collaborative, are the foundations for the introduction of new learning opportunities. This strong social dimension is critical in the first component of our framework. A number of theorists subscribe to this notion of engagement (Cobb, 1988; Lave & Wenger, 1991; Lerman, 2003; Wenger, 1999). The central idea to the work of social theorists is based on the premise that learning occurs through participation and that participation should encourage high levels of engagement and interaction. For example, Wenger (1999) indicated that meaning (for example the understanding of a concept) is most meaningfully developed through opportunities associated with one's personal life experiences or opportunities to sustain mutual engagement. If classroom practices allow students to develop mathematical ideas from personal experiences and understanding, it is much more likely that content can be introduced in meaningful ways. Such a viewpoint is frequently taken by those who adhere to the realistic mathematics model (Gravemeijer, 2010; Heuvel-Panhuizen, 2003; Widjaja, Fauzan, & Dolk, 2010). This social foundation is embedded in the way that *language* is utilised in promoting learning. Social theories associated with how experiences are scaffolded (Vygotsky, 1978); how culture influences perception (Bishop, 1988a, 1988b); and the influence of daily language on mathematics language (Adler, 1998; Setati & Moschkovich, 2010) reveal the importance of connecting personal experiences to mathematical terminology in order to ensure that sense making can be promoted.

Psychology-based theories are also influential in our learning framework. The manner in which mathematical ideas are represented is critical to sense making. Dienes (1959) argued that concrete representation and manipulatives supported students' learning as they move towards more abstract concepts and ideas. In terms of understanding mathematics ideas, concrete representations often provide the learner with the mental model of how pictures and symbols can be represented before students develop analytic reasoning. For example, Dienes' concrete manipulatives were used to represent addition and subtraction algorithms in a concrete pictorial manner. In our learning framework, this would take place in the *pictorial* component.

Mathematics representations can be classified within two systems, namely internal and external (Goldin & Shteingold, 2001). Internal representations are commonly classified as pictures "in the mind's eye" (Kosslyn, 1983) and include various forms of concrete and dynamic imagery (Lowrie & Logan, 2007) associated with personalised, and often idiosyncratic, ideas, constructs and images. External representations include graphical representations (e.g., graphs and maps), schematic representations (e.g., networks) and conventional symbolic systems of mathematics (e.g., algebraic notation or number lines). These two systems do not exist as separate entities and are seen as "a two-sided process, an interaction of internalization of external representations and externalization of mental images" (Pape & Tchoshanov, 2001, p. 119). In mathematics classrooms, visual-spatial information is commonly represented schematically or pictorially (Hegarty & Kozhevnikov, 1999), while verbal information is represented with number sentences or algorithms (Diezmann & Lowrie, 2008). The process of developing internal representations is typically visually based (or pictorial in our framework). External representations can also be pictorial—for example, a student encoding a graphic. In addition, external representations can be symbolic—for example, producing an algebraic statement.

Internal representations often involve the process of decoding information. Encoding occurs when the learners construct their own representation in order to solve a task. These techniques provide students with the opportunity to understand all the elements of a task in a manner that helps scaffold understandings (e.g., drawing two circles and dividing each into segments to better understand a ratio problem). By contrast, decoding techniques are used to make sense of information that has been given by someone else (e.g., a teacher). In such situations, the students make sense of information (e.g., a map, a pie chart, or a line graph). The capacity to encode and decode information is critical to problem solving, and is especially addressed in the *pictorial* component of the ELPSA sequence.

In our framework, the *symbolic* component involves the students' capacity to represent, construct, and manipulate analytic information in a symbolic manner. Mathematical symbols include number sentences, algebraic expressions, and other external representation that use symbolic notations. According to De Cruz and De Smedt (2013), mathematical symbols enable us to perform operations and actions that would not be possible without such supports. For example, a multiplication algorithm is a symbolic operation that is required when multiplying a 3-digit number by a 3-digit number. Theoretically, mathematics symbols are best utilised when the student has formed a sound understanding of a particular concept. Research suggests that students can experience difficulties when shifting from one symbolic representation to another (e.g., repeated addition to multiplications, and algebraic forms such as $(a + b)^2 = (a^2 + b^2)$). In this component of our learning framework, it is necessary for children to practice using symbolic operations and notations in order to develop fluency. As Uttal, Scudder, and DeLoache (1997) suggested, explicit instruction is necessary for students to use objects in a symbolic way. This commonly occurs with the use of textbooks or, more recently, in technological devices. In mathematics, algebra is often considered to be an instance where students face mathematical challenges if foundational understanding have not been developed. In our learning framework, we advocate that symbolic representation should not be presented until children have developed foundational skills. To this point, mathematical symbols are more than external representations of concepts. Symbols represent underlying mathematical concepts and understanding. If mathematical symbols are introduced too early, students are likely to use them inappropriately.

Most learning frameworks do not explicitly recognise the role of *application* in the learning process. However, the manner in which students are able to apply mathematics ideas to new situations is considered critical to the enhancement of students' mathematical literacy. In fact, international agencies, such as the Program of International Student Assessment (PISA), have stated that:

> The mathematical literacy assessment framework was written to encourage an approach to teaching and learning mathematics that: gives strong emphasis to the processes associated with confronting a problem in a real-world context; transforms the problem into one amenable to mathematical treatment; makes use of the relevant mathematical knowledge to solve it; and evaluates the solution in the original problem context. If students can learn to do these things, they will be much better equipped to make use of their mathematical knowledge and skills throughout their lives. So PISA measures not only the extent to which students can use their mathematical content knowledge, *but assesses what they know and how they apply their knowledge of mathematics to new situations* (Thomson, De Bortoli, & Buckley, 2013, p. 16; emphasis added).

There is an evidence base that suggests students do not utilise mathematics understandings developed in the classroom in out-of-school situations. An extensive study in Brazil (Nunes, Schliemann, & Carraher, 1993) found that uneducated workers were able to utilise mathematics calculations and problem-solving processes more effectively in workplaces than those people who had much more mathematics education (schooling in mathematics). Nunes et al. (1993) argued that students exposed primarily to the process of manipulating mathematical ideas symbolically could not use these representations effectively in new situations. More recently, Australia's Chief Scientist (Chubb, 2014) argued that mathematics was becoming irrelevant in the workplace because core mathematics skills, such as interpreting data and decoding graphs, were not applied in classroom situations that are connected to real-life experiences. Although problem-solving activities that are overly authentic can be very challenging for students to solve (Lowrie, 2011), exposure to activities that encourage students to apply mathematics knowledge to new situations (and related situations) both help to reinforce concepts and provide opportunities to scaffold understandings. As Boaler (1998) maintained, application-based projects support students in ways that are beneficial in and out-of-school.

The Nexus between Theoretical and Practical Aspects of the ELPSA Framework

The ELPSA framework views learning as an active process where students construct their own ways of knowing (developing understanding) through both individual thinking and social interactions with others. However, it is important not to view ELPSA as a linear process. Learning is complex and unpredictable and does not occur in a linear sequence, and thus the elements of the model should be thought of as interrelated and overlapping. It is also not to be restricted as a mathematics model.

The following analogy describes the way in which an individual could acquire an understanding of the concept "cat". An infant may hear the word "cat" whenever a small furry 'thing' is given a green bowl with something smelly in it. That process may happen every day for months (this is called ***Experience***). The infant's mum may say, "Has anyone fed the cat yet?" One extraordinary day the infant may say "cat" as the furry animal walks past (***Language*** development). A parent gives the child a hug and says, "What a clever little child. Yes, that is a cat." On a walk one day the infant says "cat" as a brown furry 'thing' walks past. "No, that is not a cat, it is a dog. Can you say DOG for me?" Twelve months later our toddler is able to point to a picture in a book and say "cat", and also point to a picture on the next page and say "dog" (***Pictorial*** representation). In Year 1, the child can write the word CAT and appreciate that cat is a pet that may come in a variety of colours and breeds (***Symbolic*** representation). In Year 3, the child understands that lions and tigers are cats, that there are domestic and feral cats, and their cats at home is called a Persian cat. By high school, the child might know the difference between a leopard and a jaguar (***Application*** of knowledge). This process of understanding the concept of "cat" could take many years to lead to a sophisticated understanding of the word. And, in fact, the application component may not be attainable during one person's lifetime.

It is important to remember that although the components of ELPSA will be discussed and presented individually, they should not and cannot be implemented in isolation but incorporated and intertwined throughout the lesson. Although we use our framework to design individual mathematics lessons, units of work and learning designs within the mathematics curriculum. It is also the case that this framework can describe concept acquisition not pertaining to the mathematics discipline.

What Is ELPSA?

The ELPSA framework follows a learning design approach, which is cyclic in nature. This design presents mathematical ideas through lived experiences, mathematical conversations, visual stimuli, symbolic notations, and the application of the applied knowledge. In this learning design, teachers are encouraged to introduce concepts from what the students know. This first component of the design process is **Experience**. Experience considers how students have used mathematics, what particular concepts they know, how they can acquire that information, and how mathematics has been experienced by individuals, both in and outside of classrooms. The experience component of the design also includes assessments, since the teacher must determine what the students know and what new information needs to be introduced to scaffold their understanding. The first component of this design process can be introduced through brainstorming, general discussions, the use of visual stimulus, and rich stories from the teacher or students. As a consequence, **Experience** is also associated with feedback and revision.

The second component of the design is associated with how **Languages** are used to promote understanding. In mathematics, mathematical language is both generic and specific. Some languages are associated with literacy, while other languages are particular to mathematics concepts (e.g., corners and angles). This component of design commonly follows Experience and focuses on both the generic and specific language required to represent mathematical ideas. This component of the design is also associated with particular pedagogy practises, since it is important for teachers to model appropriate language and for students to use this language to describe their understanding and converse with peers and teachers, to both explain and reinforce understanding.

The third component of the learning design is associated with the use of visual representations to represent mathematical ideas. **Pictures** are critical aspects of mathematics. Commonly, there are two types of pictures used in the classroom: (1) those constructed by the teachers or from learning resources; and (2) those constructed by the students. An example of the first type of picture would be the representation of different parallelograms, including rectangles, squares, and parallelograms from the textbook. These pictorial representations are used to describe two-dimensional shapes in a quadrilateral family. The second type of picture are those the students construct on paper, computer, or in their "mind's eye". Students might imagine transforming a square into a rectangle in their "mind's eye", or they might draw a diagram to solve a geometric problem. Pictures are often used to help to scaffold their understanding and to provide stimulus to complete mathematical tasks before the introduction of the symbolic notation. For example, students might cover the area of a rectangle with a small centimetre square to calculate the area of the shape. This process is the building block to understanding the concept of area, and it is necessary to introducing the formula of the area of a triangle ($A = 1/2 \times base \times height$). In most instances, students are required to decode information represented by others and encode information they represent themselves.

The next component of the learning design is the most common and frequently used in the aspect of teaching. That is, the use of *symbols* to represent mathematical ideas. This component sometimes makes mathematics different from other discipline areas, and sometimes refers to a universal language. Nevertheless, it is often the most poorly taught discipline. For example, most students are taught that $6 \times 4 = 4 \times 6$. Although the product is 24, one aspect of the symbols refers to six lots of four and the other refers to four lots of six. If students are encouraged to learn their multiplication tables by route learning, they may not always understand what the symbols actually represent. This is why it is important to progress the students through the learning cycle. Therefore, before knowing that $6 \times 4 = 24$, students should be able to draw a matrix that pictorially represents six groups of four.

The *application* component of the learning design highlights how symbolic understanding can be applied to new situations. Students who understand that area equals base times height, can apply this knowledge to new understanding associated with volume, that is volume can be represented as area times depth. The application component also provides opportunities for students to see how mathematics can be used in and out of school contexts. With respect to area concepts, there are many applications in the building industry, for example, 3-4-5 triangles associate with the Pythagoras theorem. Builders designing trusses to build the roof of the house use this knowledge.

Applications of the ELPSA Framework in the Classroom

In the following section, we outline two sequences of lessons that utilise the ELPSA learning framework. The first lesson describes the learning sequence within the geometry strand of the curriculum, specifically, developing learning opportunities around the concepts of Pythagoras Theorem. The second lesson considers learning experiences within the probability strand of the curriculum. Each lesson identifies learning activities within the five components of the ELPSA framework.

Lesson 1: Pythagoras Theorem (Grade VIII)

Experience. We need to find out what the students know about triangles and angles. For concepts of triangle, it is important to establish whether the children can identify right-angle triangles from other triangle classifications. For example, it might be helpful for students to classify isosceles triangle, from equilateral triangles and other form of triangles. By doing this the teacher is establishing children's experiences and assessing the extent to which the students can recognise triangles that have right angles (which conceptually would be necessary in order to understand the concept of hypotenuse). With respect to angle, it is necessary for the students to be able to classify right angles from angles that are obtuse and acute. These two concepts are critical to building an experience-based understanding that will eventually lead to students' acquisition of the Pythagorean formula.

Language. There are a number of critical mathematical terms that need to be developed in order to establish understanding. Terms such as *right angle* and *hypotenuse*, and a general understanding of perimeter need to be addressed explicitly. Some development will occur when establishing the student's *Experience*. Other activities should involve the teacher encouraging students to verbalise their thinking (including responses to the teacher's explicit questionings) in order to discriminate between similar sub-concepts. For example, a teacher can post questions such as: (1) What is the sum of the three internal angles in a triangle?; (2) If one angle in a triangle is 90 degrees, what could the other two angles be?; and (3) What is the relationship between internal angles of a triangle and the length of the opposite side?

As part of the questioning process, it would also be useful for small groups of students (pairs of students) to explain to one another their understanding of the challenges of calculating the perimeter of triangles. Questioning and discussion activities should provide opportunities for students to consider ways of generating solutions to problems, such as calculating the perimeter of a triangle, calculating an unknown third side or an unknown angle of a triangle, and recognising where the hypotenuse would be on a right-angle triangle. For each of these questions, students begin to realise they do not have the knowledge to complete the tasks (at that point in time).

Pictorial. Provide students with grid paper so that they can draw a number of right-angled triangles. The teachers should begin the demonstration by drawing the following picture on the board:

For the first two pictures, use the following dimensions: (1) 4cm and 3cm; and (2) 12cm and 5cm.

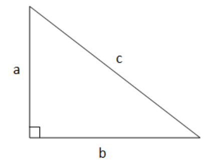
Ask the students to draw these pictures on grid paper with the appropriate scale. Then ask the students to join the pictures to make triangles and to measure the third side.

The teacher then could produce a table, which represents the sides of these shapes. The students can then draw their own right-angled triangles and record the sides of these shapes (e.g., the student might draw sides 5cm and 8cm and then have difficulty calculating the exact measurement of the third side). It is unlikely that the student will be able to produce a precise measurement and it may be the case that two students will give different answers for the third side of this triangle.

This type of disruption in thinking helps the teacher to introduce the need for a formula to ensure that accurate measurement can take place. It is often the case that the *pictorial* element of the design leads to the production of a *symbolic* formula.

Pictorial and Symbols Representations. The following symbolic formula should be introduced:

$$c^2 = a^2 + b^2$$



The teacher should draw a right-angled triangle on the board, and label it appropriately (with the algebraic notation as in the picture above). These representations are a bridge between *pictorial* and *symbolic* of the learning design. At this point in time it would also be worthwhile to generate a table to record results.

Table 1. A form for students' data record

A	***b***	***c***	a^2	b^2	$a^2 + b^2$	c^2	$\sqrt{c}$
3cm	4cm	…	…	…	…	…	…
5cm	12cm	…	…	…	…	…	…
2cm	4cm	…	…	…	…	…	…
1cm	5cm	…	…	…	…	…	…
8cm	15cm	…	…	…	…	…	…
7cm	24cm	…	…	…	…	…	…
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…

The next construct is to introduce the actual formula to the students. Formulas should be used in conjunction with pictorial representations that would allow students to easily calculate the area of particular shapes. For example, 3-4-5, 6-8-10, and 5-12-13 triangles are good examples to use.

Explicit modelling of the formula can be undertaken within the table that has been generated. The students should then draw their own triangles and complete the table using their own data. Since the students are encouraged to calculate the square root of *c* they can measure the third side of the triangle as a way of checking their answer (it is important for students to realise that this measurement must be accurate but rather an close approximation of the generated solution).

The students should then complete worked examples that require the completion of tasks that use the formula. These examples should include an unknown side (*a* or *b*), as well as examples that require a solution for side *c* (as above). Such examples are often found in textbooks or worksheets and are helpful for reinforcing and practising the newly used concept.

Application. For Batik makers and tailors, being able to ‘square material’ is important. One way of ensuring that material is ‘square’ is applying Pythagorean principles. For example, to demonstrate how carpenters and builders utilise a 3-4-5 triangle to ensure two dimensional objects or squares.

The development of a new mathematics lesson could contain all five of the ELPSA constructs. In such situations, some of the constructs would only be considered for a short period of time. It may well be the case that the constructs would be considered simultaneously. For example, in ascertaining what experiences students bring to a lesson, a teacher may design activities that involve both *Experience* and *Language* and present them together. It may also be the case that teachers focus on *Experience* only. Such situations could include a discovery or problem-based activity.

The introduction of the concept developed above (with the introduction of a new theorem) could reasonably be developed in three or four consecutive 40-minute lessons. The number of lessons would depend on the student’s foundational understandings (which would be ascertained via the *Experience* and *Language* components of the design).

More commonly, a teacher would design a lesson that focuses on two or three constructs only. For example, the teacher may introduce the lesson via pictorial representations and encourage the children to use symbolic notation throughout the lesson.

Lesson 2: Probability (Grade XI)

This sequence of lessons is based on an open-ended investigation of a well-known probability paradox. Ideally, the introductory phase of the lesson sequence should be undertaken with two mathematics classes of students (approximately 60-70 students). Thus, the *Experience* and *Language* aspects of the lesson would be introduced (potentially outside the classroom) with two groups of children being introduced to the following scenario:

“What is the likelihood of at least two children in this group sharing the same birthday?”

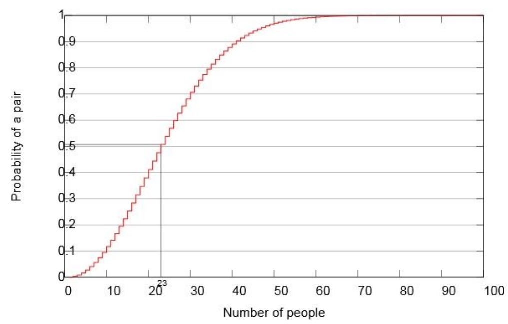
Experience. Once this scenario has been proposed to the students, an experiment should be conducted. The classroom teachers ask several students to formulate a statement based on the likelihood of the event. From our observations, we have found that the students proposed the following kinds of statements: *“There would be 1/365 chance of two children having the same birthday”; “There would be 70/365 chance of two people sharing the same birthday”; or “It would be unlikely that two children would have the same birthday”*.

The teachers should not react to these statements, however, it would be necessary to come back to the students’ statements in the *Language* component.

The teachers ask all of the children who were born in January to stand up and name the day they were born. If a common birthday is found, those students are asked to sit together and the other students are asked to sit down. This process is repeated for students born in February, followed by a series of observations until all students have being able to determine any matching of birthdays, within the 12 months. The names of the pairs of students of common birthdays are recorded and the students are asked to go back to their respective classrooms.

Language. The experimental data are recorded on the board. It is highly likely that at least one pair of students shares the same birthday; however there may be several combinations within the cohort. It is important for the classroom teacher to revisit the statements made by the students at the beginning of the experiment. Students should be encouraged to predict why their friends may have presented such statements. It is critical that the teacher models appropriate language and encourages the students to do the same. It is also necessary to allow students to provide updated ideas based on the experiment that took place in the *Experience* component of the lesson sequence.

Pictorial representation. The teacher should now present to students the following *pictorial* representation that describes the computing probability of at least two children sharing the same birthday amongst a certain number of people.



Students should be encouraged to decode the graphic and talk to their friends about what the graphic might mean. The classroom teacher should scaffold the students to interpret the information on both the *x*- and the *y*-axis and to talk about the specific information associated with the numeral "23" on the *x*-axis. The following statement should be proposed by one of the students "*If there were 23 children in a room, the probability of two people sharing the same birthday would be 0.5*". Students should be then encouraged to draw lines from the *x*- and *y*-axis that represent the experiment that took place at the beginning of the lesson sequence (e.g., drawing a line from 70 on the *x*-axis if there were 70 students in the cohort). Scaffold the students to read the graph such as "*If there were 70 children in a room, the probability of two children sharing the same birthday would be more than 0.9*".

Symbolic. The students should be encouraged to understand *symbolic* representation of the *pictorial* representation presented above. In order to simplify the task, ask the children to assume there are 365 possible birthdays that are equally likely to happen (therefore disregarding such variation as leap years).

To solve this problem, the teacher can begin from discussing a common formula for probability, that is, the sum of the probability that an event will happen and the probability that the event will not happen is always 1. For example, the chance that today might rain or might not rain is always 1 or 100%. Similarly, the probability that no two people will have the same birthday added by the probability that two people will share a birthday is equal to one.

P(at least two people share birthday) + P(no two people share birthday) = 1

P(at least two people share birthday) = 1 - P(no two people share birthday)

Finding the probability of at least two people sharing the same birthday is more complicated because this requires finding the probability that the first person will have the same birthday of the second person, or the third person, and so on. Therefore, the teacher can direct students to answer the question: *What is the probability that no two people will share a birthday*? with the following explanations.

Lets start with 2 people (N=2).

- The first person can have any birthday, leading to 365/365 or 100% chance.
- The second person's birthday has to be different because no two people will share a birthday. As a result, there only (365-1) = 364 days to choose from. So the chance that the second person has a **different** birthday is 364/365.
- The probability that both people have different birthdays is: $\frac{365}{365} \times \frac{364}{365} = 0.997 = 99.7\%$. In other words, there is a 0.3% chance that both people will share the same birthday.

If there are three people (N=3).

- The first person can have any birthday, leading to 365/365 or 100% chance.
- The second person's birthday has to be different because no two people will share a birthday. As a result, there only (365-1) = 364 days to choose from. So the chance that the second person has a **different** birthday is 364/365. Consequently, the chance that the third person has a different birthday is 363/365.
- The probability that the three people have different birthdays is: $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} = 0.992 = 99.2\%$. In other words, there is a 0.8% chance that the three people will share the same birthday.

If N = 23.

- The probability that the 23 people have different birthdays is: $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{(365-(23-1))}{365} = 0.493 = 49.3\%$. In other words, there is a 50.7% chance that the 23 people will share the same birthday.

A formula for the probability that N people have different birthdays is:

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{(365-(N-1))}{365}.$$

Therefore, the formula for the probability that at least two of N people sharing birthdays is:

$$1 - \left(\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{(365-(N-1))}{365}\right).$$

Application. An *application* of this lesson sequence could align to either: (1) further development of the *symbolic* notation; and/or (2) real-life application of probability. With respect to further development of the symbolic notation, students could be encouraged to propose the number of people within the cohort and determine each probability with the support of the graph and understanding developed in the *symbolic* component of the lesson sequences. As part of the *application* process students could construct a table similar to Table 2 presented below.

Table 2. Probability of at least two children in a group of *N* children sharing the same birthday

Number of children (*N*)	Probability *P(N)*
1	0.0
5	0.027
10	0.117
20	0.411
23	0.507
30	0.706
40	0.891
50	0.970
60	0.994
70	0.999
100	0.9999997

In this application, there is no relationship between different mathematical representations (that is, a line graph, a table, and a *symbolic* representation).

With respect to real-life application, students could be introduced to information about the weather that predicts the likelihood of rain in a given day. Data could be gathered from the Internet as illustrated in Figure 1.

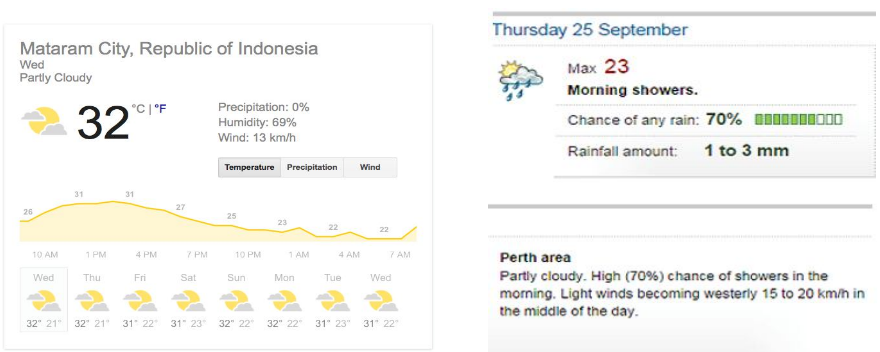


Figure 1. An example of the use of probability ideas in real-life.

CONCLUSION AND RECOMMENDATION

The ELPSA framework is presently being introduced to classroom practitioners, policy makers and university lecturers throughout one province of Indonesia. The initiative forms part of an Australian Department of Foreign Affairs and Trade project titled, "*Promoting mathematics engagement and learning opportunities for disadvantaged communities in West Nusa Tenggara, Indonesia*". Initial evidence indicates that the framework has great promise in enhancing the quality of teaching and learning in mathematics classrooms. We maintain that each component of the framework is critical for establishing sense making and mathematics understanding in classrooms, and that the component sequence provides a logical sequence to scaffold, reinforce and apply mathematics knowledge and concept development. It is important to note that the learning design will work most effectively when classroom teachers (and curriculum specialists) embrace the logic of the design—that is, present learning opportunities and activities in ways that emphasis each component of the framework. To this point, the presentation of learning activities should be seen as value adding to the design through the development of mathematically explicit ideas and practices.

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