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How Visual Imagery Contributed to College: A Case of How Visual Imagery Contributes to a College Algebra Student's Understanding of the Concept of Function in the United States

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Abstract

This investigation utilized the qualitative case study method. Seventy-one College Algebra students were given a mathematical processing instrument. This testing device measured a student's preference for visual thinking. Two students were purposefully selected using the instrument. The visual mathematical learner (VL) was discussed in this article. VL was presented with mathematical tasks to complete over the course of a semester. Each task was given to the student individually. In order to thoroughly understand VL's responses, task-based interviews were conducted and videotaped. In addition, the student was interviewed based on her response to the mathematical tasks. The tasks captured different types of mathematical functions. These included linear, quadratic, absolute value, and exponential functions.

As patterns emerged from the data, the researcher called them categories. Several emerging categories were examined in the article. Also, O'Callaghan's (1998) translating component was present during the completion of linear, quadratic, absolute value, and exponential functions.

Introduction

"Algebra, whether at middle school level, high school level, or college level often strikes fear in the hearts of students. Generation after generation have passed down the opinion that algebra is not only difficult, but perhaps also boring" (Stephens & Konvalina, 1999, p. 483). In fact after teaching mathematics courses on the university level, the researcher has found the previous quote to be true for many students. How can anyone learn College Algebra or any other type of mathematics when one starts out afraid of it? This is one of the many obstacles that the researcher attempts to overcome each semester with students. One way to try to overcome the fear of algebra is by establishing a learning environment in one's classroom where the students feel free to ask questions and share their opinions about mathematics, i.e., an algebra friendly learning community. This type of community of mathematical learners develops and grows throughout the semester.

In general, students are introduced to algebraic concepts in middle school and high school. "Some of the mathematical objects that are met for the first time in algebra are expressions, equations with unknowns, functions and variables, and monomials and polynomials" (Kieran, 1990, p. 99). Subsequently, students are expected to master the techniques of the course in college at a faster rate. Typically, Algebra I and Algebra II that are taken for two years in high school become College Algebra at a higher education institution. (Some students enroll in Algebra I in middle school.) College Algebra is offered for one semester at a community college or university.

Algebra is seen as an abstract subject to most of the students that I have taught. According to Fey (1992), "many students do not become proficient in the skills of algebra…[and] very few students acquire the understanding of algebraic ideas and methods that is required to reason effectively with symbolic expressions" (p. 1). Furthermore, students learn at various rates and in different ways. For instance, some people are visual learners (Gardner, 1983). Others possess a kinesthetic learning style. Some students are tactile learners. Many students are auditory learners.

Presmeg (1986a) defines visualizers as being "…individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual [*sic*] methods" (p. 298). According to Price (1996), visual preferences include support using "… pictures, filmstrips, computers, films, graphs, books, and magazines" (p. 10). Each of these methods may aid the understanding of algebraic concepts for the visual learner.

Presmeg (1986a) also defines non-visualizers as being "…individuals who prefer not to use visual methods when attempting...[mathematical problems which may be solved by both visual and non-visual methods]" (p. 298). Price (1996) explains that auditory preferences involve the inclusion of "…tapes, videotapes, records, radio, television, and precise oral directions when giving assignments, setting tasks, reviewing progress, using resources or for any aspect of the task requiring understanding, performance, progress, or evaluation" (p. 10). Tactile preferences include favoring the "…use [of] manipulative and three-dimensional materials; [in addition] resources should be touchable and movable as well as readable" (p. 10). Furthermore, Price (1996) states that kinesthetic learners prefer "…opportunities for real and active experiences for planning and carrying out objectives; site visits, seeing projects in action and becoming physically involved" (p. 10).

The research question for the study was developed from the following efforts. First of all, the researcher wanted to determine what understanding the algebraic concept of function meant to the visual College Algebra student. Did the students rely on mental images in order to gain understanding? If so, how did the students connect these mental images with understanding the concept of function? Could the students draw these mental images on paper?

Secondly, O'Callaghan (1998) developed a cognitive model for understanding functions. This framework included four components: "…modeling, interpreting, translating, and reifying" (p. 24). Translating was defined as "the ability to move from one representation of a function to another…" (p. 25). In addition, he explained that "the three most frequently used representations for functions are equations, tables, and graphs" (p. 25). In Lane (2006), the presence or absence of the translating component (O'Callaghan, 1998) was used to measure the mathematical learners' understanding of functions. Therefore, the following research question was created.

Research Question

- How does visual imagery contribute to a visual College Algebra student's understanding of functions?

Literature Review

This section examined research articles and studies that pertained to the concepts of functions and visualization. The section begins with a summary of literature on the concept of function, then, provides a summary of visualization studies relevant to this research.

Functions

Many of the studies regarding the concept of function examined the role of different cognitive models in the understanding of functions (Vinner, 1983; Vinner & Dreyfus, 1989; Thompson, 1994; Sfard, 1991; O'Callaghan, 1998; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). For example, the concept image and definition of a function was presented as a cognitive model to help explain how students learn (Vinner, 1983; Vinner & Dreyfus, 1989; Thompson, 1994). In fact, Thompson (1994) believed that when the concept image and concept definition were balanced, then understanding was achieved. In addition, Thompson examined the understanding of functions in terms of developing an action conception, a process conception, and an object conception. Thompson also explored the definition of function in terms of the correspondence of variables and the co-variation of quantities.

Sfard (1991) presented a different conceptual framework pertaining to functions. Instead of a student's understanding of functions being defined in terms of the concept image and concept definition (Vinner, 1983; Vinner & Dreyfus, 1989; Thompson, 1994), this model included an operational conception and a structural conception. The structural conception referred to seeing functions as "…abstract *objects*…" (p. 4). On the other hand, an operational conception included viewing functions as "*…processes, algorithms and actions…*" (p. 4).

Sfard (1991) described three ways to move from an operational conception of function to a structural conception. The first stage was interiorization. "At the stage of interiorization a learner gets acquainted with the processes which will eventually give rise to a new concept…" (p. 18). For example, "in the case of function, it is when the idea of variable is learned and the ability of using a formula to find values of the 'dependent' variable is acquired" (p. 19). The second phase was condensation. "The phase of condensation is a period of 'squeezing' lengthy sequences of operations into more manageable units." Thus, the student would still use processes, however, the concept should become more concrete. For instance, "…the learner can investigate functions, draw their graphs, combine couples of functions (e.g. by composition), even to find the inverse of a given function" (p. 19). Thirdly, reification was the last stage. This third phase involved "…conceiving the notion as a fully-fledged object…" (p. 19). For example, "in the case of function, reification may be evidenced by proficiency in solving equations in which 'unknowns' are functions (differential and functional equations, equations with parameters)…" (p. 20).

O'Callaghan (1998) developed another type of cognitive model for understanding functions. This cognitive model did not measure understanding based on a concept image and concept definition (Vinner, 1983; Vinner & Dreyfus, 1989; Thompson, 1994). Instead, the framework included four components. The components were modeling, interpreting, translating, and reifying. First of all, modeling was referred to as "the ability to represent a problem situation using functions…" (O'Callaghan, 1998, p. 25). According to the author, interpreting was considered "the reverse procedure…" (p. 25) of the first component. "Problems could require students to make different types of interpretations or to focus on different aspects of a graph, for example, individual points versus more global features" (p. 25). Translating was defined as "the ability to move from one representation of a function to another…" (p. 25). Furthermore, he explained that "the three most frequently used representations for functions are equations, tables, and graphs" (p. 25). Thus, translating could refer to moving from graphs to equations, equations to graphs, numerical tables to equations, equations to numerical tables, numerical tables to graphs, or graphs to numerical tables. "The final component of the model for functions is *reification*, defined as the creation of a mental object from what was initially perceived as a process or procedure" (p. 25). The fourth component was similar to Sfard's (1991) third stage of moving from the operational conception to a structural conception of function.

Breidenbach, Dubinsky, Hawks, and Nichols (1992) investigated how 59 math majors developed a process conception of functions that differed from Thompson (1994) and the previous authors. According to Thompson (1994), "from the perspective of students with a process conception of function [the second part of the cognitive model], an expression stands for what you would get by evaluating it" (p. 26). On the other hand, Breidenbach, Dubinsky, Hawks, and Nichols (1992) explained, "a process conception of function [the third phase of the cognitive model] involves a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object" (p. 251). Breidenbach, Dubinsky, Hawks, and Nichols presented the following three phases: pre-function conception, action conception, and process conception. According to the authors, attaining the third phase represented understanding functions.

Some of the function studies investigated college students (Vinner & Dreyfus, 1989; Thompson, 1994; O'Callaghan, 1998; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dreyfus & Eisenberg, 1983). For instance, Dreyfus and Eisenberg (1983) analyzed 84 college students' understanding of the concept of function. Dreyfus and Eisenberg (1983) specifically examined three functional characteristics: linearity, smoothness (differentiability), and periodicity. Dreyfus and Eisenberg (1983) investigated how the college students' understanding of these characteristics affected their construction of graphs of functions. In fact, "…28 % of the responses were (piecewise) linear continuations while 62 % were smooth (differentiable) continuations. The remaining 10% were neither piecewise linear nor smooth" (p. 129).

Other studies examined how high school students understood functions (Moschkovich, 1999; Schwarz & Dreyfus, 1995; Sajka, 2003; Monk & Nemirovsky, 1994). Moschkovich (1999) presented two case studies depicting how ninth and tenth grade algebra students understood the concept of function as they worked in pairs. This study focused on high school "…students' use of the x-intercept in equations of the form $y = mx + b$" (Moschkovich, 1999, p. 169). Schwarz and Dreyfus (1995) examined how ninth grade students understood functions after receiving instruction in a computer software environment. Sajka (2003) reported on how an average high school mathematics student understood functions using case study research. Sajka referred to a students' understanding in terms of developing a procept [*sic*] of function. Monk and Nemirovsky (1994) presented a case study of a twelfth grader who used visual characteristics of graphs and a technology component, called the Air Flow Device, to understand functions. Monk and Nemirovsky found the student to be concerned with the steepness of a straight line.

Some additional function studies included Eisenberg and Dreyfus (1994), Yerushalmy (2000), Slavit (1997), and Karsenty (2002). The purpose of the first study "…was to help students think of functions in a visual way, and to help us understand the obstacles they must overcome in doing so" (Eisenberg & Dreyfus, 1994, p. 47). Yerushalmy (2000) investigated the problem solving strategies of two middle school students in a function approach algebra course. Slavit (1997) analyzed how understanding was attained through the property-oriented view of function. In the final function-related study, Karsenty (2002) examined 24 adult's long term cognitive abilities on linear functions.

The gap in the literature appeared to pertain to College Algebra visual and non-visual learners. What does understanding the concept of function mean to these visual and non-visual mathematical students? Do visual and non-visual algebra learners translate from one representation of a function to another (O'Callaghan, 1998)?

Visualization. Most of the studies on visualization that were included encompassed Abstract Algebra, beginning Calculus, Calculus III, Engineering, middle school, and high school students (Zazkis, Dubinsky, & Dautermann, 1996; Aspinwall & Shaw, 2002; Aspinwall, Shaw, & Presmeg, 1997; Lean & Clements, 1981; Kirshner & Awtry, 2004; Krutetskii, 1976; Presmeg, 1986a; Presmeg, 1986b; Presmeg, 1989; Presmeg, 1992; Vinner, 1983).

For instance, Krutetskii (1976) examined research concerning 34 visual learners. Krutetskii (1976) reported the following two perspectives based on the results of the study. First of all, the absence or presence of showing a preference for the visualization of abstract mathematical concepts and having a strong development of spatial abilities "…does not determine its type" (p. 314). The second perspective was that the presence of the previously mentioned components "…showed a very high intercorrelation in our experiments" (p. 314).

Based on these perspectives, Krutetskii (1976) constructed a framework of three types of mathematical learners. They were analytic, geometric, and harmonic. The analytic mathematical learner was "…characterized by an obvious predominance of a very well developed verbal-logical component over a weak visual-pictorial one" (p. 317). For example, when given a choice between using equations and graphs, this kind of learner generally would solve mathematical problems using equations. The geometric mathematical learner was "…characterized by a very well developed visual-pictorial component, and we can tentatively speak of its predominance over a well developed verbal-logical component" (p. 321). For instance, when given a choice between expressing a mathematical relationship using equations or graphs, this student would have chosen graphs or diagrams. The harmonic mathematical learner feel confident with both approaches.

In fact, Presmeg (1989) states, "visual imagery which is meaningful in the pupil's frame of reference may lead to enhanced understanding of mathematical concepts at primary and secondary levels" (p. 21). What happens on the collegiate level? How does visual imagery impact College Algebra learners? Apparently, the gap in the research was how visualization affects College Algebra visual and non-visual students.

Methodology

This study uses a qualitative research methodology (including case study) to study the College Algebra participant's use of visual imagery in understanding functions. Two case studies of College Algebra students were investigated (Lane, 2006), however, one of the students will be discussed in this paper, the visual College Algebra student.

Participants

The population in the study came from two College Algebra courses in the fall of 2005. Together, the two sections were comprised of 71 students. The researcher taught both sections. The students attended a large[1] four-year historically black university in the South-Eastern portion of the United States.

The research involved the purposeful sample[2] (Patton, 1990) of two students using Presmeg's (1985) Mathematical Processing Instrument. This testing device measured a student's preference for visual thinking in mathematics. Therefore, one non-visual mathematical learner and one visual mathematical learner were chosen; findings from the latter are the focus of this paper.

Instruments

This investigation used the Mathematical Processing Instrument and the Mathematical Processing Questionnaire by Presmeg (1985). These tools were chosen because they measure how a student prefers to process mathematical information, i.e., visually or non-visually.

Presmeg's Mathematical Processing Instrument included three sections (A-C) of mathematics problems for students to solve. The author recommended section B only or sections B and C for college-level students. All 71 students were provided with section B of the instrument. Section B had 12 mathematical word problems to solve. Each question could be solved numerically, algebraically, and graphically. Graphical solutions or drawing diagrams were considered as visual solutions. Numerical and algebraic solutions were considered as non-visual solutions. The test was scored by adding the total of two for every visual solution, one if the problem is not attempted, and zero for every non-visual solution. The highest score possible was 24/24 (24 out of 24). The lowest score possible was 0/24 (0 out of 24). If the student's visualization score was 12/24 or higher, then he or she was considered as having a preference for visual thinking in mathematics. On the other hand, if the participant's visualization score was 10/24 or lower, then he or she was considered as having a preference for non-visual thinking in mathematics. The students were required to show their work for the solution(s), however, they were not required to use a specific method of solution over another. The participants were also asked to choose their own method of solution and turn in their papers. (See Appendix A for a copy of this instrument.)

In addition, each student was supplied with a Mathematical Processing Questionnaire (Presmeg). The questionnaire was a follow-up to the participants' responses to the Mathematical Processing Instrument. This questionnaire provided three or more solutions for the students to choose the one that was most similar to their response. After the participants completed the questionnaire, they were asked to turn in their responses. (See Appendix B for a copy of this questionnaire.)

After completing Presmeg's Mathematical Processing Instrument, 52 College Algebra students scored from 0/24 to 10/24. As a result, these 52 students were considered to have a preference for non-visual thinking in mathematics. In addition, after completing the instrument, 19 College Algebra students scored from 12/24 to 20/24. As a result, these 19 students were considered to have a preference for visual thinking in mathematics.

The Visualizer (VL) (Presmeg, 1986a) for the present study was purposefully selected from the 19 students. The participant scored 16/24 on Presmeg's (1985) Mathematical Processing Instrument. In addition, VL was extremely detailed regarding the student's answers on the instrument.

Data collection

"Qualitative data consist of quotations, observations, and excerpts from documents" (Patton, 2002, p. 47). Therefore, the data sets for the present study were interviews and document reviews. The interviews were based on the participant's responses to mathematical tasks.

The functions examined in this study included first degree, second degree, and higher order functions. For example, first degree or linear functions can be written in symbolic form as $f(x) = ax + b$ where a and b are real numbers. The graph or pictorial form of a linear function is a straight line. Second degree functions can be expressed in symbolic form as $f(x) = ax^2 + b$ where a and b are real numbers. The pictorial form of second-degree functions is called a parabola. A basic parabola can look like a bowl that is shaped like the letter "u". In addition, higher order functions can be written in symbolic form as $f(x) = ax^3 + b$ where a and b are real numbers. Higher order functions have an exponent of three or higher in their equations. Odd numbered higher order functions can be expressed in pictorial form in three parts that are connected. One portion of the graph is curved downward or decreasing. The second part of the graph remains constant. The last portion of the graph is curved upward or increasing (Lial, Hornsby, & Schneider, 2001).

The documents in this investigation were College Algebra Writing Journals, tests, College Algebra Web Homework, and a researcher journal. Bogdan and Biklen (1998) posited that documents "…can be used as supplemental information as part of case study whose main data source is participant observation or interviewing" (p. 57). The College Algebra Writing Journals included the students' feelings, beliefs, and interpretations about mathematics in their own words. They also recorded their struggles and concerns regarding College Algebra over the semester. Specifically, the struggles and concerns pertained to class assignments and/or algebraic concepts that were introduced in class. Since the course was held three days a week, the students were expected to complete a minimum of three entries per week. Each entry was at least one-half of $8\frac{1}{2}$ inches by 11 inches of a page. In addition, the visual and non-visual learners wrote an additional entry after every interview session. The interview session entries included the participants' feelings, beliefs, and interpretations about the mathematical tasks. They also recorded any struggles encountered in completing the tasks. The entry was at least one-half of $8\frac{1}{2}$ inches by 11 inches of a page.

The participants' College Algebra tests were a second source of extant data. Four chapter tests and one final examination were given. The concept of function permeated the last three tests and the final examination. As part of the directions, the students were asked to represent functions numerically, algebraically, and/or graphically on these exams.

A researcher's journal was the third document data source[3]. Specifically, the journal entries began in the fall of 2005 after the students completed Presmeg's (1985) visualization instrument. It included the time, place, and length of each interview session. The entries also included the researcher's feelings, beliefs, and interpretations regarding the mathematical task interview sessions. The idea of a researcher's journal was supported by Lincoln and Guba (1985), who indicated that "each investigator should keep a personal journal in which his or her own methodological decisions are recorded and made available for public scrutiny" (p. 210).

Data Analysis

Data analysis began through an examination of the participant's initial interview[4] session. Each transcript was analyzed before the next interview session occurred in order to look for any possible emerging patterns or themes. As any patterns/themes were found, they were investigated in the next interview session.

Measures were taken in order to ensure trustworthiness in the data analysis and thus in the findings. "The four terms 'credibility', 'transferability', 'dependability', and 'confirmability' are, then the naturalist's equivalents for the conventional terms 'intended validity', 'external validity', 'reliability', and 'objectivity'" (Lincoln & Guba, 1985, p. 300).

In the study, credibility was established by using triangulation[5] and member checking[6] (Lincoln & Guba). No assertion was considered valid unless it could be supported by two or more pieces of data. In the study, triangulation of the interview sessions and documents (College Algebra Writing Journals, tests, College Algebra Web Homework, and a researcher journal) occurred. According to Lincoln and Guba, "the concept of triangulation by different methods thus can imply either different data collection modes (interview, questionnaire, observation, testing) or different designs" (p. 306).

"The member check, whereby data, analytic categories, interpretations, and conclusions are tested with members of those stakeholding groups from whom the data were originally collected, is the most crucial technique for establishing credibility" (Lincoln & Guba, 1985, p. 314). The member checking technique was applied to the study by allowing the participant to read the results section. The student was asked to pay special attention to the overall written interpretations of her responses to the various mathematical tasks, which helped to build the case study. After that, the student provided reasons for the response. In addition, the student was encouraged to indicate any information she felt was left out of the case study that might be pertinent. All of the member check responses were reported. The biggest difference between these two forms of establishing credibility was that "member checking is directed at a judgment of overall credibility, while triangulation is directed at a judgment of the accuracy of specific data items" (p. 316).

The researcher had the "…responsibility to provide the *data base* that makes transferability judgments possible on the part of potential appliers" (p. 316). In the study, the database included descriptions of the time and context of the case study. In fact, the case study was written with thick description in order to make transferability possible.

To promote the study's dependability criterion for ensuring its integrity, the overlap methods of triangulation were used (Lincoln & Guba, 1985). This technique was chosen based on the authors' following claim which emphasized, "since there can be no validity without reliability (and thus no credibility without dependability), a demonstration of the former is sufficient to establish the latter" (p. 316). Thus, triangulation, which was discussed earlier, was used to help establish credibility and in turn as an overlap method to establish dependability. As an overlap method, the focus was on triangulation of multiple and different methods.

In order to help establish confirmability, an audit trail was maintained throughout the study. The audit trail included five of the categories that have been suggested by Halpern, 1983 cited in Lincoln and Guba, 1985. They are raw data; data reduction and analysis products; data reconstruction and synthesis products; process notes; and materials relating to intensions and dispositions. First of all, the raw data in the present study was the results from Presmeg's (1985) visualization instruments, audio and videotaped interview sessions, and the College Algebra Writing Journals and tests. Secondly, the data reconstruction was utilized in order to condense information and identify any common patterns or relationships. Thirdly, data reconstruction and synthesizing products occurred in the study by identifying and organizing common categories and/or themes, examining the participants' interpretations and my interpretations, reporting findings, and identifying connections to existing literature and/or theory. Next, the process notes, which included methodological decisions, were recorded in the researcher's journal. The fifth category of materials relating to intentions and dispositions of the study were also recorded in the researcher's journal.

In addition, the data were analyzed by using O'Callaghan's (1998) translating component for understanding functions. O'Callaghan's (1998) model contributed to the theoretical framework of the study.

Findings

Visual Imagery

In the study, the Visualizer used visual imagery in five of the mathematical tasks. Specifically, VL relied on visual imagery during the completion of tasks one, four, eight, nine and ten. These tasks included linear, quadratic, absolute value, and exponential functions.

During the completion of mathematical task # 1 (Figure 1), visual imagery contributed to the VL's demonstration of understanding functions because she relied on visual imagery regarding the linear function y = x.

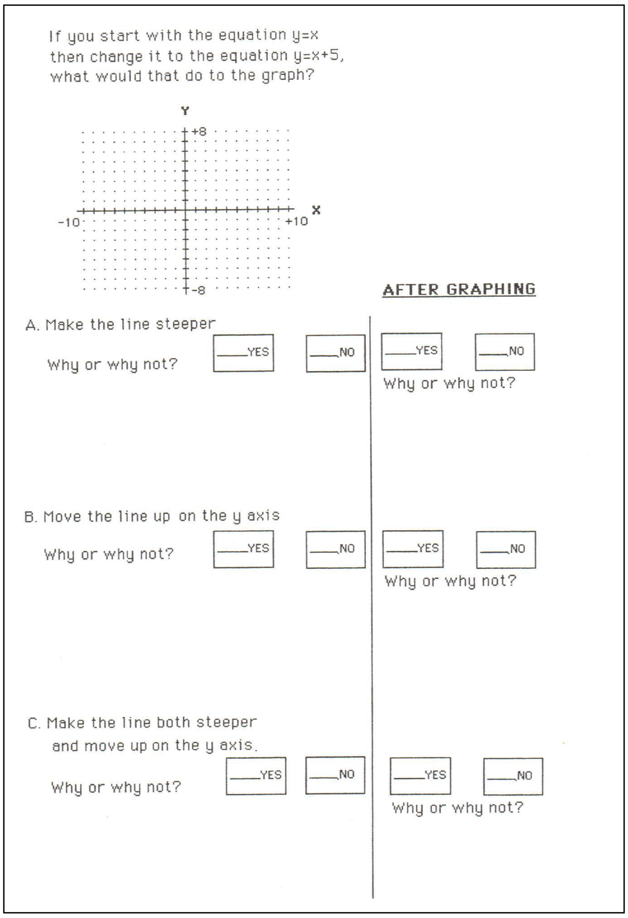

Figure1. Mathematical task # 1.

VL drew her mental image and used it to complete the task (Figure 2).

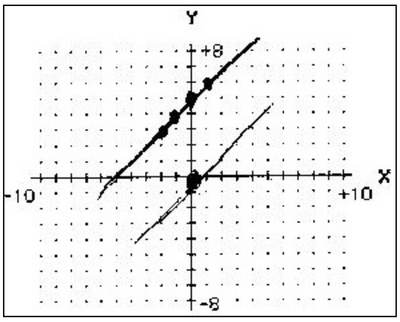

Figure 2. Visualizer's mathematical task # 1 graph.

During the completion of mathematical task # 4 (Figure 3), visual imagery contributed to the VL's demonstration of understanding functions because she relied on visual imagery regarding the quadratic function $y = x^2$.

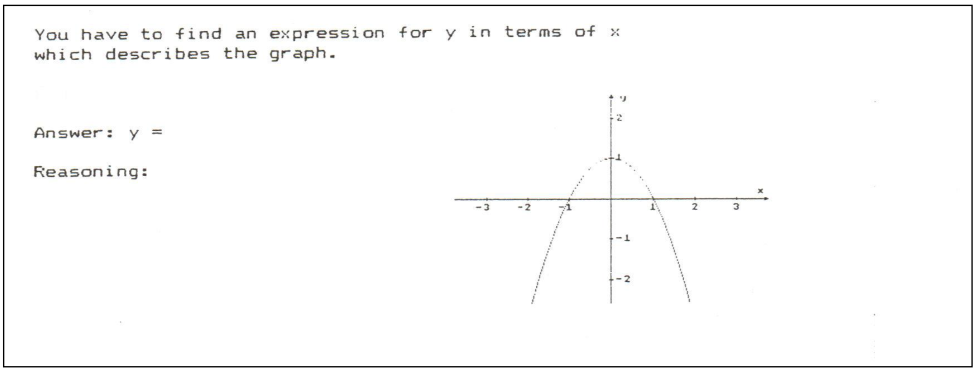

Figure 3. Mathematical task # 4.

VL drew her mental image and used it to complete the task (Figure 4).

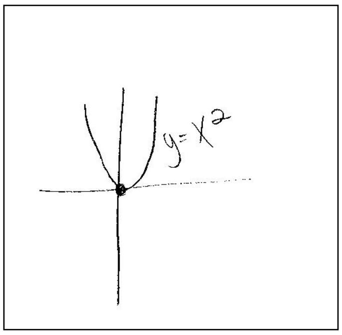

Figure 4. VL's image of $y = x^2$.

During the completion of mathematical task # 8 (Figure 5), visual imagery contributed to the VL's demonstration of understanding functions because she relied on visual imagery regarding the absolute value functions $y = |x|$, $y = |x + 1|$, and $y = |x + 1| - 2$.

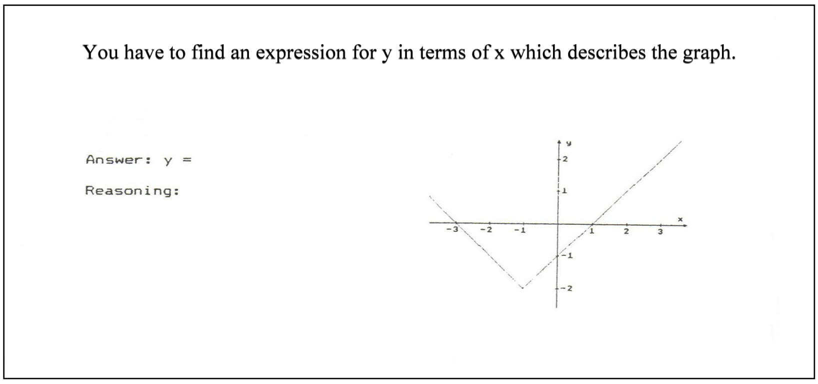

Figure 5. Mathematical task # 8.

The participant also shared her personal viewpoint of the task. "It was simple, not at all difficult. I observed the shifts in the original graph [y =$|x|$] to the present graph [provided in task # 8] and wrote the equation [y = $|x + 1|$ - 2] that I believe the graph illustrates". (Even though the graph of y = $|x|$ was not sketched on the task eight sheet, VL reported that y = $|x|$ as the original graph of an absolute value function.)

During the completion of mathematical task # 9 (Figure 6), visual imagery did not contribute to the VL's demonstration of understanding functions because she did not construct an accurate equation (symbolic form) to match the given graph (graphic form) of the function.

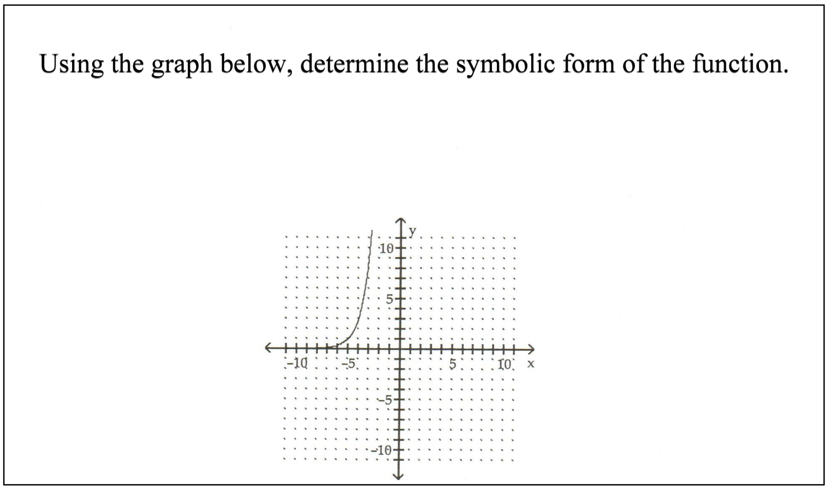

Figure 6. Mathematical task # 9.

However, VL used visual imagery to solve the problem because the participant relied on the visual imagery of the exponential function y = 2^x. In addition, VL shared her personal viewpoint about this task in the College Algebra Journal.

This task was a bit challenging to me. It reminded me of the previous task but I could not remember how I shift the graph on the x – axis instead of the y – axis.

During the completion of mathematical task # 10 (Figure 7), VL used visual imagery to solve the problem.

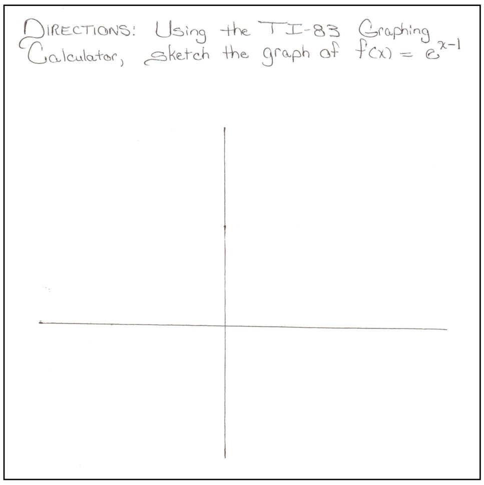

Figure 7. Mathematical task # 10.

Visual imagery contributed to VL's demonstration of understanding functions because the student relied on the image of the exponential function y = e^{x} during the completion of the tenth task.

Understanding of functions

In the study, the visual mathematical learner's understanding of functions was measured by the presence or absence of the translating component (O'Callaghan, 1998) for understanding functions. In addition, O'Callaghan's (1998) translating component was present during the completion of linear, quadratic, absolute value, and exponential functions.

Specifically, the VL translated the following linear functions: y = x and y = x + 5. First of all, VL translated the given (symbolic form) equation of y = x + 5 to its (numeric form) table of numerical values. Secondly, VL translated the numeric form of y = x + 5 and the given symbolic form of y = x to their (graphic forms) graphs.

During the beginning of the fourth mathematical task, by relying on visual imagery VL translated the symbolic form of y = x^{2} to its graphic form. VL also translated the following quadratic functions during the completion of the fourth mathematical task: y = x^{2} + 1, y = -x^{2} + 1, y = x^{2} - 1, y = -x^{2} - 1 and y = - x^{2} + 1. VL translated from equations (symbolic form) to tables of numerical values (numeric form). In addition, VL translated y = x^{2} - 1, y = - x^{2}, and y = - x^{2} + 1 from equations (symbolic form) to graphs (graphic form).

Explicitly, VL translated the following absolute value functions: y = $|x|$, y = $|x + 1|$, and y = $|x + 1|$ - 2 during the completion of the eighth mathematical task. By using visual imagery, VL translated y = $|x|$ and y = $|x + 1|$ from equations (symbolic form) to graphs (graphic form). In addition, by using visual imagery, VL translated y = $|x + 1|$ - 2 from a graph (graphic form) to an equation (symbolic form).

Specifically, VL translated the following exponential functions during the ninth mathematical task: y = 2^{x} - 5, y = 2^{x} + 5, y = (2^{x} + 1) + 5, y = -5^{x} + 1, and y = (2^{x} + 5) + 1. VL translated each function from an equation (symbolic form) to a graph (graphic form). Overall, VL did not construct an accurate equation to match the graph of the function that was provided in mathematical task # 9. This showed the absence of O'Callaghan's (1998) translating component in the final solution of task nine because the participant did not translate the given graphic form of the function in task nine to its symbolic form of y = 2^{x+5}.

During the tenth mathematical task, VL translated the given equation (symbolic form) of the exponential function f (x) = e^{x-1}. Explicitly, VL translated the symbolic form of the function to its graphic form.

Conclusions

Emerging Categories

In the report of the case study of the visual mathematical learner, as patterns emerged from the data the researcher called them categories (Lane, 2006). In this article, the categories discussed will include the patterns that emerged from the data collected pertaining to mathematical tasks one, four, eight, nine, and ten.

First of all, VL used the following three categories during the completion of mathematical task # 1. Category A was substituting specific values for the variables x and y into the equations. Category B was plotting specific points of a function on a graph. Category C was detecting a relationship between the concepts of slope and steepness.

Secondly, VL used the following four categories during the completion of mathematical task # 4. Category A was substituting specific values for the variables x and y into equations. Category E was looking for a relationship between the symbolic form and graphic form of a function. Category F was using the graphing calculator for arithmetical operations. Category G was using the graphing calculator to construct a relationship between the symbolic form and graphic form of a function.

Next, VL used two categories during the completion of mathematical task # 8. Category E was looking for a relationship between the symbolic form of a function and the graphic form. Category I was focusing on specific visual features of the graph of a function.

In addition, VL used the following three categories during the completion of mathematical task # 9. Category E was looking for a relationship between the symbolic form of a function and the graphic form. Category I was focusing on specific visual features of the graph of a function. Category G was using the graphing calculator to construct a relationship between the symbolic form of a function and the graphic form.

Finally, VL used the following four categories during the completion of mathematical task # 10. Category E was looking for a relationship between the symbolic form of a function and the graphic form. Category G was using the graphing calculator to construct a relationship between the symbolic form of a function and the graphic form. Category B was plotting specific points of a function on a graph. Category H was using various features of the graphing calculator.

To summarize, all of the Visualizer's (VL) emerging categories were listed using alphabetical letters with the corresponding mathematical task or tasks in Table 1.

- Category A: substituting specific values for the variables x and y into equations
- Category B: plotting specific points of a function on a graph
- Category C: detecting a relationship between the concepts slope and steepness
- Category D: misinterpreting the graphical representation of a function after multiplying and adding specific values to the symbolic form of a function
- Category E: looking for a relationship between the symbolic form of a function and the graphic form
- Category F: using the graphing calculator for arithmetical operations
- Category G: using the graphing calculator to construct a relationship between the symbolic form of a function and the graphic form
- Category H: using various features of the graphing calculator
- Category I: focusing on specific visual features of the graph of a function

Table 1: VL's Emerging Categories

CATEGORIES	MATHEMATICAL TASKS
A	1, 4
B	1,10
C	1
D	2 (was not discussed in this article)
E	4, 8, 9, 10

F	4
G	4, 9, 10
H	10
I	8, 9

Discussion

The results discussed in this article suggest the participant's reliance on visual imagery correlated with Presmeg (1989). The author stated, "visual imagery which is meaningful in the pupil's frame of reference may lead to enhanced understanding of mathematical concepts at primary and secondary levels" (p. 21). In the study, how the College Algebra student used visual imagery contributed to her understanding of functions.

Thompson (1994) stated, "…a concept definition is a customary or conventional linguistic formulation that demarcates the boundaries of a word's or phrase's application" (p. 24). Thus, the concept definition of a specific function describes the attributes of that function using words. "On the other hand, a concept image comprises the visual representations, mental pictures, experiences and impressions evoked by the concept name" (Thompson, 1994, p. 24). The author believed that students could be more successful in their understanding of functions when the concept image and concept definition are balanced.

In the study, VL's concept image was based on her reliance on visual imagery. When the student's use of visual imagery (concept image) and concept definition were balanced, understanding of functions was demonstrated which correlated with Thompson (1994).

Sfard (1991) mentioned three ways to move from the operational conception to the structural conception. The first stage was interiorization. "At the stage of interiorization a learner gets acquainted with the processes which will eventually give rise to a new concept…" (p. 18). For example, "in the case of function, it is when the idea of variable is learned and the ability of using a formula to find values of the 'dependent' variable is acquired" (p. 19). The second phase was condensation. "The phase of condensation is a period of 'squeezing' lengthy sequences of operations into more manageable units." Thus, the student would still use processes, however, the concept should become more concrete. For instance, "…the learner can investigate functions, draw their graphs, combine couples of functions (e.g. by composition), even to find the inverse of a given function" (p. 19).

In the study, VL depicted Sfard's (1991) condensation stage. This occurred through the student's demonstration of understanding linear, quadratic, absolute value, and exponential functions.

In the study, VL demonstrated many of the traits of Krutetskii's (1976) geometric learner. These included mathematical images from the student's mind that were described and / or drawn on paper.

Arcavi (2003) stated, "visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings" (p. 217). The Visualizer in the study displayed Arcavi's (2003) definition of visualization. The student reflected upon pictures, images, and diagrams in her mind and on paper throughout the study. In addition, VL's purpose was to try to communicate the mathematical information that she was thinking about. Specifically, this occurred on mathematical tasks one, four, eight, nine, and ten.

Significance of the study

One of the major goals in mathematics education is to ensure the success of all students in mathematics. A way of accomplishing this goal is by incorporating different kinds of learning experiences for the variety of learners in the College Algebra classroom. By determining what understanding the algebraic concept of function means to the visual College Algebra student this goal may be achieved.

It is vital that College Algebra learners see mathematics as meaningful and relevant. "Mathematics instruction must reach out to all students: women, minorities, and others who have… differing learning styles… faculty must provide a supportive learning environment and promote appreciation of mathematics" (Writing Team and Task Force of the Standards for Introductory College Mathematics Project, 1995, p. 3).

Acknowledgements

This manuscript is based on the dissertation, *How Graphing Calculators and Visual Imagery contribute to College Algebra Students' Understanding the Concept of Function*. The professor directing the dissertation was Dr. Leslie Aspinwall.

Notes

[1] Approximately, 13,000 students attended the university.

[2] The participants are selected based on a specific characteristic. In this case, the characteristic is a student's preference for visual thinking in mathematics.

[3] The researcher made entries in the journal that pertain to all of the data collection activities.

[4] The taped interview sessions were transcribed.

[5] Triangulation of data involves validating particular pieces of information with another source or method.

[6] Member checking procures confirmation that a case includes the information constructed by the participant(s) and makes corrections, if needed.

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Appendix A
Mathematical Processing Instrument

Important:

(a) Do not write on this problem sheet. Write your solutions on the solution sheet provided.

(b) For each problem, show your working as much as you can.

(c) You are required to attempt all problems.

SECTION B:

B-1. A track for an athletics race is divided into three unequal sections. The length of the whole track is 450 meters. The length of the first and second sections combined is 350 meters. The length of the second and third sections combined is 250 meters. What is the length of **each section**?

B-2. A balloon first rose 200 meters from the ground, then moved 100 meters to the east, then dropped 100 meters. It then traveled 50 meters to the east, and finally dropped straight to the ground. How far was the balloon from its starting point?

B-3. A mother is seven times as old as her daughter. The difference between their ages is 24 years. How old are they?

B-4. In an athletics race John is 10 meters ahead of Peter. Tom is 4 meters ahead of Jim and Jim is 3 meters ahead of Peter. How many meters is John ahead of Tom?

B-5. At first, the price of one kg of sugar was three times as much as the price of one kg of salt. Then the price of one kg of salt was increased by half its previous price, while the price of sugar was not changed. If the price of salt is now 30 cents per kilogram, what is the price of sugar per kilogram?

B-6. Some sparrows are sitting in two trees, with each tree having the same number of sparrows. Two sparrows then fly from the first tree **to the second tree**. How many **more** sparrows does the second tree then have than the first tree?

B-7. A saw in a sawmill saws long logs, each 16 meters long, into short logs, each 2 meters long. If each cut takes two minutes, how long will it take for the saw to produce eight short logs from one long log?

B-8. A jar of kerosene weighs 8 kilograms. Half the kerosene is poured out of it, after which the jar and contents weigh $4\frac{1}{2}$ kg. Determine the weight of the jar.

B-9. A passenger who had traveled half his journey fell asleep. When he awoke, he still had to travel half the distance that he had traveled while sleeping. For what part of the entire journey had he been asleep?

B-10. If you place a large, entire cheese on a pan of a scale and three quarters of a cheese and a $\frac{3}{4}$ kg weight on the other pan, the pans balance. How much does an entire cheese weigh?

B-11. There was twice as much milk in one can as in another. When 20 liters of milk had been poured from both cans, then there was three times as much milk in the first can as in the second. How much milk was there originally in each can?

B-12. Ten plums weigh as much as three apricots and one mango. Six plums and one apricot are equal in weight to a mango. How many plums balance the scales against one mango?

Appendix B
Mathematical Processing Questionnaire

IMPORTANT:

On this questionnaire you are asked to consider how you did the mathematical processing problems that you were recently asked to do. Every problem has three or more possible solutions.

SOLUTIONS

SECTION B:

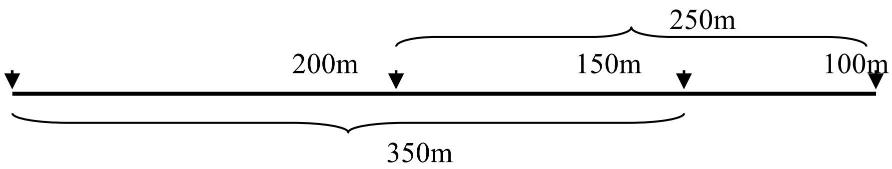
B-1. Solution 1: I solved this problem by imagining the track for the race and then working out the length of each section.

Length of third section = 450-350 = 100 metres

Length of first section = 450-250 = 200 metres

Thus length of second section = 150 metres.

B-1. Solution 2: I drew a diagram that represents the track and then worked out the length of each section.



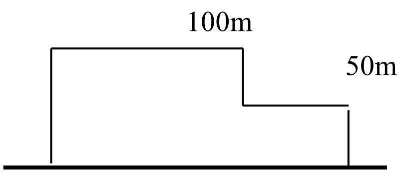
The length of the first section is 200 meters, the section section is 150 meters, and the third section is 100 meters.

B-1. Solution 3: To solve this problem I drew conclusions from the information given, and did not imagine or draw any picture at all:

Length of whole track is 450 m	x+y+z = 450
First and second sections combined is 350m	x+y = 350
Conclusion: Length of third section =450-350=100m	z = 100
Second and third sections combined is 250m	y+z = 250
Conclusion: Length of first section =450-250=200m	x = 200
Thus length of second section = 450-200-100=150m	y = 150

B-2. Solution 1: I imagined the path taken by the balloon, and then worked out the distance between the starting and finishing places. I found the distance to be 100 + 50 = 150 meters.

B-2. Solution 2: I drew a diagram representing the path taken by the balloon, and then worked out the distance between the starting and finishing places.



Distance = 100 + 50 = 150m.

B-2. Solution 3: In order to solve this problem, I noticed only the information which was important for the solution (without imagining the path of the balloon). Then the distance between the starting and the finishing places was 100m + 50m = 150m.

B-3. Solution 1: I solved this problem by trial and error:

Daughter's age:	Mother's age:	
2 years	26 years	No
3 years	27 years	No
4 years	28 years	Yes.

Thus the daughter's age is 4 years and the mother's 28 years.

B-3. Solution 2: I solved this problem by using symbols and equations, e.g.,

Let daughter's age be x years.

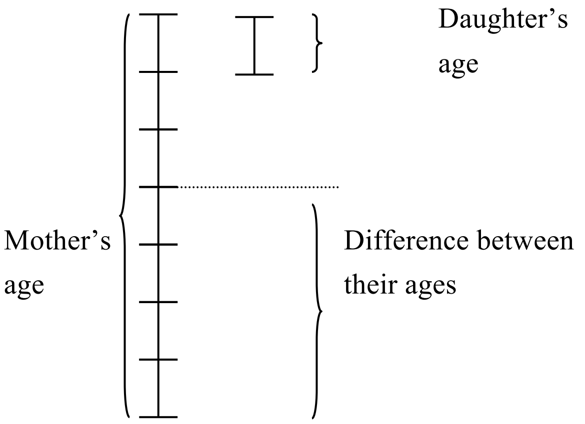
Then mother's age is 7x years.

Difference between their ages is 6x years.

Therefore 6x = 24. Thus x = 4.

Thus the daughter's age is 4 years and the mother's age is 28 years.

B-3. Solution 3: I drew a diagram representing their ages:



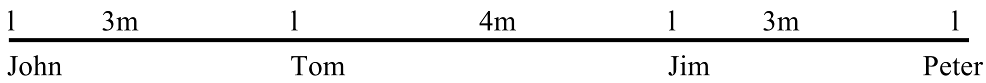
From the diagram, difference between their ages is 6 equal parts, totalling 24 years.

Thus each part represents 4 years. The daughter's age is 4 years and the mother's age is 28 years.

B-3. Solution 4: I imagined the diagram as in solution 3, and then reasoned that 6 parts represents 24 years, so one part represents 4 years (with or without using symbols). Thus the daughter's age is 4 years and the mother's 28 years.

B-4. Solution 1: I imagined the four people and then worked out the distance between John and Tom. John is 3 meters ahead of Tom.

B-4. Solution 2: I drew a diagram representing the four people, and then worked out the distance between John and Tom.



John is 3 meters ahead of Tom.

B-4. <u>Solution 3:</u> I solved this problem merely by drawing conclusions from the sentences in the problem:

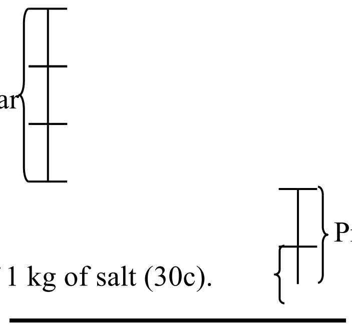
Tom is 4m ahead of Jim and Jim is 3m ahead of Peter.

Conclusion: Tom is 7m ahead of Peter.

John is 10 meters ahead of Peter.

Conclusion: John is 3 meters ahead of Tom.

B-5. <u>Solution 1:</u> I solved this problem by drawing a diagram which represented the prices of the sugar and the salt.



In the diagram it can be seen that after the price of salt was increased, the price of 1 kg of sugar ws twice the price of 1 kg of salt (now 30 cents).

Thus the price of ikg of sugar is 60 cents.

B-5. <u>Solution 2:</u> I used the same method as for solution 1, but I drew the diagram "in my mind" (and not on paper).

B-5. <u>Solution 3:</u> I solved the problem by reasoning. The price of 1 kg of salt is now 30 cents. This is $1\frac{1}{2}$ times the previous price; thus the previous price was 20 cents per kg. Thus the price of sugar is 3x20 cents, that is, 60 cents.

B-5. <u>Solution 4:</u> I solved the problem using symbols and equations, e.g.,

Suppose the previous price of salt was x cents per kg.

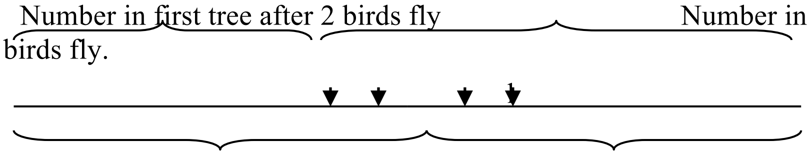
Then the price of sugar was 3x cents per kg.

After the increase, price of salt is $1\frac{1}{2}$ x cents per kg.

Thus the price of 1 kg sugar is twice the present price of salt, that is, 2x30=60c.

B-6. <u>Solution 1:</u> I solved the problem by reasoning. After two sparrows flew from the first to the second tree, the first tree had two **less** than before, while the second tree had two sparrows **more**. Thus the second tree had four more than the first.

B-6. <u>Solution 2:</u> I drew a diagram.



The second tree has four more sparrows than the first.

B-6. Solution 3: Same method as for solution 2, but I drew the diagram "in my mind" (and not on paper).

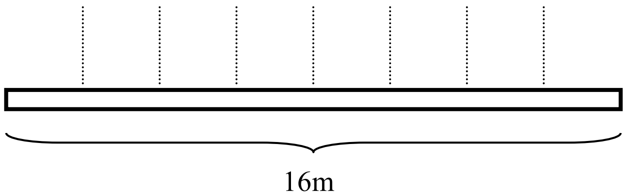
B-6. Solution 4: I solved this problem by using an example, e.g., suppose at first there are 8 sparrows in each tree. After 2 sparrows fly from the first to the second, the first tree has 6 sparrows and the second 10. Thus the second tree has 4 more sparrows than the first.

B-6. Solution 5: I solved this problem using symbols and equations, e.g.,

Let the number of sparrows in each tree at first be x.

After two sparrows fly from the first tree to the second, the first tree has x-2 and the second tree has x+2 sparrows. The difference in the number of sparrows is (x+2) – (x-2) = 4.

B-7. Solution 1: To solve this problem I drew a diagram representing the long log being cut into small logs.



From the diagram, 7 cuts are needed to produce 8 short logs. Thus time required is 7x2 = 14 minutes.

B-7. Solution 2: As in solution 1, but I "saw" the diagram in my mind.

B-7. Solution 3: I solved the problem by reasoning. If the long log were more than 16 meters long, one would need 8 cuts to produce 8 short logs. But the last cut is not needed, so 7 cuts are required. Time taken is 7x2 = 14 minutes.

B-8. Solution 1: I solved this problem using symbols and equations, e.g.,

Let the weight of the jar be x kg.

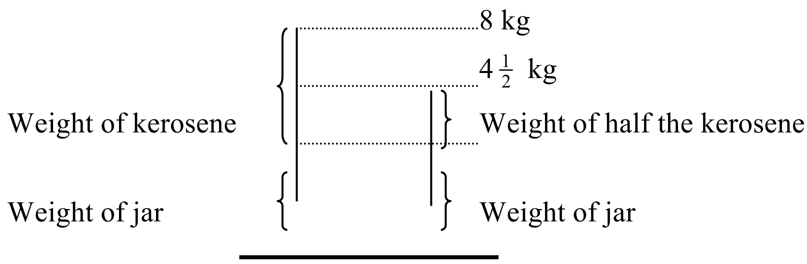
Then the weight of kerosene is (8-x) kg.

So the weight of half the kerosene is $\frac{1}{2}$(8-x) kg.

Then $x + \frac{1}{2}(8-x) = 4\frac{1}{2}$. Thus x = 1.

Thus the weight of the jar is 1 kg.

B-8. Solution 2: I drew a diagram representing the respective weights.



From the diagram, weight of half the kerosene is $8 - 4\frac{1}{2} = 3\frac{1}{2}$ kg.

Thus weight of kerosene is 7 kg, and weight of jar is 1 kg.

(Or directly: Weight of jar is $4\frac{1}{2} - 3\frac{1}{2} = 1$ kg.)

B-8. Solution 3: As in solution 2, but I "saw" the diagram in my mind.

B-8. Solution 4: As in solution 2, but without any diagram or image at all.

B-9. <u>Solution 1:</u> I drew a diagram representing the distance traveled.

Half his journey Distance he slept

Half distance he traveled while sleeping

From the diagram, if the whole journey is 6 parts, he slept for 2 parts, that is, one third of the entire journey.

B-9. <u>Solution 2:</u> As in solution 1, but I "saw" the diagram in my mind.

B-9. <u>Solution 3:</u> I solved this problem using symbols and equations, e.g.

Let the distance for which he slept be x units.

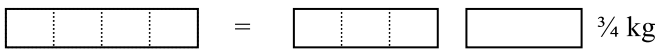
When he awoke, the remaining distance was $\frac{1}{2}$ x units.

Then (x + $\frac{1}{2}$ x) constitutes half the journey.

So the whole journey was $2(x + \frac{1}{2}x) = 3x$ units.

Thus he slept for one third of the journey.

B-10. <u>Solution 1:</u> I solved this problem by drawing a diagram representing the objects.



Removing three quarters of a cheese from both scale pans, one quarter of a cheese balances a $\frac{3}{4}$ kg weight. Thus a whole cheese weighs $4x\frac{3}{4}$, i.e., 3 kg.

B-10. <u>Solution 2:</u> As in solution 1, but I "saw" the diagram in my mind.

B-10. <u>Solution 3:</u> I solved this problem using symbols and equations, e.g.,

Let the weight of a cheese be x kg.

Then $x = \frac{3}{4}x + \frac{3}{4}$. Therefore x = 3

Thus the weight of a cheese is 3 kg.

B-10. <u>Solution 4:</u> I reasoned without using a diagram or image:

One quarter of a cheese weighs $\frac{3}{4}$ kg. Thus a cheese weighs 3 kg.

B-11. <u>Solution 1:</u> I solved this problem using symbols and equations, e.g.,

Let original amounts of milk be x liters and 2x liters.

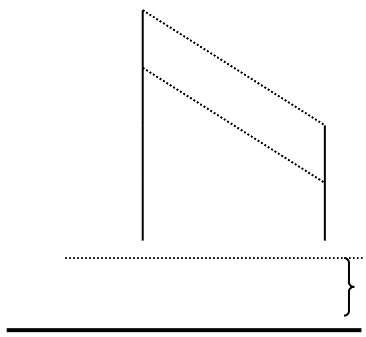
Amounts after pouring out are (x-20) and (2x-20) liters.

Then $3(x-20) = 2x-20$.

$x = 40$.

Thus the original amounts of milk were 40 liters and 80 liters.

B-11. <u>Solution 2:</u> I drew a diagram representing the amounts of milk.



20 liters (same amount poured from both cans).

From the diagram, for the first can to contain three times as much as the second after pouring, amount remaining in second can must be 20 liters. Thus original amounts were 40 liters and 80 liters.

B-11. <u>Solution 3:</u> As in solution 2, but I “saw” the diagram in my mind.

B-12. <u>Solution 1:</u> I used symbols and equations, e.g.

Let weight of plum be x units and weight of apricot be y units.

Then weight of a mango is (6x+y) units.

Thus $10x = 3y + (6x+y)$

So $x = y$

Then weight of a mango is 6x+x, i.e., 7x units.

Thus one mango balances the scales against 7 plums.

B-12. <u>Solution 2:</u> I solved this problem by drawing diagrams representing the weights.

balanced

(10 plums) (3 apricots, 1 mango)

balanced

(10 plums) (3+1 apricots, 6 plums)

From each scale pan remove 6 plums. Then 4 plums will balance 4 apricots. Thus 1 plum will balance 1 apricot. One mango balances 6 plums and one apricot, which is thus equivalent in weight to 7 plums.

B-12. <u>Solution 3:</u> As in solution 2, but I “saw” the diagram in my mind.

B-12. <u>Solution 4:</u> I solved this problem by reasoning (without imagining any picture).

One mango balances 6 plums and 1 apricot.

Thus 10 plums balance 3 apricots, 6 plums, and 1 apricot.

Thus 4 plums balance 4 apricots.

Thus (from first line) one mango balances 7 plums.