Procedural and Conceptual Knowledge: Adults Reviewing Fractions

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Abstract

In the United States a majority of the students who enroll in community colleges require a review of secondary math before they are eligible for college level mathematics. In the pre-algebra course, that has a high drop-out rate, the most difficult topic for students is fractions. In order to better understand the fraction concept Kieren separated it into five related sub-constructs. In Kieren's model the foundational sub-construct of part-whole provides a path to the sub-constructs of ratio, operator, quotient and measure. This model was previously investigated and found effective by Charalambous & Pitta Pantazi (2007) with children and Baker, Dias, Doyle, Czarnocha & Prabhu (2009) with adults. This article extends the earlier study of Baker et al with adults by considering an alternate pathway to competency with operator and measure through multiplication and addition of fractions. This path is based upon Piaget's theories in which concepts develop through reflection upon corresponding procedures. The goal is to show that the path through direct extension of part-whole knowledge and the path through procedural knowledge work together and complement one another in explaining students' competency with operator and measure. An open question of whether measure is part of the multiplicative structure of fractions is also addressed using a mixed methodology of quantitative and qualitative analysis.

Keywords: Fractions, adult education, sub-construct, operator, measure

Introduction

It is increasingly common for prospective college students in the United States, especially those entering community colleges with open admission, to take entrance exams in mathematics.

If they fail they must enroll in remedial courses that review middle and high school level mathematics. These courses have a high dropout rate and thus serve as a barrier for adult students exploring the option of a college education (Hagedorn, Siadat, Fogelo, Nora & Pascarella, 1999). In these remedial courses fractions are typically the cause of more student angst than any other topic. One explanation for students' difficulties with fractions is the multifaceted nature of this concept. In an effort to clarify the different aspects of the fraction construct, Kieren (1976) maps out a fraction schema composed of five sub-constructs, the foundational sub-construct of part-whole knowledge and four secondary sub-constructs: ratio, operator, quotient and measure.

An extension of this model by Behr, Harel, Post & Lesh (1993) that relates the four secondary sub-constructs to corresponding procedural knowledge is often used to support an educational approach based upon the direct extension of students' part-whole knowledge to the four secondary sub-constructs in order to give meaning to the corresponding procedural knowledge. Reform minded educators follow this approach because they believe the traditional fraction curriculum has too much focus on procedural algorithms and prefer an emphasis on Kieren's sub-constructs as the starting point for instruction.

The quantitative analyses by Charalambous & Pitta-Pantazi (2007) with children, and Baker, Dias, Doyle, Czarnocha & Prabhu (2009) using an identical set of exercises with adults, both confirm Kieren's underlying hypothesis that part-whole knowledge should be used as a basis for competency with the remaining fraction sub-constructs. However, with regard to the Behr et al (1993) extension of the Kieren model to procedures (Kieren-Behr model) the work of Charalambous & Pitta-Pantazi (2007) and Baker et al (2009) highlight cognitive differences between children and adults. For children the approach of operator and measure enabling proficiency with the corresponding procedures of multiplication and addition was suitable.

In contrast adults experienced more difficulty with operator and measure than multiplication and addition. This indicates that operator and measure are not necessary for proficiency with multiplication and addition.

In this present study[1] with adult students, conceptual knowledge of operator and measure is taken not as a path to procedural knowledge but instead as the goal of instruction. In this sense it is the converse hypothesis to whether conceptual knowledge enables procedural knowledge that is being tested. Thus, using statistical analysis, the primary objective of this study is to determine which pathway is the most effective in explaining students' competency with the operator and measure concepts: Through direct extension of part-whole knowledge, through the corresponding procedural knowledge of multiplication and addition, or do these pathways work together and complement one another? Like the prior empirical studies on development of fractions knowledge (Charalambous & Pitta-Pantazi (2007); Wilkins & Norton (2009, 2010 and 2011); Baker et al (2009)) this quantitative investigation is subject to the limitations that come with trying to understand processes by analyzing a written set of exercises without direct observation of the learning mechanisms involved.

However, this limitation does not render the results invalid indeed, as eloquently stated by Wilkins & Norton quantitative analysis is an effective and appropriate methodology for testing hypothetical models of learning (2009). These authors cite Kilpatrick (2001) as having called upon mathematics educators to test using quantitative analysis, "… hypotheses that are derived from qualitative research" (2011, p.386).

The statistical analysis of the quantitative results in this study, like those mentioned previously, seeks to overcome this limitation through the following premise. Given a positive and significant correlation between two variables when there is a significant difference between them, it is reasonable to conclude that the variable students find easier, concept X, has substantial potential to be used in acquiring the variable they find more difficult, concept Z. This can be expressed as, X implies Z or X is necessary but not sufficient for Z.

This premise is what allows Charalambous & Pitta-Pantazi (2007) in their empirical study using statistical analysis to conclude that, "… the part-whole interpretation of fractions should be considered a necessary, but not sufficient condition for developing an understanding of the remaining notions of fractions…"' (p.310). Then using this premise they relate conceptual understanding of the sub-constructs to procedural knowledge, "… a profound understanding of the different interpretations of fractions can uplift students' performance on tasks related to the operation of fractions…" (p.311).

In the methodology section the basic premise underlying this statistical analysis is extended to include two independent variables, part-whole and procedural knowledge of either multiplication or addition. Then analysis of variance is used to determine which of these variables has more influence in explaining variations in the corresponding conceptual knowledge of operator or measure; whether these variables work together and supplement one another is also tested. If part-whole is the dominant influence this will support for the Kieren-Behr model.

On the other hand, if procedural knowledge is the dominant influence, this will support learning theories based upon Piaget in which concepts develop through reflection upon procedural knowledge (Sfard 1991; Czarnocha, Dubinsky, Prabhu & Vidakovic, 1999). The third possibility is that conceptual knowledge of part-whole and procedural knowledge of multiplication and addition are both significant and complement one another. This result would support the model of Gray, Pinto, Pitta & Tall (1999) which proposes two separate cognitive pathways to concept development through reflection upon either existing concept knowledge or procedures.

One area in which the study of Charalambous & Pitta-Pantazi (2007) did not support the Kieren-Behr model was the lack of correlation between measure and addition of fractions. Investigation into this lack of correlation was left for future research by these authors. A reason for the weak or non-existent correlation may be found in the open question posed by Wilkins & Norton (2009); whether measure is part of the multiplicative structure of fractions. For this reason, a secondary objective is to investigate the role of part-whole, addition and multiplication in promoting measure. Quantitative methods will be used to determine whether measure is closer to the multiplicative or additive structure of fractions. Acknowledging the limitations of this technique, a transcript of a small group session is reviewed to provide insight into these relationships.

Theoretical Foundation

The Kieren-Behr Model

In Figure 1 the reader can view the distinction between conceptual knowledge in part-whole, the four secondary sub-constructs in the middle row (Kieren, 1976) and the procedural knowledge of addition and multiplication of fractions represented in the bottom row (Behr et al, 1993).

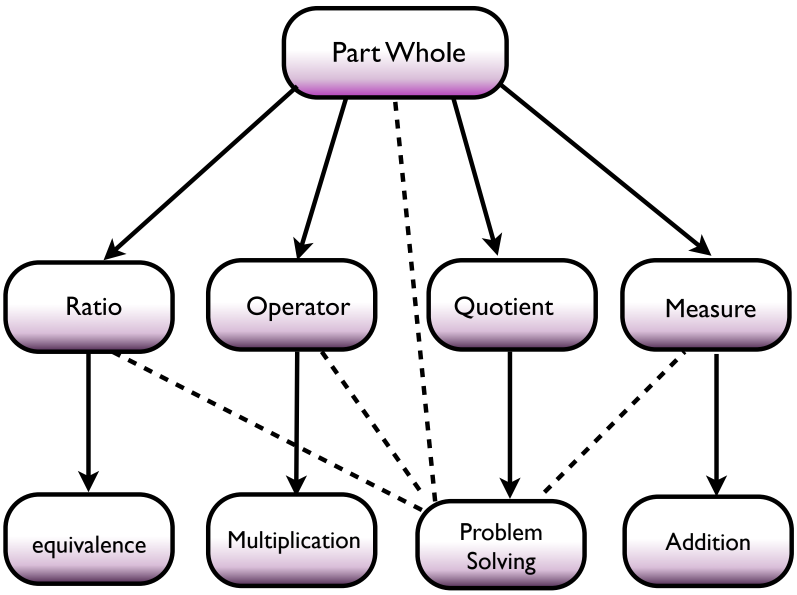


Figure 1. Kieren- Behr model, Excerepted-from Behr et al, 1993, p.100

The hierarchical structure of Figure 1 suggests that part-whole knowledge will imply competency with the four secondary sub-constructs, a proposition that was confirmed by Charalambous & Pitta-Pantazi (2007) and Baker et al (2009). The structure of Figure 1 also suggests that competency with sub-constructs such as operator and measure will imply proficiency with the corresponding procedures. For this reason, a narrow interpretation of Figure 1 leads to the conclusion that knowledge of operator and measure is necessary for proficiency with multiplication and addition of fractions.

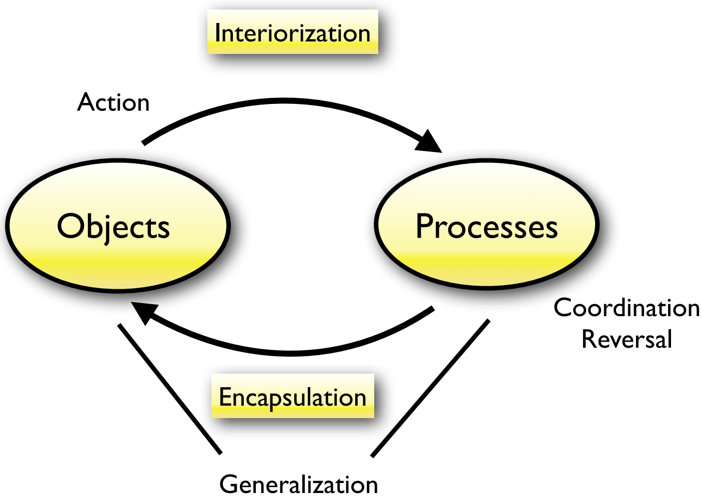
Educational and Developmental Approaches

Haapasalo & Kadijevich (2000) outline two distinct pedagogical methods; the Educational Approach and the Developmental Approach to mathematics based upon the relationship between procedural and conceptual knowledge. In the Educational Approach concepts enable procedural knowledge and pedagogy for fractions that follows this approach focuses on extending students' part-whole knowledge to the secondary sub-constructs as suggested by the Kieren-Behr model (Figure 1). The Developmental Approach is based upon the enrichment of conceptual knowledge through reflection upon procedures. Haapasolo & Kadijevich propose that learning theories based upon Piaget in which concepts develop through reflection upon procedural actions support this approach.

The Piaget based model-APOS

Dubinsky describes the APOS (action-process-object-schema) model of concept development based upon the work of Piaget as a cycle that begins with action upon existing concepts, "first, an action must be interiorized … this means some interior construction is made relating to the action. An interiorized action is a process" (Dubinsky, p107, 1991).

The process is then reflected upon and abstracted into a new object, thus concepts develop through reflection upon procedures.



Excerpted from: Figure 13, Schemas and their construction, p.107 Dubinsky 1991

Figure 2. Concept Development-Dubinsky APOS

In Figure 2, the sequence of arrows labeled interiorization and encapsulation follow the transformation of a learner from procedural action on existing concepts to reflection upon these procedures and the resulting development of new concepts. It is important to note that there is no arrow representing direct reflection upon existing conceptual knowledge, thus a narrow interpretation of Figure 2 results in the conclusion that procedural knowledge is necessary for concept development. In this sense the APOS model can be seen as support for the Developmental Approach.

Two Separate Cognitive Pathways to Concept Development

Some educators characterize the restriction that concepts arise through refection upon procedure as a narrow interpretation of the work of Piaget. They claim Piaget would allow for mental actions associated with reflection upon existing concepts as well as procedures. (Gray et al, 1999; Berger, 2005; Gilmore & Inglis, 2008; Tall, 1999; Gray & Tall, 2001).

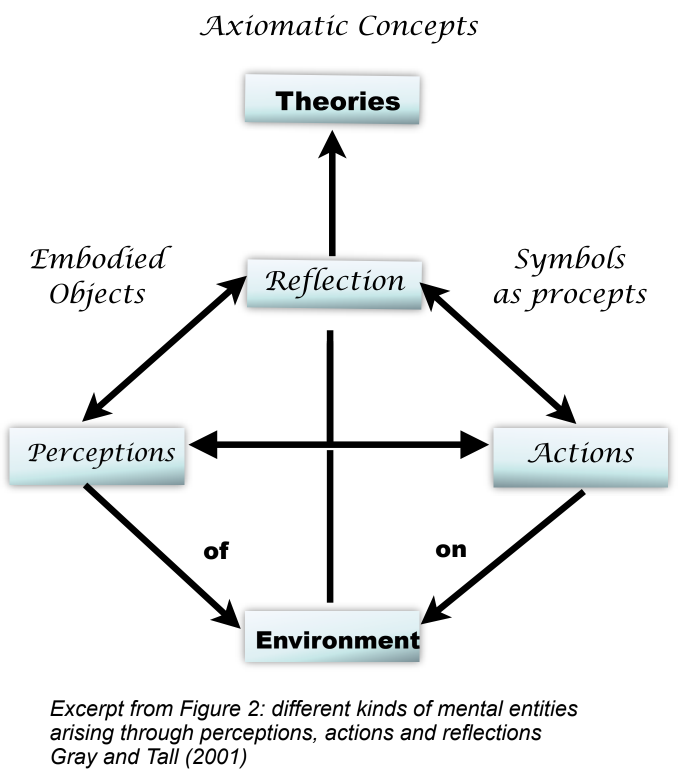


Figure 3. Two Separate Pathways to Concept Development-Gray and Tall

Figure 3 depicts two pathways to development of concept and theories of learning. The path on the left represents perceptions and reflection upon existing concepts and is similar to the pathway from part-whole to the secondary sub-constructs in Figure 3. The path on the right represents reflection upon actions and is similar to the pathway to concepts depicted in Figure 3.

Previous Research with Children: Operator

The results of Charalambous & Pitta-Pantazi (2007) show a strong positive correlation between the part-whole sub-construct and the operator sub-construct. On the other hand the correlation between operator and multiplication, although significant was relatively weak.

This result suggests the Educational Approach of direct extension of part-whole to operator is more appropriate than the Developmental Approach using multiplication as a path to operator.

Mack (2000) follows the Educational Approach to investigate the effectiveness for children of the direct extension of their part-whole knowledge to the operator concept. She concludes that children draw heavily on their knowledge of partitioning during problem solving with operator and multiplication and that their understanding of the operator concept deepened over time. One problem she noted with the traditional algorithm for multiplication was that it, "… dominated their thinking …"" (p.329).

Thus, the quantitative analysis by Charalambous & Pitta-Pantazi (2007) and the qualitative analysis by Mack (2000) both validate the Educational Approach for children.

Previous Research with Children: Measure

In a longitudinal study focusing on Kieren's sub-constructs, Lamon (2007) refers to the measure sub-construct as the strongest because of its connection to the others. However, the work of Charalambous & Pitta-Pantazi (2007) indicates measure is the most difficult of the sub-constructs. Furthermore it does not correlate significantly with addition of fractions. They leave this lack of association between measure and addition as an open question for future research.

Wong & Evans (2008) confirm the difficulty children have relating measure concepts with the symbolic addition algorithm.

The results of Charalambous & Pitta-Pantazi (2007) show that although the correlation between part-whole and addition was significant it was relatively weak. Herman et al (2004) confirm that children experience difficulty relating their part-whole pictorial imagery to the addition algorithm in a meaningful way. This leads them to conclude that the pathway of extending part-whole to object status is cognitively different than that to the addition algorithm.

Adults Reviewing Fractions

Research in learning fractions is for the most part restricted to children however, the large number of students in the U.S. who enter community colleges requiring a review of this topic suggests a need to explore the cognitive differences between children and adults. Adult students have previous experience, often negative, with the concepts and procedures of fractions. Many are returning to college after years of work or family life and their ability to recall fraction procedures can be a challenge.

The study of Baker et al (2009) followed the work of Charalambous & Pitta-Pantazi (2007) and investigated adults reviewing fractions within the framework of the Kieren-Behr model (Figure 1). The results show that although the direct extension of part-whole to operator and measure was appropriate, multiplication and addition were more readily understood than operator and measure. This result suggests that competency with operator and measure is not necessary for proficiency with multiplication and addition and lead to the present investigation into whether adult students have the potential to use procedural knowledge to become competent with operator and measure.

It is important to note that the exercises were geared towards children, and the use of more difficult exercises is a consideration before concluding the Educational Approach is inappropriate for adults. It also important to note that although the direct extension of part-whole knowledge to operator and measure may not be effective when the goal is taken to be procedural knowledge, these concepts can be understood as the goal of fraction pedagogy.

In this present study working with a new set of exercises appropriate for the adults (Appendix) and the assumption that conceptual development is the goal of mathematics pedagogy, the question becomes which approach to operator and measure demonstrates more potential as a pathway to concept competency? The Educational Approach through direct extension of part-whole or the Developmental Approach through corresponding procedural knowledge?

The Sub-constructs of Part-Whole, Measure and Operator

The definition and exercises used to evaluate the fraction sub-constructs (Appendix) are taken as in Charalambous & Pitta- Pantazi (2007). However, exercises were included from the adult curriculum to insure the exercise sets were a valid measure of the content these students were required to be proficient with. These exercises are from a different perspective thus factor analysis is used to verify the components of each set of exercises. In addition a reliability test is done to determine the internal consistency of each exercise set.

The part-whole sub-construct interprets the symbol notation $\frac{p}{q}$ to represent the partitioning of a whole entity either continuous or discrete into q equal shares and then taking p out of the q shares of a quantity. The part-whole sub-construct is used as a foundation for developing rational number sense in the mathematics curricula and generates much of the language used for describing fractions (Behr et al, 1983). The part-whole sub-construct was evaluated through exercises that involved translating a pictorial image into a fraction as well as the inverse processes of shading in objects to represent a given fraction.

All the exercises for part-whole were used in Charalambous & Pitta-Pantazi (2007). The operator concept is associated with applying a function to a quantity. It is synonymous with the process of taking a fraction of some quantity, thus the operator sub-construct interpretation of $\frac{p}{q}$ involves multiplication or expansion by p and division or contraction by q. Exercises used to evaluate the operator sub-construct included two sets of exercises both of which were used in Charalambous & Pitta-Pantazi (2007). The first component contains the input-output box in which the output is a fractional amount of the input quantity. The second component contains exercises that translate taking a given fraction $\frac{p}{q}$ of some quantity into appropriate statements of multiplication by p and division by q. Also included in this sub-construct were exercises based upon the adult curriculum. These included taking a fraction of a quantity, taking a fraction of a fraction of a quantity and the inverse process. These exercises are identified in the appendix.

Charalambous & Pitta-Pantazi consider the measure concept to be related to the relative size of a number. In particular this involves an understanding of partitions of the unit interval into segments and iterations of this segment. In this sense, measure involves an application of the part-whole concept to determine the placement of the fraction $\frac{p}{q}$ on an interval with a designated unit through dividing the unit into q equal parts and then iterating this process p times. Thus, the measure sub-construct can readily extend the part-whole process to include improper fractions.

Although the measure sub-construct is frequently evaluated through placement of a fraction on the number line Charalambous & Pitta-Pantazi suggest that, "to fully develop the measure personality of fractions students also need to master the order and equivalence of fractions."' (p.310)

For this reason, the exercises used to evaluate measure include placement on the number line (Charalambous & Pitta-Pantazi, 2007) and two exercises from the adult curriculum that involve ordering and consideration of the relative size of fractions. These exercises are identified in the appendix.

Measurement as Multiplicative Reasoning

Thompson & Saldanha (2003) place measure and the other sub-constructs within the context of multiplicative reasoning. Lamon's (2007) intuition that Kieren's sub-constructs are part of the multiplicative structure inherent in solving proportions leads her to ask, "what are the links between additive and multiplicative structures?" (p.662). Wilkins & Norton are more specific in asking, "… whether the measure sub-construct supports multiplicative reasoning, as the operator sub-construct does" (2010, p.181). The consideration of measure as part of the multiplicative instead of the additive structure (Figure 1) is supported by the lack of correlation between measure and addition for children (Charalambous & Pitta-Pantazi, 2007).

Objectives

The first research objective is to determine which of the three hypothetical pathways demonstrates the most potential benefit for students' competency with operator and measure; through direct extension of part-whole, through procedural proficiency or through the integration of these factors. These pathways are tested using statistical analysis to determine which has more influence in explaining variations in competency with operator and measure. If part-whole is the dominant influence this will support the Educational Approach, the Kieren-Behr model (Figure 1) and reform pedagogy with its focus on concepts. If procedural proficiency is the dominant influence this will support the Developmental Approach, the APOS model (Figure 2) and the traditional curricula focus on procedural proficiency. On the other hand, if both variables part-whole and procedural proficiency are significant and work together this will support the separate cognitive pathways approach advocated by Gray & Tall (Figure 3) and indicate that reform and traditional pedagogy have common ground that needs to be explored.

The measure sub-construct has been called the most powerful sub-construct and yet the relationship between measure and the procedural factors of multiplication and addition of fractions remains an open question. The second research objective is to employ both quantitative and qualitative analysis to determine whether measure is closer to the additive (Figure 1) or multiplicative (Wilkins & Norton, 2010) structure of fractions.

Methodology

Research Questions

Research Question 1: Which is the most significant and influential factor in predicting variation in students' competency with the operator and measure sub-constructs?

- Part-whole knowledge (Educational Approach, Figure 1)
- Procedural proficiency of multiplication and addition of fractions (Developmental Approach, Figure 2)
- Do part-whole knowledge and procedural proficiency with multiplication and addition complement each other in explaining competency with operator and measure? (Figure 3)

Research Question 2: Which is the most influential predictor of variation in competency with the measure sub-construct, part-whole knowledge, addition of fractions, or multiplication of fractions?

Setting

The following study is based on quantitative statistical analysis of data involving fraction concepts and procedures collected over several semesters from 333 adult students enrolled in pre-algebra courses taught by six professors of Mathematics at Hostos Community College (HCC) and Bronx Community College (BCC) both urban community colleges in the City University of New York (CUNY) system. The student body at these community colleges is predominately female (70%-80%) and minority (85%-95%) and is the mathematically weakest group of students applying to the CUNY system. These students have failed both the algebra and pre-algebra placement exams in mathematics and are not eligible to take college level mathematics course until they pass these courses. At these community colleges the pre-algebra course lasts fourteen weeks and it covers real numbers such as decimals and fractions, proportions, percent and an introduction to algebra.

Quantitative Analysis

Six professors of mathematics acting as teaching researchers at HCC and BCC gave 333 adult pre-algebra students sets of exercises to be completed as homework. The exercises included tasks on the concepts or sub-constructs; part-whole, operator, and measure as well as the procedural factors of multiplication and addition of fractions (Appendix). They were taken from the exercise sets used by both Charalambous & Pitta-Pantazi (2007) with children who used quantitative analysis to verify the Kieren-Behr model of Figure 1. However additional exercises from the adult curriculum have been added to make the sets correspond to the appropriate level for these adult students.

In the quantitative analysis for this study, variables may be either concepts or procedures. The sub-construct part-whole will be an independent variable and labeled X. The other independent variable will be the procedural factor of either multiplication or addition and will be labeled Y.

The dependent variable is either operator or measure and will be labeled Z. Although statistical analysis using correlations does not provide direct observation of an individual's development, it demonstrates how strong a relationship is between variables for a group of students and thus verifies the potential for a hypothetical pathway of learning. In this study, the hypothetical pathways are from X to Z (Educational Approach, Figure 1) or from Y to Z (Developmental Approach, Figure 2). The quantitative analysis of correlations between variables used in this study is based upon the assumption that the mean scores of two variables are significantly different (T-test) and there is a positive and significant correlation between them.

In this case the underlying premise is that students' knowledge of the easier concept X or factor Y will precede and be used to acquire knowledge of the more difficult concept Z. Thus knowledge of X will imply knowledge of Z, this will be written as, $X \Rightarrow Z$. Furthermore, the square of the correlation coefficient r^2 indicates the percent variation of Z explained by X.

This will be written as $X \Rightarrow Z$ (r^2%). For example, if $X \Rightarrow Z$ (25%) then 25% of the students' grade on Z is determined by how they did on X. In particular, if a class of students is proficient in X then about 25% of the class is competent with Z.

Statistical correlations are sufficient to test the relationship between two variables however to consider the effect that two independent variables X, Y have on a third dependent variable Z a multiple linear regression analysis or analysis of variance (ANOVA) will be used.

In this study an underlying assumption for such a model is that each independent variable correlates significantly with the dependent variable and there is a significant difference between the mean score of each independent and dependent variable. The F-value or ratio is an indicator of the strength of the relationship between the independent and dependent variables and the p-value determines whether the model is significant. Assuming the model is significant, the relevant question becomes, what is the interaction between these variables?

The first indicator of how the independent variables interact in the model is the significance value of each variable. When both independent variables X, Y are significant they work together for predicting or explaining the dependent variable Z. In this case, the two independent variables can be considered as separate pathways to the dependent variable, this will be written as $X \ \& \ Y \Rightarrow Z$.

Although an ANOVA provides a correlation coefficient (r-value) for a multiple independent variable model because of the interaction between these variables the ANOVA also provides an adjusted r^2 value that is slightly lower than and used in place of the r^2 value to determine the variation of the dependent variable explained by the independent variables working together. This will be written as $X \ \& \ Y \Rightarrow Z$, (adj-r^2) to indicate that X and Y together explain the given percent of variation in Z.

The second indicator of how the independent variables interact is the beta value. The beta value is a measure of how much influence each independent variable has in predicting or explaining variation in the dependent variable. More specifically a beta value of 0.5 for X indicates that for every unit change of a standard deviation of X, there is a corresponding 0.5 or 50% change of a standard deviation in Z.

The exercise set used in Baker et al (2009) was identical to that used by Charalambous & Pitta- Pantazi. However, in this study exercises from the adult curriculum have been included to the sets evaluating operator, measure, multiplication and addition. These exercises were included because they are similar to questions used on college readiness exams the students must pass to exit from remedial mathematics. Following Cramer, Post & delMas (2002) principal component analysis will be used to determine number of components or factors in each subset and the Kaiser-Meyer-Olkin measure of sampling adequacy is found to demonstrate the acceptability of the factor analysis. Factor loading values from the component matrix and commonality values are included with in the appendix. Ideally each set of exercises should be one component or factor however, in order to maintain some resemblance and congruency to the earlier work of Charalambous & Pitta-Pantazi (2007) and Baker et al (2009) the original set will be left intact and not decomposed into subsets even when factor analysis reveals it could be.

Thus, the objective of factor analysis is not to decompose the existing exercise set but to understand what the components are and to ensure that when exercises from the adult curriculum are included to the original set they do not increase the number of components or factors. As in Cramer, Post & delMas (2002) a reliability test (Cronbach's alpha value) is given to determine the extent to which the exercises in each construct are related to one another, this test provides an overall index of the repeatability or internal consistency of the scale as a whole. Thus, the exercise set that is deemed acceptable as a result of the factor and reliability analysis can be viewed as a coherent or reliable set of exercises with potentially several components. Furthermore, these components existed in the original set of exercises used in previous research and any exercise included from the adult curriculum will fit into one of the existing components.

Qualitative Analysis

Quantitative techniques will be used in both the first and second research question. The first question involves testing hypothetical pathways to concept competency developed from well established models of learning that have been extensively used as theoretical foundations for qualitative research in mathematical education thus the quantitative methodology is appropriate to clarify the discrepancies between these models. The second research question has not been widely studied therefore it will be analyzed through a mixed methodology. Quantitative analysis is used to determine the probable relationship between the additive and multiplicative structure and the measure sub-construct.

Then qualitative transcripts of a small group study session are reviewed for evidence of the underlying learning mechanisms behind the statistical relationships observed.

The small group session (3-5 students) was conducted by a group leader (teacher-researcher) as part of a teaching-research experiment[2] on problem solving following the Educational Approach. The transcripts were recorded and reviewed and edited by another member of the research team. The research objective of the group sessions was to determine how they think and to understand the route they use to answer a given problem.

Results

Principal Component Analysis

Factor analysis or more accurately principal component analysis was used to determine the number of factors in each sub-construct. Ideally each sub-construct would be one factor however, as these exercises sets are formed by including exercises appropriate for adults with those developed for children by Charalambous & Pitta-Pantazi (2007) and used by Baker et al (2009) it would make comparisons to earlier work problematic if groups of exercises from these original sets were dropped for the sake of consolidation. Thus factor analysis is used to understand the components of these original set of exercises and set a criteria that any exercise included cannot increase the number of factors in the original set of exercises. The Kaiser-Meyer-Olkin measure of sampling adequacy is used with preferred value of at least 0.7 and cutoff point of 0.6 is used to determine whether the factor analysis is appropriate Kaiser (1970).

The Cronbach's alpha with preferred value of 0.7 but cutoff point of 0.6 (Cramer et al, 2002) is used to determine whether the exercise set is reliable or internally consistent.

The acceptability of the factor loading values and commonality values according the criteria used by Cramer et al (2002) are listed in the appendix.

The 11 exercises used to assess student understanding of Part-Whole are taken from Charalambous & Pitta-Pantazi. The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.83 which is relatively high and suggests the factor analysis is appropriate. The exercises that involved direct translations from a picture to a fraction were one component. The exercises that involved both a translation from a picture to a fraction and then the inverse process represented another component. A reliability test was performed and the Cronbach's alpha value was 0.8 and thus the Part-Whole exercises demonstrate good reliability and internally consistency.

The operator sub-construct was assessed through 8 exercises. 6 of these exercises were used by Charalambous & Pitta-Pantazi (2007) and Baker et al (2009). Factor analysis revealed there were two components the first involved exercises using input-output boxes or function boxes in which the output was a fraction of the input. Also included in this factor was an exercise involving taking a fraction of a quantity. The second component involved the interpretation of taking a fraction of a quantity into language statements involving multiplication and division of the numerator and denominator. The two exercises included in this present study involved an iteration of taking a fraction of a quantity and an inverse input-output box. Factor analysis revealed these three exercises all fit in with the first component of input-output boxes. The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.63 which is slightly below standard acceptability but above the cut off value of 0.6. The Cronbach's alpha value was 0.64 which is not spectacular but above the 0.6 cut off point indicating this exercise set is sufficiently reliable.

The 8 exercises that made up multiplication and division of fractions contained 4 exercises from Charalambous & Pitta-Pantazi (2007) and Baker et al (2009) and 4 new exercises. Factor analysis revealed all 8 exercises were in one factor or component. The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.74 is in the average or standard acceptable range. The Cronbach's alpha value was 0.62 which is rather low although but above the cutoff point used by Cramer et al (2002).

The Measure sub-construct was evaluated by 7 exercises 5 from Charalambous & Pitta-Pantazi (2007) with 2 additional exercises included. As stated early the 2 exercises included were an attempt to extend the assessment of measure beyond the use of the number line to include the relative size or order of the fractions as suggested by Charalambous & Pitta-Pantazi (2007). The original set of 5 exercises used by Charalambous & Pitta-Pantazi contained two components or factors one in which students located the fraction on a number line with some given segmenting and two exercises that involved placement of the unit on a number line that contained only a fractional quantity and 0 labeled. Factor analysis revealed that two additional exercises involving ordering or the relative size of a fraction were both in the first component. The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.66 which is slightly below the average or standard of 0.7 and is what can be termed the mediocre but still above the cut off value 0.60. The Cronbach's alpha value was 0.66 which is rather low well above the cutoff point 0.6 used by Cramer et al (2002).

The 7 exercises that made up addition and subtraction of fractions contained two components. The first contained 2 exercises with addition of proper fractions used by Charalambous & Pitta-Pantazi (2007). The second component contained 5 addition and subtraction with mixed numbers or both whole numbers and mixed numbers 3 of which were added for this study. The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.78 which is relatively high and indicates factor analysis is appropriate. The Cronbach's alpha value was 0.74 which is in the standard range and indicates the exercises are internally consistent and reliable.

Student Performance, Mean and Standard Deviation

Paired T-tests reveal that the mean scores in Table I for each pair of variables are significantly different. Thus part-whole knowledge of fractions is significantly easier for students than measure and operator, which suggests the Educational Approach is appropriate. These results are in agreement with the earlier findings of Baker et al (2009) with adult students and Charalambous & Pitta-Pantazi (2007) with children.

Variable	x*	SD
1) Part-whole	0.67	0.23
2) Operator	0.53	0.27
3) Measure	0.47	0.27
4) Multiplication	0.64	0.28
5) Addition	0.57	0.29

(*) n= 333

Table 1. Mean scores and standard deviations on sub-constructs

The mean scores in Table I also confirm the result of Baker et al (2009) that the procedural factors of multiplication and addition are significantly easier for students than operator and measure. This suggests the Developmental Approach may also be appropriate.

Furthermore, in this study the procedural exercises were more rigorous than those used previously in Baker et al (2009) which were designed for children.

Thus, the research question of whether part-whole knowledge or procedural knowledge is more significant and influential in predicting students' competency with the operator and measure is valid.

Operator

The square of this correlation between part-whole and operator r^2 =0.203 indicates that 20.3% of the variation in students' competency with operator, represented by Z, can be explained by their competency with part-whole, represented by X, X ⇒ Z (20.3%). The square of the correlation value between multiplication, represented by Y, and operator demonstrates that Y ⇒Z (37.2%).

Variable	Part-whole	Multiplication	Operator
Part-whole	1.00	0.36**	0.45**
Multiplication		1.00	0.61**
Operator			1.00

Pearson Correlations r, **Significant at 0.01 level (2-sided) n=333

Table 2. Correlations r-values. Operator.

This suggests the Developmental Approach through multiplication is a more effective path to competency with operator than the Educational Approach through part-whole.

As both part-whole and multiplication are significantly easier than and good predictors of operator the next step is to test whether they supplement one another. For this reason a multiple regression analysis or analysis of variance ANOVA with independent variables, part-whole X, multiplication Y, and dependent variable operator Z, was conducted to determine whether these two paths work together to increase operator competency. The results with 2 independent variables and n=333 students are F (2, 333) = 127, p<0.001. The adjusted r^2 reveals that 43.1% of the variation of the operator is explained by these two variables X & Y ⇒Z (adj-43.1%). The use of both variables results in a 15.8% increase in student competency with operator over that obtained using only multiplication Y ⇒ Z (37.2%). Thus, an integration of the Educational and Developmental Approaches has potential benefit for increasing students' competency with operator.

Predictor Variable	Beta	p-value
1) Part-whole	0.27	p<0.001
2) Multiplication	0.52	p<0.001

Table 3. Beta and significance values. Operator.

The beta values in Table III indicate that multiplication Y has more influence than part-whole X in explaining variation in operator Z, this reinforces the Developmental Approach and the APOS model with its focus on reflection upon procedural knowledge as more effective than the Educational Approach. However, the p values in Table III indicate that both part-whole and multiplication are highly significant variables in explaining variation in operator. This supports the two separate pathways approach and thus provides further evidence that the Educational and Developmental Approaches should be integrated. The following diagram represents a visual schema for competency with operator based upon part-whole and multiplication.

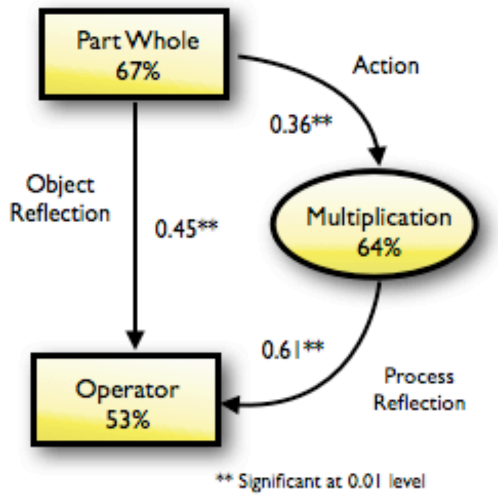


Figure 4. Adults' Operator Schema. Percents reflect mean scores. Decimal values (arrows) represents correlations. The r-value for the combined pathways to operator is 0.66

The two separate pathways to operator competency visually expressed in Figure 5, are different than the results for children which support the Educational Approach. The teacher-researchers in this study agree that a class lesson about operator based upon multiplication and supplemented with part-whole is readily understood by adult students. However, even though exposed to such lessons, evidence of students using their part-whole knowledge when they are independently solving operator exercises is very rare. Thus, although students demonstrate a potential increase of competency with operator through a pedagogy that integrates part-whole and multiplication, and they will readily follow such a lesson plan they will almost exclusively use arguments based upon multiplication when solving operator exercises. In this way they are similar to the children observed by Mack (2000).

Measure

Table I indicates that part-whole (67%) was the more readily understood than measure (47%). Table IV demonstrates a positive and significant correlation between them ($r = 0.49$) and the r^2 value indicates 24% of the variation in measure Z is explained by variation in part-whole knowledge X, X ⇒ Z (24%).

Variable	Part whole	Add	Multiply	Measure
Part-whole	1.00	0.44**	0.36**	0.49**
Addition		1.00	0.61**	0.41**
Multiplication			1.00	0.40**
Measure				1.00

Pearson Correlations, **Significant at 0.01 level (2-sided), n=333

Table 4. Correlations between part-whole, addition, multiplication and measure

The square of the correlation between addition and measure indicates that only 16.8% of the variation of measure is explained by that of addition Y, Y ⇒ Z (16.8%). These results suggest that part-whole is a more dominant influence than addition in explaining measure and thus for adults and children the Educational Approach is more effective. However, these results show a potential for adults, that addition may supplement direct extension of part-whole. For this reason, a multivariate linear regression ANOVA is used to test whether the independent variables of addition Y and part-whole X work together in explaining students' competency for the dependent variable of measure Z. The results with 2 independent variables and n=333 students are $F(2, 333) = 65.9$, $p<0.001$. The adjusted r^2 value reveals that 28.1% of the variation of the measure sub-construct is explained by these two variables X & Y ⇒ Z (adj-28.1%). This quantifies as 17.1% the increase in student competency with measure obtained over using only part-whole X ⇒Z (24%). This supports an integration of the Educational and Developmental Approaches to competency of measure.

Predictor Variable	Beta	p-value
1) Part-whole	0.38	p<0.001
2) Addition	0.24	p<0.001

Table 5. Beta and significance values. Measure.

The beta values in Table V show that part-whole is a more influential predictor of measure than addition. This confirms that the Educational Approach in more effective than the Developmental Approach. However, the p values indicate that both part-whole and addition are significant variables in explaining variation in measure. This provides support for the two pathways model of Gray & Tall and again suggests a potential benefit for the integration of the Educational and Developmental Approaches for competency with measure.

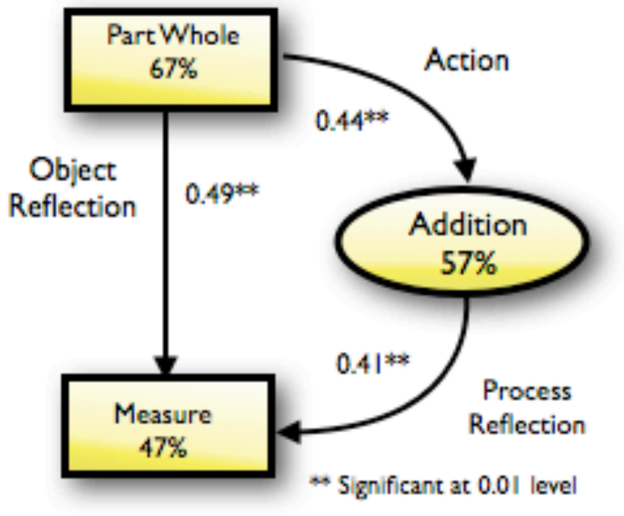


Figure 5. Adults' Measure Schema. Percents reflect mean scores. The decimal value (arrows) represents correlations. The r-value for the combined pathways is 0.53

In Figure 5 the arrow labeled object reflection represents the direct extension of part-whole to measure. This represents the Educational Approach and is the most effective pathway to competency with measure. The arrow labeled process reflection represents the supplemental pathway through addition. The teacher-researchers in this study, based upon anecdotal classroom evidence, suggest that measure has a lot of potential for assisting in problems involving fractions and percent. However, students do not easily relate the addition algorithm to measure.

Multiplicative Structure and Measure-Quantitative Analysis

The correlation between multiplication and measure r=0.40 is very close to the correlation between addition and measure r=0.41 (Table IV). Thus, it is reasonable to ask whether measure is closer to the multiplicative or the additive structure of fractions. The first part of the investigation involves quantitative analysis of variance ANOVA with three dependent variables; part-whole knowledge X, addition Y_1 and multiplication Y_2 to predict students' competency with measure Z. In order to determine which factor has the most influence on measure. The results with 3 independent variable and n=333 students $F(3,333) = 48.9$, $p<0.001$, with adjusted r^2 that reveals 30.2% of the variation of the measure sub-construct is explained by these three variables, $X \& Y_1 \& Y_2 \Rightarrow Z$ (30.2%). This represents a 25.8% increase over the 24% of variation in measure explained using only part-whole.

Predictor Variable	Beta	p-value
1) Part-whole	0.36	p<0.001
2) Addition	0.13	p=0.03
3 Multiplication	0.19	p=0.01

Table 6. Beta and significance values. Measure.

The beta values suggest that while part-whole is by far the most influential predictor of measure however, multiplication and addition are significant. The fact that the beta value for multiplication is higher than that for addition supports the hypothesis that measure is closer to the multiplicative structure of fractions. This is one possible reason for the lack of correlation between measure and addition found in Charalambous & Pitta-Pantazi (2007).

Multiplicative Structure and Measure-Qualitative Analysis

The following small group session was designed to introduce elementary concepts of algebra while simultaneously reviewing fractions. The group leader (GL) was following the Educational Approach of extending part-whole knowledge to measure and then relating this to the addition algorithm. The conversation was recorded and edited by a member of the research team. This transcript exemplifies the relationship between part-whole knowledge, addition of fractions and the measure concept. The students involved were given the pseudo-names: Sandy, Laura and Fran.

The group leader (GL) introduces the problem: $\frac{1}{2}x + \frac{1}{4}x = 1\frac{1}{8}$ and asks the students how they would proceed. There is no response so GL draws a circle using part-whole knowledge of fractions to stimulate discussion. Students follow the lead partitioning the circle up into halves and quarters, representing them correctly as ½ x or ¼ x. The GL then draws a number line with 0 and 1 and labels 1 as 1x, and asks where ½ x is located, after they correctly respond the GL asks them to locate ¼ x.

0 ½x 1x
←--|--------|----------|------→

GL:	Where would ¼ x go on the number line?
Laura:	It would go between the 0 and ½ x
GL:	Could you be more exact?
Laura:	It would go exactly in the middle.
GL:	Why?
GL:	(turning to the other members of the group). How do we know that this position is ¼ x?
Fran:	Because half a dollar is two quarters.
Laura:	Because four of them make up a whole.
GL:	How do you know four of these segments make up the entire segment?
Sandy:	Because it is half the ½ x (After further discussion the group accepts this explanation).

Laura's explanation that four parts make up the whole can be interpreted as additive reasoning, assuming she iterated the ¼ x segment four times. Sandy's explanation can be viewed as evidence of splitting, she appeared to understand that two halves make a whole and thus half of a half would require twice this amount. Wilkins & Norton (2010) characterize such thought as being an early instance of multiplicative reasoning with fractions.

	(After some discussion, the group work continues with the following diagram).
GL:	Can you represent ½ x + ¼ x on the number line?
	(After receiving no answer she rephrases the question).
GL:	Okay, we know where ½ x and ¼ x are placed on this number line, now where would we place the value of the expression: ¼ x more than ½ x on this number line?
Fran:	(Points to the correct position).
GL:	Okay and why do you know this is correct?
Laura:	We count: 1, 2, 3 (pointing to three ¼ x segments as she counts).

GL: Okay but what did you change the ½ x into while you were counting?
(The students appear to intuitively understand but unable to articulate the reason, they are quiet so presently the GL continues, redirecting the question to the group)
GL: How many quarter x's did Laura use for ½ x?
Laura: I like um (pause), I used two.
GL: Why?
Sandy: Because it (pointing to the ½x) is two quarter x's (The others accept this explanation).

The group work continues, relating these measure concepts to the addition algorithm. Sandy who previously demonstrated the ability for splitting was able to articulate the equivalence of ½ x with $\frac{2}{4}$ x, this suggests a link between splitting and the multiplicative reasoning required in equivalence of fractions. Laura uses the visual representation of ¼ x and she iterates this three times to find the sum, this confirms her tendency towards additive reasoning specifically, her use of iteration to understand measure concepts. According to Wilkins & Norton (2011) the ability for splitting develops after and is more advanced than partitioning and iterating. As Laura demonstrated the ability for iterating but not splitting and could not articulating a reason why two iterations of ¼ x are equal to ½ x this supports their claim.

The measure tasks (Appendix) require both an understanding of the splitting and iteration processes. These findings suggest that students who have developed beyond iteration and understand splitting are more effective in relating measure to fractional equivalence and thus the addition algorithm. This suggests multiplicative reasoning, which is closely related to splitting, may be a prerequisite for relating measure to fractional equivalence and hence the addition algorithm.

If multiplicative reasoning is required for relating measure to the addition algorithm and such reasoning comes after the ability for iteration this would explain why multiplication is more influential in predicting measure than addition.

Conclusion

In this study, quantitative analysis is used to investigate the potential of two hypothetical pathways to competency with the operator and measure sub-constructs for adults reviewing fractions. The first pathway is through the direct extension of part-whole knowledge (Educational Approach, Kieren-Behr model). The second path is through the corresponding procedural knowledge of multiplication and addition of fractions (Developmental Approach, APOS-model). Baker et al (2009) and Charalambous & Pitta-Pantazi (2007) have previously used quantitative analysis to test the Educational Approach of direct extension of part-whole knowledge to sub-construct knowledge of: ratio, operator, quotient and measure with resulting proficiency with procedural knowledge.

Charalambous & Pitta-Pantazi (2007) conclude that for children knowledge of these different sub-constructs implies proficiency with fraction procedures. Baker et al (2009) conclude adults are more proficient with multiplication and addition than operator and measure.

This suggests that for adults these concepts are not necessary for proficiency with procedural knowledge while leaving open the question of which hypothetical pathway to conceptual competency of operator and measure demonstrates more potential; the Educational Approach with its focus on direct extension of part-whole or the Developmental Approach with its focus on procedural knowledge.

The results of this study demonstrate that part-whole knowledge and procedural knowledge of multiplication and addition represent separate pathways that complement one another in improving students' performance on tasks related to operator and measure.

In particular, when part-whole knowledge and procedural proficiency were used together as independent variables there were substantial increases in the percent of operator and measure explained over that explained by either variable alone. This suggests that an ideal pedagogy would integrate the Educational and Developmental Approaches to benefit students' concept competency. This answers the second research question by verifying the separate cognitive pathways model of Gray & Tall (2001).

Although both part-whole knowledge and procedural knowledge were significant variables in predicting operator and measure there were noticeable differences in the results for each concept. In the case of operator, multiplication was much more influential than part-whole.

This suggests that for adults, the Developmental Approach of procedures enabling concept knowledge as supported by the APOS model is more effective than the Educational Approach.

In the case of measure, part-whole was more influential than addition this supports the Educational Approach of direct extension of part-whole knowledge and the Kieren-Behr model of concept development. Thus, the answer to the first research question of which pathway to concept competency demonstrated more potential depends upon the concept under consideration.

In summary, these contrasting results between operator and measure demonstrate the need for flexibility in models of concept development as provided by the two separate pathways approach in the Gray & Tall model. Thus in addition to answering the first research question these results also highlight the substantial cognitive differences between children and adults reviewing fractions. For children the Educational Approach to operator was more appropriate while for adults the Developmental Approach was more suitable. Adults demonstrated a potential benefit through the integration of the Educational and Developmental approaches for both operator and measure that children did not. Thus concept development appears to be effected by the concept under consideration as well as the age-status of the learner. For this reason fraction pedagogy and curricula for adults needs to be developed independently from that for children and to provide maximum benefit for students should be tailored to the concept under consideration.

In regards to the open question concerning the relationship between measure and addition this study found the correlation between these variables to be significant and relatively strong. Although part-whole was the most influential predictor of measure the increase in percent variation of measure explained when addition is included with part-whole is noticeable. Thus adults see a relationship between measure and the additive structure of fractions which they have the potential to use in becoming competent with measure.

In regards to the second research question concerning the relationship between the additive and multiplicative structures in supporting measure, the results suggest that measure is best understood as an extension of part-whole. That being said, measure is closer to the multiplicative structure than the additive structure of fractions. One possible reason measure is closer to the multiplicative structure than the additive structure is suggested by the qualitative analysis. The student who used iteration to relate measure concepts to addition was not able to articulate her use of fractional equivalence. In contrast the student who understood the concept of splitting was able to articulate fractional equivalence. Splitting is closely related to multiplicative reasoning and it appears to be necessary for effectively relating measure to fractional equivalence and hence the addition algorithm. For this reason, an individual's ability to relate measure to addition may be dependent on their multiplicative reasoning.

Future Research and Pedagogy

The result that measure is part of the multiplicative structure of fractions is unexpected given its connection to the additive structure in Figure 1. Further research is required to determine if all of the sub-constructs are part of the multiplicative structure of fractions.

This would provide support for the work of Thompson & Saldanha (2003) as well as Lamon's (2007) intuition in using the Kieren sub-constructs as a basis for proportional and multiplicative reasoning.

In this study, the integration of the Educational and Developmental Approaches through part-whole and procedural proficiency demonstrates potential benefit for students' concept competency. However, Mack (2000) raises the pedagogical issue of student reluctance to reflect upon mental processes when a procedural algorithm is available. Her observation highlights the obstacles in developing a curriculum that successfully integrates these pathways. Although the development of such a curriculum remains a future endeavor we share some thoughts about how to design a pedagogy that integrates measure and addition while presenting students with a clear need for measure independent of the addition algorithm.

One pedagogical approach designed to emphasize the measure sub-construct is to alternate layers of addition and measure tasks in the curriculum. In this approach a critique of the group session is that it began and ended with the addition algorithm and this places to much emphasis on procedures. This approach would simply add another layer of measure tasks after the addition exercises. For example, asking students to place the improper fraction $\frac{7}{6}$ on a number line labeled with; 0, 1, 2 and the fraction $\frac{1}{3}$. This would require the coordination of splitting and iteration to extend part-whole knowledge to measure and demonstrate to students the importance of these skills separate from addition of fractions. Another approach is design tasks that integrate measure and addition rather than layer them. For example, asking students to locate or place the value of the expression: $\frac{1}{6}$ more than $\frac{2}{3}$ on the number line. This task can be accomplished through the use of addition and measure concepts or exclusively through the measure concepts.

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End notes

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(2) Problem Solving in Remedial Mathematics: A jump start to reform, CUNY College Collaborative Incentive Research Grant Program (2010), Czarnocha, Prabhu, Dias & Baker.

Appendix: Exercise Set

Following the work of Cramer et al (2002) factor loadings of each exercises which tell us how closely the exercise correlates with the component it has been placed in is listed and the expectation is that each exercise will have a factor loading or correlation value of at least 0.4.

The range of commonality values for the exercises-variables which indicates the percent of a variable's total variance that can be explained by the factors is listed.

Ideally this value should be at least 0.4 preferably 0.5 and all exercises by the present authors are at the 0.4 level or above. However we take 0.2 as a cutoff for variables that were used in earlier studies in order not to change the sub-constructs so much that these results reflect upon those obtained previously.

Addition & Subtraction

Factor analysis revealed there were two factors for addition and subtraction of fractions. The first component included addition of proper fractions.

1) (†) $\frac{2}{8}+\frac{5}{8}=?$

2) (†) $\frac{5}{8}+\frac{4}{5}=$

The second component contained the addition and subtraction of mixed number problems.

3) (†) $8\frac{1}{2}-2\frac{2}{3}=?$

4) (†) Subtract 13 $\frac{5}{7}$ from 20

5) (†) $4\frac{1}{6}-3\frac{3}{4}=$

6) (†) $5\frac{5}{8}+4\frac{3}{40}=?$

7) (†) Find the sum of $3\frac{2}{5}+5\frac{5}{6}$

Note, exercises 3, 4, & 5 which involve regrouping or borrowing and carrying over were added by the present authors to the earlier set used by Charalambous & Pitta-Pantazi (2007) and Baker et al (2009).

(†) factor loading value above 0.4 or 16% of the variable explained by the factor (Cramer et al, 2002)

The commonality values which indicate the percent of the variables explained by these factors ranged from 30% to 71% all but one above 40%.

Multiplication and Division

Multiplication and division contain one component with 7 exercises all of which have at least a 0.4 loading factor value.

1) (†) Simplify: $3\frac{1}{5}\div 4$

2) (†) $4\div\frac{5}{8}=?$

3) (†) $\frac{3}{4}\times\frac{1}{5}=?$

4) (†) $2\div\frac{1}{2}=?$

5) (†) $6\frac{3}{4}\times 4\frac{3}{7}=?$

6) (†) $\frac{1}{6}\times 24=$

7) (†) Find $\frac{3}{5} \times \frac{5}{8} \times 4000$

Note, exercises #1, 2, 7 & 8 were not in the earlier study of Charalambous & Pitta-Pantazi (2007) and Baker et al (2009).

(†) Factor loading value above 0.4 or 16% of the variable explained by the factor (Cramer et al, 2002) Since there is only one factor the commonality values tell use the same thing as the factor loading values these values ranged from 20% to 40%.

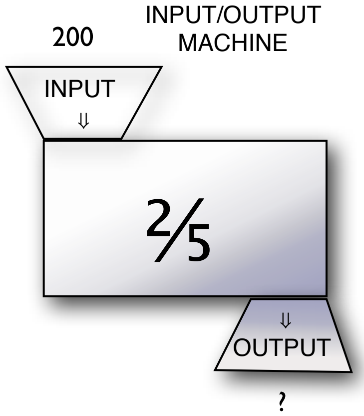
Operator

There were two components to the operator sub-construct the first contained the input out boxes and taking a fraction of a quantity. The second component involved comparisons of statements about taking a fraction of a quantity and their equivalent statements using multiplication and division.

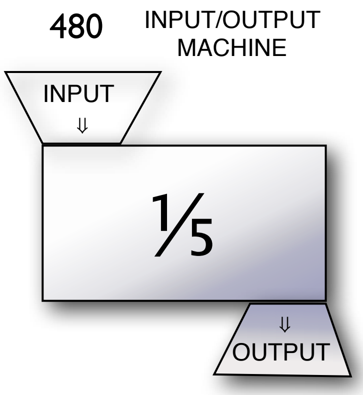
Component #1

1) (†)The following diagram represents a machine that outputs $\frac{2}{5}$ of the input number. If the input number is 200 then what is the output number?

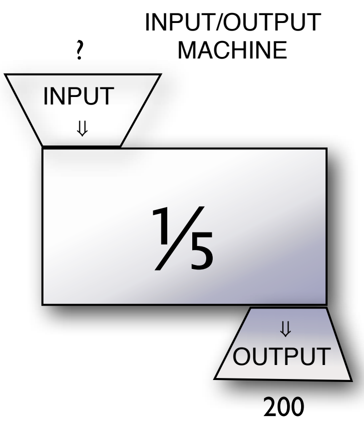
(Charalambous and Pitta-Pantazi, 2007, appendix)



2) (†) The following diagram represents a machine that outputs $\frac{1}{5}$ of the input number. If the intput number is 480 then what is the output number?



3) (†) The following diagram represents a machine that outputs $\frac{1}{5}$ of the input number. If the output number is 200 then what was the input number?



4) (†) Find $\frac{4}{5}\,of\,\frac{7}{8}\,of\ 40{,}000$

5) (†) Find half of 1 ½ hours

Component #2

6) (†) Taking ¾ of a number is the same as dividing the number by 4 and multiplying this result by 3. True/False?

7) (†) If we divide a number by 7 and multiply by 28 this is the same as multiplying by the fraction $\frac{1}{4}$ True/False?

8) (†) If we divide a number by 2 and multiply by 5 this is the same as multiplying by the fraction $\frac{2}{5}$ True/False?

Note, exercise #3 involving the inverse and item #4 involving the iteration or composition of taking a fraction of a quantity were not in the earlier studies of Charalambous & Pitta-Pantazi (2007) and Baker et al (2009).

(†) Factor loading value above 0.4 or 16% of the variable explained by the factor (Cramer et al, 2002)

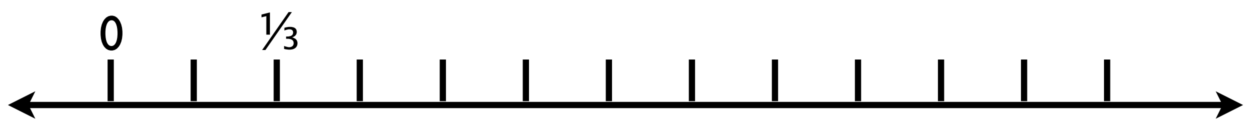
The commonality values which indicate the percent of the variables explained by these factors ranged from 22% to 71%.

Measure

In the measure sub-construct of a fraction, "first, it is considered a number … a quantitative personality of fractions, viz., how big the fraction is. Secondly, it is associated with the measure assigned to some interval…a unit fraction is defined 1/a and used repeatedly to determine a distance from a preset starting point...this means that students should be capable of locating a number on a number line and, conversely, be able to identify a number represented by a certain point on the number line." (Charalambous & Pitta-Pantazi, 2007, pp.299-300)

Component #1

1/2/3) Locate the following numbers on this number line, #1) (†) $\frac{1}{6}$ #2) (†) $\frac{4}{3}$ #3) (†) $\frac{5}{6}$



Component #2

4) (†) Locate the number one "1" on the number line below: Charalambous and Pitta-Pantazi, 2007, appendix)

5) Locate the number one “1” on the number line below: Charalambous and Pitta-Pantazi, 2007,appendix)

Component #1 and #2

6) (†)Which of the following fractions is closest to one?

a) $\frac{2}{3}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) $\frac{5}{6}$

7) (†) Circle the smallest fraction.

$A)\ \frac{2}{11}\quad B)\frac{3}{13}\quad C)\frac{1}{5}\quad D)\frac{4}{23}\quad E)\frac{5}{26}$

(†)Factor loading value above 0.4 or 16\% of the variable explained by the factor (Cramer et al, 2002) All except #5 satisfied the 0.4 loading factor requirement. The factor loading value for #5 was only 0.35 however, since it was part of the original set of exercises and increased reliability it was used in this set. Exercises #6,7 are listed in component #1 but could also be considered in component #2 that is they were found in both components. The commonality values range from 20% which to 70% all except #5 were above 45%.

Part-Whole

“The part-whole sub-construct of fractions is defined as a situation in which a continuous quantity or a set of discrete objects are partitioned into parts of equal size.” (Charalambous & Pitta-Pantazi, 2007,p.296)

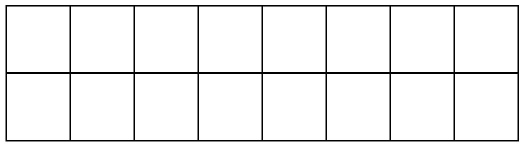
Part-Whole contained two components one that involved translations between pictures representing fractions to symbolic fraction notation and then back to an equivalent picture and the other involving one step translations between pictures and symbolic fraction notation.

Component #1

The first set of problems involved two steps. The first involved naming the fraction given by a picture diagram and the second involved reversing this process.

1a) (†)What fraction of the given stars is circled?

1b) (†) Shade the number of boxes, so that $\frac{3}{4}$ of the total boxes are shaded



2a) (†) Given a 2x3 rectangular array of boxes with one box shaded represent this as a fraction.

2b) (†) Then the student is given a collection of 24 equal objects and asked to shade in the appropriate number to represent the fraction obtained in 2a.

3a) (†) Three out of four equal objects are circled and the student is asked to identify what fraction of the objects are circled.

3b) (†)Given a 4x4 square array of boxes the student is asked to shade in the appropriate number of boxes that represents the fraction obtained in 3a.

4a) (†) Given a 2x3 rectangular array of boxes with 4 shaded the student is asked to determine which fraction is represented.

4b) (†) Given 24 equal objects with 4 circled the student is asked to represent the fraction of objects circled.

5a) (†)Given 16 equal object with 4 of them circled the student is asked what fraction of objects are circled.

5b) (†)Given a 2x2 square array of boxes the student is asked to shade in the appropriate number of boxes to represent the fraction obtained in 5a.

6a) (†) Given a 2x12 rectangular array of boxes with 4 shaded in the student is asked to represent the fraction shaded.

6b) (†)Given 6 equal objects the student is asked to shade in the appropriate number of boxes to represent the fraction obtained in 6a.

Component \#2

7) What fraction of the balls shown below are 7's? (†)

❼⓿✪

❼❼✪

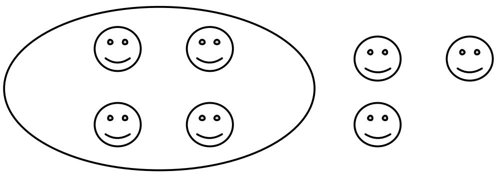
❼⓿✪

❼❼✪

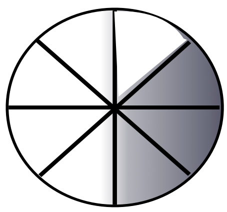
8) What fraction of the total number of 7's in the previous problem is represented by these three 7's? (†)

❼❼❼

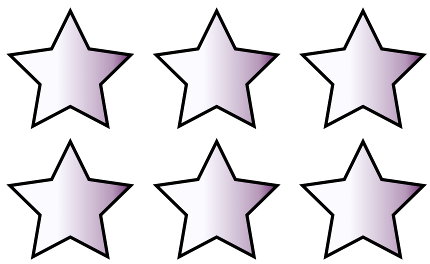
9) (†) Write the fraction of circled objects to total objects



10) (†) What fraction of the circle is shaded?



11) (†) If the following set of stars represents ⅔ of all the stars that Mark sees in the sky how many stars does he see?



(Charalambous and Pitta-Pantazi, 2007, appendix)

(†) Factor loading value above 0.4 or 16% of the variable explained by the factor (Cramer et al, 2002)

The commonality values ranged from 25% to 71% with only the inverse exercises #11 less than 57%.