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An Alternative Route to Teaching Fraction Division: Abstraction of Common Denominator Algorithm

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Abstract

From a curricular stand point, the traditional invert and multiply algorithm for division of fractions provides few affordances for linking to a rich understanding of fractions. On the other hand, an alternative algorithm, called common denominator algorithm, has many such affordances. The current study serves as an argument for shifting curriculum for fraction division from use of invert and multiply algorithm as a basis to the use of common denominator algorithm as a basis. This was accomplished with the analysis of learning of two prospective elementary teachers being an illustration of how to realize those conceptual affordances. In doing so, the article proposes an instructional sequence and details it by referring to both the (mathematical and pedagogical) advantages and the disadvantages. As a result, this algorithm has a conceptual basis depending on basic operations of partitioning, unitizing, and counting, which make it accessible to learners. Also, when participants are encouraged to construct this algorithm based on their work with diagrams, common denominator algorithm formalizes the work that they do with diagrams.

Keywords: Teaching fraction division, abstracting common denominator algorithm, curriculum development

Introduction

Arithmetic operations, and teaching and learning of them have always been an interest for mathematics education community. In his historical analysis, Usiskin (2007) pointed out that operations (especially on fractions) still preserve its importance in school mathematics and they should be given enough emphasis. Division is one such operation that has taken considerable attention by many researchers. The attraction to this operation is partly because of its complexity. This complexity is caused by the fact that division requires a meaningful organization of a variety of interconnected relationships (Thompson, 1993). In other words, division can be considered as a relationship between three quantities (dividend, divisor, and quotient) and an invariant relationship exists

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among these three quantities (Post, Harel, Behr, & Lesh, 1991). Here, the invariant relationship is meant to describe the multiplicative relationship between divisor and dividend, divisor and quotient, and dividend and quotient. Abstractly thinking about these relationships among the quantities in a division situation is difficult even for most teachers (Simon, 1993), which is one of the reasons why division takes considerable attention by many researchers.

Division is a complex operation to conceptualize and treatment of it within fractional domain makes it even more complicated for learners (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Ma, 1999; Sowder, 1995). The fact that division of fractions require conceptual proficiency in both division and fraction concepts (Armstrong & Bezuk, 1995) makes this area of mathematics problematic in the upper elementary and middle grades. One of the reasons for such problem is the fact that fractions, as part of the rational number set, itself has several different interpretations (Kieren, 1993) and division acting on that set makes this area more problematic. Therefore, division of/by fractions deserves a special attention in school mathematics.

Even though this topic deserves a special attention in school mathematics, research studies point out that teachers' understanding of this topic is not strong enough and they are not well-equipped to teach it conceptually. Teachers' understanding of division in fractional domain is closely associated with remembering a particular algorithm, invert and multiply algorithm (Ball, 1990), which is very poorly understood (Borko et al., 1992; Zembat, 2007) and dependent on rote memorization without conceptual basis (Li & Kulm, 2008; Simon, 1993). Teachers are not able to provide concrete examples or any rationale for invert and multiply algorithm (Ma, 1999). In fact, making sense of such an algorithm and conceptualizing it using the inverse relationship between multiplication and division is very difficult (Contreras, 1997; Tzur & Timmerman, 1997). In spite of this, a majority of teachers use it as a primary way to teach their students division of fractions (Ma, 1999). Most of the traditional mathematics textbooks make their introduction to division of fractions with this algorithm too. When explaining her previous experiences on teaching fraction division with the invert and multiply algorithm, a participant teacher from Sowder and her colleagues' (1998) study commented that

> "[…] one of my students said, "why do you flip it and why are we multiplying? This is division." And she [referring to the student teacher] says "Because I just told you to do it." And I sat there and thought, "Boy that was a wonderful question, and that was a very common answer." And I don't know how I would […] have to […] think about it to give more concrete examples." (p. 46)

Teachers' lack of necessary mathematical background to delve into the rationale for algorithms such as invert and multiply algorithm (hereafter abbreviated as IMA) is one side of the issue whereas feasibility of this algorithm is another. From a curricular stand point, the traditional IMA for division of fractions provides few affordances for linking to a rich understanding of fractions. On the other hand, an alternative algorithm, called common denominator algorithm (hereafter abbreviated as CDA), has many such affordances as explained in the following section.

The current study serves as an argument for shifting curriculum for fraction division from use of IMA as a basis to the use of CDA as a basis[1]. This was accomplished with the analysis of learning of two prospective elementary teachers being an illustration of how to

[1] Note that in countries like US or Turkey, education authorities (e.g., National Council of Teachers of Mathematics for US and Ministry of Education in Turkey) make curricular recommendations to teach CDA but most textbooks ignore them and used IMA as an initial basis to teach fraction division.

realize those conceptual affordances. The following section elaborates on the affordances and constraints of both algorithms through use of a mathematical analysis.

Meaning of CDA and IMA – Affordances They Provide?

Sharp and Adams (2002) indicated that IMA does not give learners enough opportunity to invent their own algorithm because of its complex and algebraically situated mathematical structure. This is not to say that IMA should not be included in school mathematics at all. On the contrary, as Sharp and Adams (2002) stated, learning of it should be delayed until after learners gained enough experience about the conceptual and procedural basis for division of fractions. Moreover, as a result of their synthesis of the extensive literature review in this area Sharp and Adams (2002) pointed out that CDA is most useful in developing a meaning for arithmetic as detailed below; it is meaningful and easier to be based on whole numbers whereas IMA as given in schools encourages learners to memorize it since learners find little sense in the procedure. The meaning of CDA is detailed below.

Given that the denominators of the dividend and divisor are relatively prime, an algebraic interpretation of CDA can be given as follows:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A \times D}{B \times D} \div \frac{C \times B}{D \times B} = \frac{A \times D}{C \times B}$$

Such an interpretation suggests that CDA with the above restrictions includes two phases: finding the common denominator for the dividend and divisor, and dividing the numerators. Details of this algorithm are given below.

Considering the division operation as a multiplicative comparison of two quantities, in other words as measuring one quantity with respect to other, requires making such a comparison/measurement on a common basis. This is not easily done if the dividend and divisor have different denominators. For example, comparing 1/2 to 4/5 is much more difficult for the problem (4/5)÷(1/2) than comparing 1/4 to 3/4 for the problem (3/4)÷(1/4) since latter one has a common basis, namely fourths, to compare divisor and dividend (e.g., 3 of the 1/4 can go into 3/4) whereas the first one does not have such a common basis because different denominators (e.g., fifths and halves) suggest different size-units to compare. Therefore, transforming the given two quantities into a form that enables one to make a direct multiplicative comparison between dividend and divisor is necessary (i.e., turning (4/5)÷(1/2) to (8/10)÷(5/10)). Once the two quantities are of the same type (with same denominators), division operation that is given in the fractional system can be interpreted as if acting in whole number system, which means dividing the numerators. For example, after turning (4/5)÷(1/2) to (8/10)÷(5/10) the question of 'how many 5/10 are in 8/10?' is same as 'how many 5 are in 8?' since we compare same size units, namely tenths). Therefore, teaching CDA provides learners an opportunity to tie their experience in this area to their whole number division knowledge.

The IMA, on the other hand, requires students to understand concept of inverse as part of the group theory as explained below and it depends on the use of multiplication instead of addition. There are two versions of applying IMA detailed below. In the first version one needs to understand that in order to find the answer for $(A/B) \div (C/D)$, the divisor, C/D, needs to be eliminated through multiplication of the inverse of divisor. Therefore, both the dividend, A/B, and the divisor, C/D, are to be multiplied by the inverse of divisor, D/C. In the second version one needs to understand that dividing A/B by C/D is equivalent to finding a number of C/D that is equivalent to A/B and understand the multiplying by inverse. Both versions of IMA are quite similar and seem to be hard to conceptualize by students (Sharp & Adams, 2002).

Version 1: $\frac{A}{B} \div \frac{C}{D} = \frac{\left[\frac{A}{B}\right] \text{x} \left[\frac{D}{C}\right]}{\left[\frac{C}{D}\right] \text{x} \left[\frac{D}{C}\right]} = \left(\left[\frac{A}{B}\right] \text{x} \left[\frac{D}{C}\right]\right) \div 1 = \left[\frac{A}{B}\right] \text{x} \left[\frac{D}{C}\right]$

[Version 1 using group theory: $\frac{a}{b} = \frac{a \times b^{-1}}{b \times b^{-1}} = \frac{a \times b^{-1}}{e} = [a \times b^{-1}] \div e = a \times b^{-1}$]

Version 2: $\frac{A}{B} \div \frac{C}{D} = X \Rightarrow \frac{A}{B} = X \times \frac{C}{D} \Rightarrow \frac{A}{B} \times \frac{D}{C} = X \times \frac{C}{D} \times \frac{D}{C} \Rightarrow X = \frac{A}{B} \times \frac{D}{C}$

[Version 2 using group theory: $a \div b = X \Rightarrow a = X \times b \Rightarrow a \times b^{-1} = X \times b \times b^{-1}$ $\Rightarrow X \times e = a \times b^{-1} \Rightarrow X = a \times b^{-1}$]

Even though the current literature points to the ways in which students and teachers reason about division of fractions and related algorithms, a limited number of studies suggested ways to think about developing a solid understanding of algorithms. An articulation of what it takes to abstract algorithms and a detailed description of the associated processes are necessary to design effective instruction. The current study uses an approach to help prospective teachers develop an understanding of CDA by referring to some of the activities (partitioning, unitizing[2], and counting) that are already available to them. In so doing, it investigates the following research question: *What are the conceptual affordances of CDA as reflected in the learning of two prospective elementary teachers*? The purpose of this study is not to generalize the findings gained from two participants to whole population of teachers. Instead, the purpose is to analyze the learning of two prospective elementary teachers being an illustration of how to realize the conceptual affordances that the CDA provides. The theoretical framework guiding this research is detailed below.

Theoretical Framework

Reflection on Activity-Effect Relationship framework (Simon, Tzur, Heinz, & Kinzel, 2004) and Piaget's (2001) description of different types of abstraction were used to design the instructional sequence and to explain participant prospective teachers' development of CDA in this study. In their framework that explains conceptual advancements, Simon and his colleagues (2004) proposed a model based on individuals' own (mental) activities and their reflections on those activities.

According to this framework, in a given problem situation, the learner is the one who sets the *goal*, which is the desired outcome toward which an activity is carried. For instance, a given problem would be "a cake requires 1/8 kg of sugar, how many cups of

[2] The term *unitizing* here refers to "the size chunk one constructs in terms of which to think about a given quantity" (Lamon, 1996, p.170). For example, turning a word problem modeling (3/4)÷(1/3) into (9/12)÷(4/12) by considering 1/3 as a unit of 4(1/12-unit)s and 3/4 as a unit of 9(1/12-unit)s are examples of unitizing 1/3 and 3/4. Through such unitizing one can reinterpret the given situation in the word problem in light of these new quantities.

sugar can be made with 3/4 kg of sugar?" and learner may be asked to solve it using diagrams only. The learner then may set the goal for this problem as "how many 1/8 are in 3/4?" This goal setting is dependent on the learner's available understandings. For example, here the learner may have the understanding of meaning of fractions, understanding of division of whole numbers and counting. Once the goal is set, then the learner pursues it based on his or her activity. Here, *activity* is considered as a mental action engaging the learner in service of reaching the set goal. To reach this set goal, the learner calls on an *activity sequence* (sequence of actions to reach the goal) that is already a part of his or her current conceptions. For our sample problem an activity sequence may involve: drawing 3/4, repartitioning 3/4 to make 1/8s, and counting number of 1/8s. As the learner engages in the activity (sequence), she or he attends to the *results* of it. For the sample problem situation the result of repartitioning 3/4 gives 6/8. Since the learner is the one who sets the goal, the assumption in this framework is that she or he can judge what results get the learner closest to the goal and what results cause deviation from the goal. Each attempt of going through the activity sequence and attempting to the results of it is recorded mentally as an *experience*. The learner mentally compares these records of experience, which results in his or her recognition of pattern(s) or regularities. For our sample problem the learner may think that the first question asks about number of 1/8 in 3/4, the second question asked about number of 3/5 in 9/4, etc., and realize the pattern that "so all questions asks number of one quantity within another." Through reflection on these regularities and patterns, the learner makes an *abstraction* that all such problems ask for the number of one quantity within another.

Here, abstraction is considered as the mechanism of constructing relationships in Piaget's (1971) terms. Piaget (2001) identified two types of abstraction: *empirical abstraction* that is ranging "over physical objects or material aspects of one's own actions" (Piaget, 2001, p.30), and *reflecting abstraction* that is the abstraction of the effects of actions (Piaget, 1983), abstracting the relationships between actions (Piaget, 1964), or abstracting the properties of action coordination (Piaget, 2001, p.30). According to Piaget, reflecting abstraction is the process by which new, more advanced conceptions develop out of existing conceptions.

In designing the current study, the aforesaid theoretical constructs were used for two distinct purposes. First, the analysis of fraction concepts serves to chart the learners' conceptual development through the process of instruction. Second, constructivist theorizing informs the pedagogical approach used in the study. In this sense, this study used a theory-based instruction design that only took into consideration what participants already had available as knowledge and helped them learn conceptions that were more complex than the ones they already had. The instruction was used as a main source of facilitating conceptual development of CDA.

Method

This study was based on a teaching experiment for which I benefited from Steffe and Thompson's (2000) teaching experiment methodology. In the current study, during the data gathering process, I acted as the teacher-researcher instructing two prospective elementary teachers and benefited from three other doctoral students who helped in observing the sessions, data gathering, and partial on-going data analysis. These outside observers witnessed the occurrences that took place in the teaching sessions.

The study consisted of ten teaching sessions and (pre- and post-) clinical interviews. The overall goal of the part of the study reported here was to promote and study participant prospective elementary teachers' conceptual development of the CDA to better understand the conceptual affordances provided by CDA. Therefore, this article basically draws on the analysis of the last two teaching sessions that was mainly designed to promote an understanding of the CDA. A detailed description of all sessions is provided in subsequent sections of the article.

Participants and Selection Criteria

The participants were two prospective elementary teachers from a northeastern U.S. university, who were in the fourth year of their elementary teacher certification program. One of the important factors that affected the selection of participants was the volunteers' knowledge of mathematics. I looked for volunteers who had a very basic understanding of: (a) Fractions, including what a fraction was, knowing how to name, show and represent them, and knowing what numerator and denominator meant; (b) Carrying out basic arithmetic operations on whole numbers and knowing what they meant; (c) Equivalent fractions. In addition, they were not to know about CDA for fraction division. Volunteers' initial understandings were assessed through one-on-one interviews and the ones who met the above criteria were invited to participate in the study. Two of them agreed to commit to the study for a whole semester.

There are several reasons for working with such a limited number of participants. Tracing the conceptual development of learners is very hard in classroom settings since those settings are comprised of a variety of different variables. Having limited number of participants helped me focus on their progress more closely as they engaged in the given task sequence and as they reflected on that sequence. It may be feasible to engage a classroom of learners in a task sequence but it is hard to investigate what aspects of the given task sequence caused difficulty for individuals, how individuals reason about those tasks, or how they reflect on those tasks. In addition, with a few number of learners it is more convenient for the teacher-researcher to facilitate participants' thinking, have them listen to each other, analyze and question each other's solutions, and purposefully reflect on what they did. By having only two participants, I had very few variables left at hand with respect to teaching and learning, and more time to zoom in on the aspects of participants' conceptual development of CDA. This approach is also supported by Steffe (1991) and Simon, Saldanha, McClintock, Akar, Watanabe, and Zembat (2010).

Data Sources and Data Collection

The data consisted of videotapes and audiotapes of the teaching sessions and of one-on-one interviews, the participants' written work produced during the teaching sessions and during the one-on-one interviews, and the field notes taken during and after the teaching sessions. Two interviews, the pre-interview and post-interview, were conducted to gain insight into participants' available mathematical understandings.

The participants (with the pseudonyms, Nancy and Wanda) agreed to meet twice a week, each for two hours and the teaching sessions were completed in five weeks. I designed the teaching sessions to be conducted in a particular format. Specifically, the participants were constantly encouraged to share their ideas, make conjectures, and justify those conjectures. They were not to use any arithmetic operation or algorithm unless they were told to do so. In all the teaching sessions, they were limited to diagrams and the available materials as primary sources for reference[3]. I then modified this sequence, as needed, in response to my analyses of the students' mathematical activity.

[3] Some of the initial ideas for the teaching sequence came from a set of problems designed by Prof. Martin A. Simon and then I further developed that sequence by drawing on participants' development throughout the teaching experiment. Prof. Simon was my PhD dissertation advisor at the Pennsylvania State University (USA) by the time I collected this data.

Throughout the teaching sessions, one of the three co-researchers operated a digital camcorder and an audio recorder, while at least one of the other co-researchers observed the sessions from a secluded corner of the room where she or he did not interfere with the recording or the implementation of the sessions. The focus for the observers was to capture participants' work as much as possible for analysis and to keep field notes pertinent to the important moments that transpired in the sessions. I myself taught the sessions without any interruption from the other researchers.

Tasks for Teaching Sessions 1-8 and Participants' Abstractions

As previously mentioned, there were total of ten teaching sessions. What follows is a brief description of these ten sessions and the participants' available abstractions before the last two teaching sessions.

First two sessions were about helping participants develop an abstraction of quotitive situations as division with fractions. The first teaching session included four main sections: Section 1 consisted of five real world problems modeling quotitive division, which need to be solved using diagrams only; Section 2 consisted of a problem asking about the commonality of the previous five problems and writing a generalization describing the commonality; Section 3 included a problem asking about whether the provided two real world problems (one modeling multiplication of fractions, another modeling division of fractions) fit the generalization provided by participants without actually solving them; Section 4 asked participants to create their own word problems that fit the generalization they already described. Throughout this session, the participants' work was limited to diagram use and the word "division" was absent. Even though the participants went through all the given tasks successfully in the first session, they were not able to create their own word problems modeling division of fractions at the end of the session. One of them created a problem that modeled a whole number division that is not appropriately structured whereas the other participant created a multiplication problem. This result pushed me to modify the tasks for the second session.

The second session, therefore, included a task sequence that helped participants make an abstraction of multiplication with proper fractions as quantification of the part of a given quantity in terms of the given quantity (e.g., $(1/4)\times(3/7)$ means how big $1/4$ of $3/7$ is). What followed this sequence in the second teaching session was another task sequence to help participants to abstract division of fractions, which was quite similar to the sequence given in the first teaching session. Then the participants were asked to make a comparison between the two sets of activities (one for multiplication with fractions, another for division of fractions) once they went through those. In this way, they had the opportunity to compare the activity sequences for both operations, made generalizations for those operations and compare those based on the activity sequences they went through. At the end of these two sessions, the participants' abstraction of division becomes dependent on quotitive situations whereas before the sessions it was about arithmetic relationships between dividend and divisor that gives quotient. In other words, they now considered the quotitive division (of fractions) as modeling quotitive situations and as an investigation of the number of one quantity within another quantity as opposed to a simple arithmetic operation that helps them find the missing factor, quotient, given two other factors; dividend and divisor. Namely, they can now think about questions like $(3/4)\div(1/2)$ with the understanding that it models 'how many $1/2$ are in $3/4$?' instead of thinking about that question as finding the value of X in $3/4\div1/2=X$.

Sessions 3, 4, and 5 included tasks to help participants make a distinction between partitive and quotitive division situations in fractional settings, which was not very helpful to them in terms of working toward CDA. However, these three sessions revealed the importance of understanding the relationship between divisor, fractional part of the quotient and remainder.

The sessions 6, 7, and 8 were about development of a solid understanding of remainder in whole number setting, moving to fractional setting and the abstraction of the relationship between divisor, quotient, and remainder by developing a sense for divisor as an intensive quantity (items/group) that connects the two extensive quantities (items, groups), dividend and divisor. These sessions used a similar format as the others but this time participants' attention was directed toward two different but related ways to interpret the results of division word problems: results only having quotient, results having whole number part of quotient and remainder. In doing so, I had the participants initially start working on contextual problems and then move to context-free problems because of the importance of realistic situations in developing mathematical concepts (Sharp & Adams, 2002; Streefland, 1991; Perlwitz, 2004). At the end of session 8, the participants had a solid understanding of division of fractions (as abstraction of quotitive situations), a sound understanding of a difference between partitive and quotitive division in fractional settings, and a solid understanding of remainder in both whole number and fractional settings.

As a result, upon entering the algorithm sessions (sessions 9 and 10), participants already had the abstraction that division of fractions means an investigation of the number of one quantity within another. They also had the notion that division is a multiplicative comparison of two quantities to get a third one. In addition, they abstracted the idea that there is a network of multiplicative relationships among the divisor, remainder, and quotient. They also had an abstraction of the role and meaning of equivalent fractions, referents (the dividend refers to the quantity at hand, divisor means quantity per group and quotient refers to number of groups) and coordination of referents. They went into sessions 9 and 10 with all of these abstractions.

Tasks for Teaching Sessions 9 and 10 with their Conceptual Analyses

The tasks given to the participants during the last two sessions were based on context-free problems. The problems consisted of division of fractions for which the denominators of the fractions were relatively prime numbers (see Figure 1).

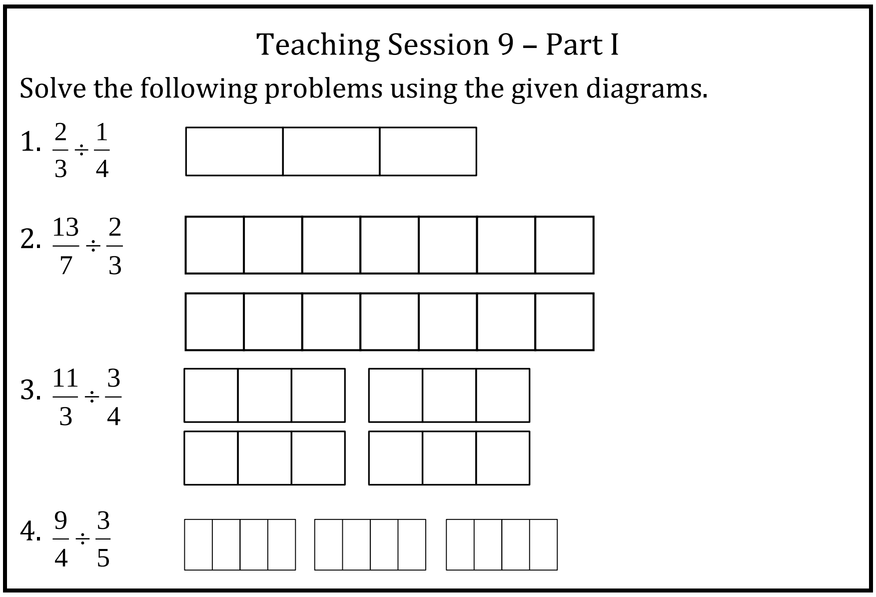

Figure 1. Illustration of tasks for Teaching Session 9 – Part I.

The purpose in doing so was to help participants not to get distracted by irrelevant solution strategies (e.g., partitioning the quantities vertically versus horizontally) and by some intermediate steps that would hamper the developmental process.

In the last two sessions, the main purpose was to have participants abstract CDA and then investigate the development of such an abstraction. Therefore, Teaching Session 9 and 10 served the purpose of helping the participants coordinate the understandings mentioned above in such a way that they would develop an algorithm based on their activities.

Teaching Session 9 - Part II

For the following problems, do not draw diagrams. Instead, write down in words each step that you would do if you were to draw diagrams.

Problem 1: $\frac{7}{2} \div \frac{2}{3}$	
Steps You Would Take	*Results of Those Steps*
1.	

Problem 2: $\frac{5}{8} \div \frac{2}{7}$	
Steps You Would Take	Results of Those Steps
1.	

Problem 3: $\frac{14}{15} \div \frac{3}{8}$	
Steps You Would Take	Results of Those Steps
1.	

Figure 2. An illustration of tasks for Teaching Session 9 – Part II.

Problems of Part II in Teaching Session 9 (see Figure 2) included the same activity sequence as in Part I but this time the participants were not to solve the given problems using physical drawings. Instead, they need to solve them mentally benefiting from mental diagram work. The fractions used in each question were bigger and messier than the ones used in previous ones. The purpose for not allowing participants to solve the problems with physical drawings was to help them mentally reflect on the activity sequence they had and move them toward an algorithmic thinking about the sequence.

For each problem, the participants were to think about every step they would go through to solve the problems, as if they were using diagrams, and the results of each step. In this way, they were to think about what should be drawn first and then to note the corresponding result of that action, and continue in that manner. Meanwhile, they were not allowed to physically draw any diagram or use any formulae. This way of operating was important in order to help participants develop an anticipation of the activities they would go through and the associated results. Helping them develop such anticipation was thought to be useful. That is, helping participants reflectively think about the activity sequence and associated results and make an abstraction would be possible.

As the fractional quantities got bigger and messier, the participants were encouraged to think about how their activities affected the size of the dividend and divisor, and the overall goal. They were to learn two things in this process: (1) knowing that common denominator results in same size units for divisor and dividend; (2) knowing that when the dividend and divisor are based on same size partitions, quantifying the number of partitions (that make up the divisor) within the dividend is same as dividing the numerators.

Teaching Session 10 – Part I				
Problem 1	$\frac{3}{2} \div \frac{2}{5}$	What is the goal in this question?	What needs to be done?	For what purpose?
	STEP 1 STEP 2 ...	...	...	...
Problem 2	$\frac{8}{3} \div \frac{3}{4}$	What is the goal in this question?	What needs to be done?	For what purpose?
	STEP 1 STEP 2 ...	...	...	...
Problem 3	$\frac{22}{5} \div \frac{2}{3}$	What is the goal in this question?	What needs to be done?	For what purpose?
	STEP 1 STEP 2 ...	...	...	...

Figure 3. An illustration of tasks for Teaching Session 10 – Part I.

Teaching Session 10 consisted of Part I and Part II: three initial problems in Part I, followed by another similar three in Part II. For Part I, similar to the previous session, participants were asked to think about the steps they would take if they were using diagrams mentally in solving the given fraction division problems. In solving the problems, they were to answer several questions as illustrated in Figure 3. For each step, they were to identify the specific goal, the action to be taken and the purpose of that action. If the change in the type of the quantities affected the goal, they needed to restate the goal in the appropriate column. For example, when the common denominator for the given fractions was found, the initial overall goal, finding for example number of 2/5 within 3/2, was to be changed to "finding the number of 4/10 within 15/10." The purpose for following such a method was to encourage them to think about why they were doing what they were doing rather than having them go through additional similar type problems. In addition, in the previous session, they were changing their goals by basing their discussion on the numeric results (by unitizing the dividend and divisor) without paying attention to the nature of that change in goals. Such structuring of the questions was to help them reflectively think about the change in the overall goal and its affects in the solution process.

Note that Part II of Teaching Session 10 was similar to Part I except that the given fractions required messy calculations (e.g., [21/38]÷[7/98]). Since participants made the necessary abstractions for CDA once they completed Part I, there was no need to apply Part II and therefore, it was skipped. Thus, Part II is not included in this paper.

Throughout Teaching Session 10 the participants were allowed to use calculators to find the result of messy calculations once they talked about what they need to do.

Therefore, even though the numbers get messier for each subsequent question, because of calculator use, they were not to deal with the calculations but the methods they would employ to find the results. In this way, the participants also had the opportunity to reflect on the meaning of the activities in the activity sequence they were going through.

Data Analysis Procedure

Analysis of the abstractions participants had prior to last two teaching sessions, the retrospective analyses of the last two teaching sessions and analysis of the post-interviews helped me characterize conceptual advances of the participants. The reason for mostly drawing on the analyses of the last two teaching sessions for this paper is because they are specifically related to the development of CDA.

In analyzing the data, I initially identified parts where the participants did not have a certain understanding and then I located the places where they had that understanding. Then, benefiting from the aforesaid theoretical constructs (e.g., goal setting, activity, activity sequence etc.), I explained the learning trajectory of the participants by using evidence throughout the data. I also investigated the reasons for such shifts in understanding. In explicating on the learning of participants, I identified places where the participants only focused on the numeric aspects of the given tasks and where they reflectively abstracted concepts as well as the nature of shifts in between.

In doing so, I constantly tried to formulate hypotheses about the participants' evolving understandings and made claims, and tried to support those with the data at hand. The ones for which I was able to provide considerable support were then stated as claims. Once the claims were made, I also looked for counter evidence for such claims. When a hypothesis was generated or a claim made, I searched throughout all the data to check to see whether there was contradictory evidence. Finally, using the collection of claims I had, I organized them to help model the participants' evolving understandings pertinent to CDA. Throughout this process all these categories and claims were discussed with a PhD mathematics educator and continuously reviewed and revised.

Results

Participants' Work with a Particular Activity Sequence in Session 9

To help the participants develop a sense for the CDA and how it functions, they were given four context-free problems in teaching session 9 as illustrated in Figure 1. They had gone through the similar sequence previously but this time the main focus of the session was on the given task sequence to develop an algorithm. The participants solved all four problems using very similar solution processes without any difficulty in about ten minutes. Both participants initially solved the problems alone and then one of them explained her solution on the board with a follow-up discussion. What follows is one example [for (9/4)÷(3/5)] to exhibit participants' approaches and their thinking processes about the mathematical relationships hidden in the problems.

> W: [*drawing three rectangles and partitioning each into four pieces vertically as in Figure 4.1*] Okay, so we have our dividend. I am not using these three [*pointing to the shaded three pieces in Figure 4.1*] because we only have nine fourths [*pointing to unshaded parts in Figure 4.1*]. And another [*partitioning each whole rectangle into five pieces*

horizontally as in Figure 4.2]. Okay. So I divided [*each whole rectangle*] into fifths because we want to know how many three fifths are in nine fourths. So, um, since these are fifths [*pointing to horizontal sections in the first rectangle of Figure 4.2*], we want to count by three fifths so here is one thing of three fifths [*circling the upper three rows of the first rectangle as in Figure 4.3*] -

R: Uh huh.

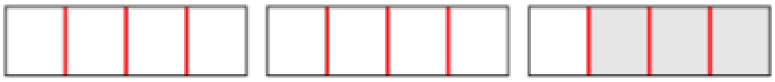

Figure 4.1

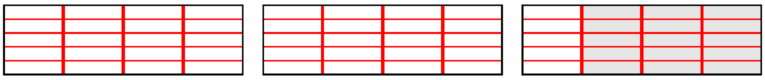

Figure 4.2

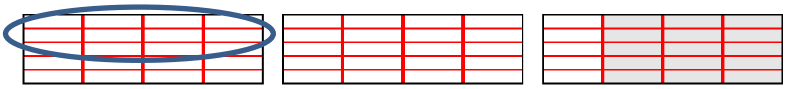

Figure 4.3

Figure 4. Representation of participants' drawings for (9/4)÷(3/5).

W: And here is two things of three fifths [*marking the bottom two row of the first rectangle together with the utmost row of the second rectangle*], and here is three things of three-fifths [*circling the second, third and fourth row of the second rectangle together*] and we don't have enough so there is three [*referring to 3 circled divisor groups*]. And we don't have enough to make, um, another three-fifths so in three fifths, there is twelve of these little things [*pointing to the pieces of the size 1/20 in the first marked 3/5-group*]. And we only have nine [*pieces of the size 1/20 unmarked*], so there is nine twelfths of another three-fifths left. And -

R: What's the? Okay and what?

W: And, um, why divide like something into twelfths when you can have it simpler [*referring to 9/12 and its simpler form 3/4*] so there is three, it could be three [*and*] three fourths instead [*considering the answer as 3-and-3/4*].

As seen in the above episode, Wanda set her goal as to find the number of three fifths within nine fourths. The analysis illustrating the activities and the corresponding results Wanda (and also Nancy) generated to reach that goal in this problem and in all other problems of Part I of Teaching Session 9 was given in Table 1.

Because of their appropriate choice in referents and their accurateness in referring to the important multiplicative relationships (between the divisor and quotient, and remainder and fractional part of the quotient), the participants followed this activity sequence and attended to the associated results without any trouble. In addition, this sequence was similar to their previous experiences in the prior sessions but here the focus was to be on developing an algorithm, which will be further investigated in the following sections.

When the participants had doubt about the parts of their activity sequence, they either reminded themselves about the overall goal for the problem (looking for number of a within *b* for a problem like *a*÷*b*) or they checked the referents for dividend, divisor, and quotient to decide on what to focus on. Such adjustments within the task sequence helped them organize their thinking in approaching the given problems more appropriately.

Table 1. Participants' activity sequence for Part I of Teaching Session 9.

Mental/Physical Activities	*Corresponding Results*
(1) Draw unit wholes (as rectangles), partition them, and shade out the necessary (vertical) partitions to identify dividend	Diagrammatic representation of dividend
(2) Partition horizontally each unit whole to allow for marking divisor-size groups	Diagrammatic representation of divisor
(3) Unitize divisor and/or dividend according to the new partitioning	Diagrammatic representation of the unitized dividend and divisor that have same denominators
(4) Identify full divisor groups within dividend by either numbering partitions that makes up a group with the same numeral, or by grouping the partitions first and numbering each group as a single whole	Numeric result of whole number part of quotient
(5) When there is not enough partitions to make another divisor group, multiplicatively compare the number of remainder partitions with the number of partitions that make up a divisor group	Numeric result of fractional part of the quotient
(6) Identify quotient using the results of activity 5 and activity 4	Numeric result of quotient

Note that even though Wanda and Nancy followed such an activity sequence, we cannot assume that they reflectively think about their sequence in the course of solving the problems. Therefore, the second part of Teaching Session 9 was given to have them consciously reflect on that sequence.

Developing the CDA

Once the participants solved the first problem of Part II in Teaching Session 9 (see Figure 2), I asked them to tell me what they wrote for each activity and the corresponding result, and then I was only drawing what they directed me to draw on the board without any interruption. After the first problem, the discussion was about the activities and the corresponding results without going into the actual drawings. Their solutions consisted of two-way partitioning (horizontal and vertical partitioning of the wholes making up the dividend) and they paid considerable mental attention to the referent units and the involved multiplicative relationships among the divisor, dividend, and quotient (e.g., quotient refers to the number of divisors within dividend).

Participants were able to anticipate the results of the hypothetical activities to be taken in representational world without physically working in that environment. For example, for problem 3 of Part II [(14/15)÷(3/8)], the participants explained what steps to take and the corresponding results appropriately. In this section, the sentences within the quotations are actual wordings of the participants. Since the participants either accepted or explained the rationale of each other's actions, I used the pronoun "they" instead of individual names in this section.

They first identified their overall goal as figuring out the number of 3/8 in 14/15. They then indicated that they should "draw a whole […] then divide the whole into fifteen equal parts vertically, […] shaded out one fifteenth" to get the dividend, 14/15. They then pointed out that they needed to "divide the whole into eighths horizontally" which results in 120/120. In this new diagram, 14/15, the dividend, becomes 112/120 and the 3/8 becomes 45/120. Then, using the 3/8 (i.e., 45/120) as a unit, they stated they will, in fact, be "count[ing] the number of one hundred twentieths in three eighths," so they were set to "mark off every 45/120 in 112/120."

Here, there is a shift in their initially set goal. Their overall goal was to find number of 3/8 in 14/15 but now it takes the form of finding number of 45/120 within 112/120. This adjustment in the fractions and in the overall goal was enabled by unitizing both quantities of 3/8 and 14/15 in terms of 1/120ths – in other words, they made the denominators common. This also helped them unconsciously turn their initially set overall goal into the goal of finding number of 45/120 in 112/120. This shift in the overall goal was natural for the participants since they were only focusing on the numeric aspects of the problem as opposed to reflectively thinking about the activity sequence they were going through. Otherwise, they could have figured out the algorithm at this point.

Once they had the unitized dividend and divisor, they started counting the number of 45/120 in 112/120. Perhaps, by benefiting from the numeric relationship between 45 and 112, they realized that there were two-whole 45/120 in 112/120 with a remainder of 22/120. They then interpreted 22/120 as 22/45 of another whole group of the size 45/120 by multiplicatively comparing 22/120 to 45/120. That is, they measured 45/120 by using 22/120. This measurement resulted in 22/45. As a result, they announced the quotient as "2 and 22/45."

As seen in this solution method, even though the numbers were increased, the participants were still able to think mentally about the activity sequence they had (see Table 1) and the associated results (see Table 1), and applied it efficiently to the question of (14/15)÷(3/8). However, they were still working numerically and not attempting to think about ways to consider the problems with an algorithm.

At times, when the participants thought they were having difficulty, they reminded themselves about the overall goal for the problem and refocused themselves on the solution process. However, they were able to execute the activity sequence they already had mechanically. In addition, they did not reflect on the parts of the activity sequence to formulate a way to think about the fraction division problems more efficiently since they did not have an abstraction of the CDA yet. This was also because they were only solving problems having dividend and divisor with relatively prime denominators. The Teaching Session 9 ended at this point.

Abstracting the Numeric Aspects of the Algorithm

Previously, they went through certain activities but they did not question the rationale for those activities without my prompts. The task sequence given in Part I of Teaching Session

10 was intended to have them reflect on those rationales for their actions. What follows is an example of how they went through Problem 2, (8/3)÷(3/4). Note that the sentences given within quotations are what participants said during the session. They initially identified their overall goal as "How many three fourths are in eight thirds?" and mentioned that they needed to "draw three wholes and divide [each] into thirds. […] shade out one third […] to get 8/3." At this point, they knew that "you still have the same goal." After restating their goal, they continued as "split them into fourths horizontally […] to make groups of three fourths." Now, their overall goal was changed to "find the number of nine twelfths in thirty two twelfths" for which they needed to "group together nine twelfths as many times as possible."

This description suggests that the participants went through certain mental activities as follows:

1. They initially identified the dividend (drawing enough rectangles, partitioning them, and marking enough of them to identify dividend);
2. They identified the divisor (repartitioning the dividend and grouping enough partitions within the dividend to identify divisor);
3. They counted the number of divisors within the dividend (grouping a number of partitions that make a full divisor group, counting such full groups), and if there was a remainder, they made a multiplicative comparison between the divisor and leftover by measuring the leftover with the divisor;
4. Finally they noted the result of that comparison as the fractional part of the quotient.

Once they identified the dividend and then the divisor by unitizing the dividend, they actually found the common denominator of the divisor and the dividend. When they counted the number of divisor groups (certain number of partition groups) within the dividend (total number of partitions in dividend), they actually counted a number of partitions within total number of partitions, which was same as dividing the numerators of the dividend and divisor. However, they did not seem to pay enough attention to these facts yet.

As they went through this sequence, they began to see some numeric pattern among the results. By looking at (8/3)÷(3/4) and (32/12)÷(9/12), Nancy mentioned

> N: Well, I don't know if it is just coincidence but it's thirty two over nine [*referring to the result, 32/9*] and there is a thirty two, like you can cross out the twelfths and then there would be thirty two divided by nine.

When encouraged to think about what it means to "cross out those twelfths," Nancy's response was

> N: Well, since you are both being divided by the same thing, can you just divide them by each other?

whereas Wanda confirms "it works." This realization was based on their attention to the numeric patterns among the results of their activities since they also agreed that they did not know why there would be such numerical pattern. The rule they used, at this point, consisted of finding the common denominators and canceling out those common denominators. They derived this rule from the numerical pattern by comparing the numeric results of the activities they went through for several problems, but they did not know the rationale for such a rule yet.

For Problem-3 of Part I of Teaching Session 10, (22/5)÷(2/3), they went through a similar activity sequence and generated a result mentally. When they were asked about the reason for changing the nature of dividend and divisor (through unitizing), they reasoned as in the following episode:

> N: So you are working with the same like the wholes that are divided into same number of parts.
>
> W: Hmm hmm.
>
> R: Like in this case, fifteenths?
>
> N: Yeah. Instead of working with fifths and thirds.
>
> R: So this [*pointing to 66/15*] tells us what?
>
> N: That tells us what twenty-two fifths [*is*].
>
> R: Twenty-two fifths and two thirds [*writing 2/3 next to 10/15 on the board*]. Why didn't we focus on these [*pointing to 2/3 and 22/5 in (22/5)÷(2/3)*] and moved to here [*pointing to 66/15 and 10/15*]?
>
> W: What she said.
>
> N: Because there, it was just hard to figure out like equate thirds and fifths together.

The above episode suggests that they seemed to understand the rationale for finding the common denominators as generating same size units on which the divisor and dividend were based. This understanding seems to be resulted from their reflection on the change in units and the unitizing process for the initial dividend and divisor. This is not to say that they did not know the rationale for equivalent fractions previously. Instead, they were beginning to pay attention to the shift from one form of dividing fractions, (22/5)÷(2/3), to another, (66/15)÷(10/15), and reflecting on that shift. And this shift became meaningful by calling on their understanding of the equivalent fractions. Hence, they became more conscientious about the role of unitizing in dividing fractions.

Focusing on the Rationale for the Explored Numeric Pattern

When the participants went through the individual activities and the results of those, the numerical values they encountered for the divisor and dividend seemed to have the same denominators. They realized that the denominators of both quantities were being equated numerically.

As illustrated in the last episode, it was coming together for Nancy as she made some reflection on the activities and the results associated with those. She came to realize that the comparison between 22/5 and 2/3 was not as easy as the comparison between 66/15 and 10/15. In one case, there was no common ground to compare the two fractions whereas in the other case there was a common denominator. In the first case, identification of the multiplicative relationship between dividend and divisor was almost impossible in a diagrammatic approach whereas in the latter case it was easier for them to think about the involved multiplicative relationships among divisor, dividend, and quotient. When I asked them about the relation between 66/15 and 22/5 in a diagrammatic environment, Nancy reacted as,

> N: Well, you divided the twenty two fifths into thirds, so there are three times as many pieces. […] we have five fifths but you divide each fifth into three parts. […] and so the fifths, you have fifteen and so that twenty two pieces, we have sixty six pieces. But because both of them stay the same, I mean.

Wanda also supported this argument. In this setting, participants realized that repartitioning an already partitioned quantity such as 22/5 by a certain factor (e.g., 3) requires a relative proportional increase between numerator and denominator. Their call on equivalent fractions was important since it was the basis for understanding the rationale for finding a common denominator. This realization on participants' part seemed to be because of their attention to and reflection on the purpose of changing the form of the dividend and divisor by keeping the sizes constant. Prior to such attention provided to them in the task sequence in Figure 3, they were just mechanically going through the activity sequence without reflecting on the pieces of it and the role of equivalent fractions. However, with my prompt, they were encouraged to reflect on the rationale for adjusting the given quantities and adjusting the overall goal. The next step for participants was to develop an understanding of the second part of the algorithm: dividing the numerators.

Making Sense of Dividing the Numerators

Nancy and Wanda observed that dividing the new equivalent forms of dividend by divisor would give the same result as dividing the numerators if the denominators were same. They initially were thinking about a canceling method with which they had familiarity from probably their early schooling. However, I encouraged them to think back to their diagram activity so they could abstract an understanding of why this relationship existed.

> R: Why are we dividing sixty six by ten? You are saying we are canceling these out, how does it appear in the diagram?
>
> W: I don't know. We kind of know we are working in fifteenths so.
>
> R: Okay, you are working with fifteenths but why would you divide sixty six by ten?
>
> N: Well, because there are sixty six total pieces that we're working with. And we are grouping ten pieces together.
>
> W: As many times as we can. [*Nancy repeats what Wanda said*]
>
> R: Okay. How is it related to sixty six divided by ten?
>
> N: Because that would be the same thing as dividing sixty six by ten.
>
> R: What does sixty six divided by ten tell us?
>
> W: It says how many groups of ten are in sixty six.
>
> R: Okay [*writing what Wanda said on the board*].
>
> […]

N: Like everything is in fifteenths. Like both when we look at sixty six, it's sixty six fifteenths in the whole thing. And we want groups of tens, ten fifteenths, so.

R: So you are trying to figure out number of ten fifteenths in sixty-six fifteenths, which is same as -

N: How many tens are in sixty six.

Based on the diagrammatic representation of grouping 10/15 partition within 66/15, they seemed to think that the actions both divisions required were the same. In each case, there was a grouping action of ten pieces. And therefore, they thought that both divisions resulted in the same answer. They considered the denominator as the common size of the pieces. The question of "How is (66/15)÷(10/15) related to 66÷10?" encouraged them to think about the relationship between those two expressions. But then, since they were working with same size pieces, this realization led them to think about the process for both division cases as investigating the number of 10 objects within 66 objects of the same size. The diagram in a sense was hiding this fact since they were counting 10-piece groups within 66 pieces. In doing so, the size of each piece (1/15) was being hid by the diagram unless questioned. However, their focus on the relation between the use of common size pieces (1/15) and the nature of grouping activity within diagrammatic work (10-piece groups within 66 pieces) helped them reflect on what was being hid behind the diagrammatic representation (looking for 10-piece groups within 66 pieces is same as looking for number of 10/15 within 66/15). Their fluency in solving the subsequent problem, (23/24)÷(3/7), also suggested that they abstracted the rationale for dividing fractions. The subsequent problem was (23/24)÷(3/7) and they needed to solve it mentally by identifying the activities they would go through with the associated results, and they did it successfully.

Above examples of participants' work from the last teaching session suggests that once they explained their activities and the results they would get from those activities, they mainly pointed to two outcomes: finding the common denominators and dividing the numerators. This realization came from their treatment of the activities to generate dividend and divisor as single entities. They knew that their initial goal was to determine dividend even though it might include several steps to reach that goal. The next goal for them was to identify the divisor even though it might mean a new set of activities. Once the dividend and divisor were determined this way, their new goal, which was an adjustment of the old one, was to identify the multiplicative relationship between the unitized divisor and unitized dividend. However, this time such identification was easier since both quantities were based on same size partitions. The partitioning they did so far to figure out divisor and dividend resulted in two quantities of the same type to be multiplicatively compared. At this point, they abstracted the relationship that the identification process was about simplifying the multiplicative comparison between the given two quantities (divisor and dividend). They also had dividend and divisor as two single entities to be compared. And the problem at this point was to make sense of that multiplicative comparison. Conceptualizing the divisor and dividend as single entities led them to abstract the multiplicative relationship between those two quantities as manifestations of finding one object (of a certain size) within another object (of the same size), which was about division of numerators. The last teaching session ended at this point.

The Participants' Understanding of CDA as seen in Post-Interviews

The results of post-interviews also showed that Wanda and Nancy had an abstraction of the algorithm and its fundamental pieces. Even though the post interviews were conducted three weeks after the teaching sessions ended, the participants seemed to hold the necessary understandings required for articulating the meaning and functioning of CDA. During the post-interview, the participants were asked a question that consisted of an algorithm for a specific example as follows:

Question: Mary claimed that to divide two fractions, you change all mixed numbers into improper fractions, find common denominators, and then divide the numerators. For example, (3 $^{4/5}$)÷(2/3)=(19/5)÷(2/3)=(57/15)÷(10/15)=57/10=(5 $^{7/10}$). Will this method always work?

In order to answer such a question, they needed to know that there was a reduction of fractional division to the whole number division. And they needed to know that this reduction was possible by making both the divisor and dividend quantities having same units.

Wanda was aware that the first part of the algorithm was about equivalent fractions and she explained it as:

> W: Because nineteen fifths and three and four fifths. Although they are in different forms, they still represent the same amount of something. […] And since those represent the same amount, you need to put them in like the same proportion so that you can see them like side by side as equal things. So […] finding the common denominator would do that.

For Wanda, the reason for finding the common denominators was to "compare the quantities because we know that they are the same size pieces." As seen through her wording, she referred back to her equivalent fractions understanding. In a sense, she was also referring back to the diagrammatic approach she would use for such a division problem. In this way, she knew that finding equivalent form of a fractional quantity did not affect the size of the quantity at all. In this manner, to Wanda, it was possible to turn the given dividend and divisor into improper fraction mode. And since the goal for the division problem stayed the same, this change would not affect the result. She also seemed to be aware that she needed to make a multiplicative comparison between the two quantities, and the comparison could easily be done when the involved quantities are based on the same size fractional units.

Nancy's reaction was not much different from Wanda's in interpreting the common denominator step:

> N: […] if you look at the same whole, fifths are smaller than thirds. So you can't really compare fifths and thirds. But fifteenths, I mean if you find the common denominator so you do change the numerators but they remain equivalent, like the new numerators here, they are equivalent to the prior fractions but now you have the same base. So the fifteenths are the same size as these fifteenths. So you don't even really have to worry

> about the size of them, just the number of things that are being divided. Like, the fifty seven divided by ten.

Here, Nancy referred to the fact that changing the number of partitions proportionally for a fraction did not affect the size of the fraction.

When the issue was to explain the rationale for the second part of the algorithm, Wanda based her rationale for dividing the numerators on the fact that she used the same size pieces. Wanda seems to think about her diagrammatic approach and about the activities that she would go through for such a step. Since she already identified the dividend and divisor, she seemed to know that the division 57÷10 was conceptually and procedurally same as the division (57/15)÷(10/15). In addition to Wanda, Nancy also pursued a similar reasoning to make sense of the division of numerators, the second part of the algorithm.

As a result, their answers to the post-interview tasks showed that the participants had an understanding of the common denominator algorithm and were able to provide the rationale for each step of the algorithm. They stated that having a common ground for both given fractional quantities was a way to reduce the complexity in the given fraction division problems. They also knew that in this way, one could think about the quantities (divisor and dividend) as objects of a certain size. And, as long as the size of the objects matched with each other, they would think about the investigation of number of one object within another in different ways ("number of 2/3 in 3-and-4/5" ≡ "number of 10/15 in 57/15" ≡ "number of 10 in 57"). In this sense, the appearance of the object did not affect the overall goal and functioning of the operation for the problem.

Discussion and Conclusions

The current study contributed to the current mathematics education literature by analyzing the learning of two prospective elementary teachers as being an illustration of how to realize the conceptual affordances provided by CDA. This is further explained below.

Conceptual Affordances Provided by CDA: The process of developing CDA consists of several developmental steps that are based on learners' activities that they held before the instruction. First, it requires a multiplicative comparison between the given two quantities. To do such a comparison there needs to be a simplification of the given quantities if they are not easily comparable to each other. This simplification process is based on identifying the given quantities and unitizing them to make them refer to the same referents (by referring to understanding of equivalent fractions) so that they can be easily multiplicatively comparable. This type of unitizing results in the modification of the initially set overall goal. If the initial given problem is (3/4)÷(3/7), for example, then after simplification process one gets (21/28)÷(12/28) which leads one to modify the initial goal according to these newly unitized quantities as: "How many 12/28 are in 21/28?" which is same as, "How many 3/7 are in 3/4?" Modification of the overall goal sheds light on the multiplicative comparison to be done between the unitized divisor and dividend. Here, one other developmental step is that the multiplicative comparison between 21/28 and 12/28 is the same as the one between the numerators 21 and 12 since both comparisons are based on the same overall goal of finding number of 12-partition groups in 21-partition (same size) groups. Use of diagrams in this sense reduces the division on fractional dimension to division on whole numbers dimension. The result of such realization takes care of developing a sense for the second part of CDA, dividing the numerators. Since the investigation of 12/28 within 21/28 is based on same size fractional units (1/28), the same investigation can be considered when looking for 12 units within 21 units of the same size. In other words, it requires one to think about both fractional quantities as objects to be compared multiplicatively.

As a result, the CDA has a conceptual basis depending on basic operations of partitioning, unitizing, and counting, which makes it more inventible by participants since these operations are already available to them. CDA differs from IMA in this manner. Also, when participants are encouraged to construct this algorithm based on their work with diagrams, CDA formalizes the work that participants do with diagrams. In working with diagrams, learners need to go through a well-articulated activity sequence that they could refer to, whenever needed. Reflecting on the purpose of each activity in the activity sequence is essential for reflective abstraction. Otherwise, one only thinks about generating ways to think about the transition between different steps as opposed to reflective abstraction. Therefore, it is important for the learner to keep in mind the goal of each step and make a comparison based on the tri-set: goal-activity-result. This kind of reflection results in thinking about the algorithm independently of its numeric base.

Fostering the Development of CDA: The development of CDA consisted of two sub-steps. The first one was to help the participants develop an understanding of the rationale for using same size units to multiplicatively compare two given fractional quantities. Then the next step was to help them develop an understanding of the idea that dividing two fractional quantities had the same structure as dividing numerators of those two quantities as long as the quantities were all based on the same size partitions. The reason for choosing this algorithm was that it represented the activity participants pursued in diagrammatic setting. In going through the activity sequence that they had, there was not too much curtailment and CDA was inventible based on participants' activity. This is consistent with J. Gregg and D. Gregg's (2007) finding about accessibility of CDA with measurement interpretation of division.

To help participants develop these two sub-steps for an algorithm, the designed task sequence engaged them in mentally solving the given division of fractions problems as if they were using diagrams. This type of work helped them come to a point where they anticipated what to do next and focus on what to pay attention to. In this way, they were encouraged to think about their thought processes to make an abstraction. By going through the activity sequence they already had from the previous sessions, in light of diagram use, they were also encouraged to think about the reason as to why the given fractional quantities transformed into another form for which understanding of equivalent fractions plays an important role. In this way, they realized that the purpose was to have equal size partitions so that the multiplicative comparison between the divisor and the dividend was easily identifiable. Then, based on their diagram work, they realized that they were counting a certain number of partitions within some total number of partitions, which was equivalent to thinking about dividing the numerators of the fractional quantities at hand.

Acknowledgements

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