Flawed Mathematical Conceptualizations: Marlon's Dilemma

By Lauretta Garrett

Abstract: ***Adult developmental mathematics students often work under great pressure to complete the mathematics sequences designed to help them achieve success (Bryk & Treisman, 2010). Results of a teaching experiment demonstrate how the ability to reason can be impeded by flaws in students' mental representations of mathematics. The earnestness of the subject's efforts and the frequent detours his learning took create a vivid portrait of what happens in the lives of students for whom "the dream stops" at developmental mathematics (Bryk & Treisman, 2010, p. 19). Results provide teachers with a clearer picture of what is needed to help their students build mathematical understanding.***

Many technological applications in use in developmental mathematics courses focus on procedural fluency rather than on conceptual understanding.

Step into a developmental mathematics classroom at any community college or university and you will find a diverse population of students all of whom have something in common: they have not developed enough mathematical understanding to be successful in college mathematics courses. Among these students are adults coming back to school, hoping to develop a new career and a new life for themselves. Such students may have graduated from high school 10 or 15 years ago. They may be back in the course for the second, third, fourth, or fifth time. They may also bring nonacademic problems into the learning environment, including multiple demands on their time (Caverly, Collins, DeMarais, Otte, & Thomas, 2000). Developmental mathematics students face a dilemma when they work under great pressures in their personal lives while attempting to complete the mathematics sequences designed to help them get ahead (Bryk & Treisman, 2010).

Nonacademic challenges aggravate the dual issue of students' lack of belief in their ability to succeed in mathematics, and an inability to accurately identify factors that have limited their success in previous attempts (Hall & Ponton, 2005). Adult students may feel certain they are incapable of doing well while being unable to give a good reason why. Thus they have no specific set of issues to conquer. They most likely started to fall behind in their school mathematics at a point in their education where they began to learn algebra, the primary mathematical content on which they are assessed for placement (Epper & Baker, 2009). Once students enter the developmental mathematics pipeline, they may find themselves in a situation where the instruction starts at chapter one and rushes forward without regard to their level of conceptual understanding; as a result they are never able "to get on the train" (Galbraith & Jones, 2008, p. 31). Demands on these students' time makes it difficult for them to participate in enrichment programs that might help remedy the situation (Caverly et al., 2000).

In addition to their other challenges, adult students may have difficulty finding the time to become comfortable with the use of required technologies (Epper & Baker, 2009). Attention to issues surrounding digital literacy in developmental mathematics courses is vital because technology is often used to enable student learning outcomes (Caverly et al., 2000). If not managed well, technology can become yet another thing to learn and another set of procedures to memorize (Herman, 2007). To overcome this perception, students need time to master technology and engage in discourse that facilitates conceptual connections. Any technology used in developmental mathematics courses must be wisely chosen and appropriately integrated to support student efforts in building understanding and taking control of their own learning (Brothen, 1998).

Many technological applications in use in developmental mathematics courses focus on procedural fluency rather than on conceptual understanding, due in part to the demands of the market (Epper & Baker, 2009). Such technology often follows a behaviorist model, only providing students with superficial levels of knowledge (Caverly et al., 2000). It may act as a virtual textbook or tutor, but it will not become an effective tool in the hands of students. Software as an effective tool allows students to choose and control which representations to display, and through its use they have a greater chance of reaching more complex levels of understanding (Caverly et al., 2000).

Instructors must take into account the influence of such representations on student understanding as they plan for the use of technology. Each student must have a conceptual understanding of standard, discipline valued representations in order to be successful in mathematics course work. Conceptions of mathematics involve mental imagery and other *internal representations* of mathematics that may or may not conform to

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standard representations and discipline valued meanings (Goldin, 2003). As previously noted, the quality of students' internal representations of mathematics can be characterized in three ways; valid, useful, and enduring (Garrett, 2010). Internal representations are considered *valid,* if they accurately reflect the mathematics they seek to represent and are flexible enough to allow additional mathematical ideas to be built upon them. Valid internal representations are also accompanied by sound mathematical habits of mind. Internal representations may be considered *useful* if they are accessible for reasoning and sense-making, communication of new ideas, and building new understanding; *enduring* representations remain with the student in various situations apart from the environment in which they were initially developed. Enduring internal representations will be carried forward, built upon, and refined over a period of time.

An example of an invalid internal representation is the conception of the graph of a quadratic equation as a parabola that must pass through the x-axis at two different locations. Such prototypical visual images are at odds with other different but also valid visual images of the same concept (Mourao as cited in Presmeg, 2006). These invalid internal representations can lead to manipulation of external representations without attached meaning. Adult students cannot be successful in building complex levels of mathematical understanding unless they possess valid, useful, and enduring internal representations of mathematics.

A qualitative study was designed to gain insight into the effect of the use of technology on adult developmental mathematics students' internal representations of mathematics. The purpose of the study was to observe and analyze their interactions with technological representations and the technological, verbal, and hand written representations that resulted. The central research question was "How can technology best be used to address adult learners' needs and help them build valid, useful, and enduring internal representations of mathematics?" (Garrett, 2010, p. 85). In order to gain insight into students' internal representations, case studies were conducted as part of a teaching experiment to allow the researcher to experience the subjects' ways of thinking and reasoning firsthand (Steffe & Thompson, 2000). The observations of students' interactions with and production of representations provided insight into their internal representations, as suggested by Goldin (2003). One of the cases in the original study resulted in unexpected findings about the nature of the conceptual challenges adult students face, giving rise to the emergent question: What was it about the subject's thinking that hampered his ability to learn more about standard mathematical representations through the use of technology?

Methods

Participants and Setting

The subject of the case study to be discussed was Marlon, a 53-year-old African American male, formerly in the military, who was taking the second of three developmental mathematics courses at a midsized university in the southeastern United States. He had taken the first developmental mathematics course twice before passing it. He dropped out of high school in his final year and then earned his GED while serving in the military. The exact reasoning for his dropping out of high school was not clear. He said, "I tried to take my subjects very seriously but ...trying [to] raise kids at that time I just fell out." He could not remember much about his high school mathematics classes, saying that he was "probably more into sports" at that time. He said at one point that he had loved mathematics when he was growing up but that "you really have to practice it all the time." At the time of the study, he found mathematics challenging, particularly after having been out of school for a while, which he said made learning harder.

Students' interactions with and production of representations provided insight into their internal representations.

The study took place at a midsized university in the southern United States that not only offered advanced degrees but also had a large developmental mathematics population. At the time the data was collected, the total enrollment was 8,179, with about 60% female, 40% male. About 32% of the enrolled students were African American and 57% Caucasian; 638 students were enrolled in basic studies. Students were placed in basic studies if their mathematics SAT score fell below 460 for traditional students or 500 for nontraditional students or their ACT mathematics score was below 19 for traditional students or below 21 for nontraditional students. A mathematics score of 37 on the Compass exam was required to exit basic studies.

The developmental mathematics program, housed in the basic studies department, included Math 98 and Math 99. In Math 98 students reviewed basic mathematics and were reintroduced to algebra and linear functions. In Math 99 they reviewed those topics and also looked at quadratic functions. Another course at the developmental level, Math 100, was offered through the mathematics department. Math 100 topics included real numbers, equations and inequalities, functions and graphs of functions, and systems of equations. Marlon was enrolled in Math 99 and listed Math 98 as a previous course. He had some experience with technology but not with Geometer's Sketchpad (Key Curriculum Press, 2006).

Procedures

To gather qualitative data about Marlon's thinking, a sequence of interviews was conducted in order to gain insight into the way in which his internal representations of functions were affected by his interactions with technology, specifically the algebraic features of Geometer's Sketchpad (Key Curriculum Press, 2006). The use of teaching experiment methodology allowed an examination of what Marlon might think in addition to what he did think going into the study (Steffe, 1991). It also allowed the researcher to become "an actor" who was constructing models of what was occurring in Marlon's mind as a result of the researcher's actions (Steffe, p. 177). Researcher actions were based on Marlon's actions and included "on the spot" decisions based on what was happening in the experiment and the emerging model of his thinking (Steffe, p. 177).

Recording devices. In order to capture as much information as possible about the work Marlon was doing, the interviews were recorded in three ways. Software was installed to record everything that happened on the computer screen during the session. A camera was placed on a small tripod on the desktop to record the work the subject did on paper. In addition, another camera was set up across the room on a large tripod to record the interactions between the subject and interviewer. Recordings captured both audio and video. These recordings allowed the collection of rich data, and care was taken in the transcriptions to note mouse movements and the timing of the events within the episode. The transcription technique was inspired by Campbell's (2003) examination of students' mouse movements as a way of gaining insight into their thinking, which he called *dynamic tracking.* He included actions within braces { } and timing within brackets [ ] to better capture that information. Marlon participated in seven semistructured interviews for a total of 6 hours and 47 minutes, and Campbell's transcription technique was applied.

Reliability and validity. In order to improve the validity of the data, I rephrased and repeated questions and encouraged Marlon to explain what he was thinking (Kvale, 1996). He was encouraged to talk continually as if he was thinking out loud to ensure that his thinking was accurately interpreted (Koichu & Harel, 2007). Only minimal field notes were kept since my focus was on responding to the circumstances that arose in the teaching experiment. More detailed reactions were recorded later

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in a descriptive journal. Memoing, journaling, and multiple recordings added to the reliability of the work through triangulation (Tuckett, 2005): journaling provided a fresh memory of what happened during a session and recordings allowed key portions of the resulting transcriptions to be checked for accuracy. Inter-rater reliability tests were also conducted to determine the appropriateness of codes assigned during the original data analysis. Total agreement over three sessions in recognizing the four major themes (mathematical content and thinking processes, representational ideas and issues, influences and uses of technology, and other) was 81.94%.

Reliability was also strengthened through the use of a pilot study. For the pilot study, a male in his early 20s, Steve, participated in 5 sessions for a total of 3 hours and 34 minutes. He had been out of school about 4 years and did not remember taking Algebra 1. The pilot study allowed technological procedures to be practiced and also helped refine instrumentation, helping me to adjust tasks to the appropriate mathematical level.

Overarching structure. Initial questions in the first session were designed to put Marlon at ease and later questions were designed to include more specific examinations (Kvale, 1996). The methodology employed in the study resulted in rich data cataloging Marlon's thinking and his struggle to demonstrate and build mathematical understanding. The purpose of the study was to observe the effect of technology use on adult students' understanding of representations associated with functions. The hope was to gain insight into the ways in which a student can build upon his or her own understanding of standard representations through the use of technology. Marlon's ability to benefit from technological explorations was hampered by qualities of his thinking and understanding that may be common to other adult students. An examination of some of the interactions that occurred during the teaching experiment will highlight the specific challenges he faced.

Results

This examination of Marlon's thinking is structured chronologically to help the reader gain insight into the mathematical journey that Marlon was experiencing. The results will then serve as evidence for a discussion of the mathematical dilemma he faced. Marlon's journey through the teaching experiment began with an opportunity to demonstrate the mathematics he knew coming into the study. Following this, he had the opportunity to examine pictorial examples of functional relationships. These served as a conceptual foundation for technological explorations of discipline valued representations.

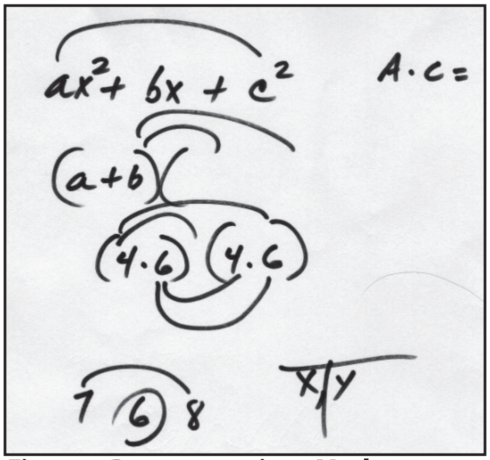

***Figure 1.* Representations Marlon created to show mathematics he remembered. Arcs indicate that the FOIL method was applied to the multiplication problem.**

Showing Mathematics He Knew

I began my work with Marlon by asking him to tell me about himself and his past experiences. He described the challenges of coming back to school, stating that "[Y]ou have to really adapt yourself ...[and] get back into ... learning different concepts. Being out for so long a period of time makes it harder." He attributed his lack of success in his first attempt to take Math 98 to not studying sufficiently, saying that he "didn't really open up [his] math book." Once he began sharing more freely and some rapport seemed to have been established, I asked him to share some mathematics that he remembered. He gravitated toward algebraic topics he was currently studying and attempted to remember procedures he had been taught. He recalled the first outside inside last (FOIL) method of multiplying two binomials, but he misapplied it to a multiplication problem, as can be seen in Figure 1.

Looking at Dot Patterns

Later in the first session, I asked him to analyze the patterns found in two handouts "Looking at patterns" and "Looking at dot patterns" seen in Figure 2. This allowed me to learn something more about his mathematical thinking and provided a context for mathematical ideas to be investigated in later sessions with the software. His exploration of "Looking at dot patterns" began in the first session and continued into the second session. He examined it carefully and determined that since the odd patterns had the leg of the T lined up with the center dot of the base but the even numbered pattern had the leg of the T lined up with the space between two dots in the base, that he would consider the odd and even steps separately as if they were two different patterns. He was confused about whether or not to count the center dot in the base of the odd patterns as part of the leg or not. His inconsistency in doing so caused confusion as he analyzed the odd patterns. As a result, he focused on just the even patterns, which he could more accurately analyze. I allowed him to choose his

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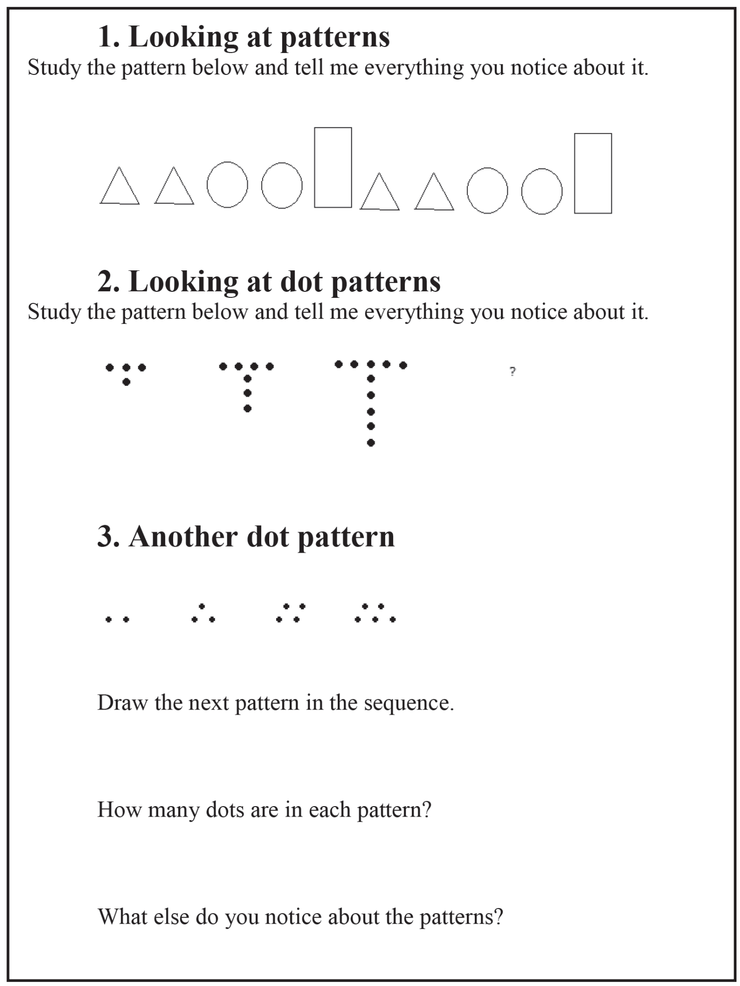

***Figure 2:* Initial tasks used to provide a foundation for the study of functions.**

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method of analysis so that I could learn more about his thinking. He came up with a reasonable way of thinking about the number of dots in the legs of the even numbered patterns. He gave the base of the T in his fourth pattern 6 dots and the leg of the T in his fourth pattern 5 dots. He had added 2 to the number of dots in the leg of the second pattern to get the number of dots in the leg of the fourth pattern. This reasoning did not fit the overall pattern as it would conventionally be analyzed, but it made sense to Marlon based on his observations of the first three patterns. Figure 3 shows Marlon's work with some of his statements added about the even numbered steps as he saw them. He had at first "estimated" that step 10 would have 13 dots in its leg. He later reasoned logically to determine that he should remove the last two dots so that the leg would only have 11 dots (see Figure 3).

Marlon dealt with his confusion about the odd numbered patterns by focusing on an aspect of the mathematics he could understand. Within his chosen focus, he was able to observe patterns, solve problems, reason, and make sense of things. He had valid mathematical ideas and was able to make some sense of what he was seeing.

Looking at Another Dot Pattern

Because I wanted him to have a conceptual understanding of an entire pattern before beginning the technological work, I gave him "Another dot pattern" as a follow up to his work with "Looking at dot patterns." The task associated with that pattern and a portion of Marlon's work with it is seen in Figure 4.

He seemed to work with this task more fluidly, aided by the tabular representation that was included this time. After he had made five entries in the table based upon the five illustrated dot patterns constituting the first five steps of the pattern, I asked him what he would put on the next line of the table of values. He noted he would put 6 in the left column and 7 in the right column, explaining that "looking at the pattern here everything is in numerical order, and I notice that the next one here {*indicating the right hand column*} follows 2 and it's also starting from 2 in numerical order." It was uncertain whether the statement transcribed as "follows 2" meant "follows the number 2" or "follows also." It did seem clear that he was noticing that the right hand column started at 2 and that when looking down the column, the numbers were in numerical order. Later he gestured from the left to the right hand column in explaining why the eighth step would have 9 dots. These gestures, used to show where he was looking for his information, were followed by the statement, "It's just adding one." The tabular representation seemed to help him to see these relationships.

Graphing Points

In session two, I felt that Marlon had a conceptual understanding of "Another Dot Pattern," and so I introduced him to the software. He was allowed to explore the tools and then shown how to use the graph menu to open up a coordinate plane. On that plane he was encouraged to place a point using the point tool and then to plot a point using the graph menu. Once these two objects were created, he was encouraged to use the selection arrow tool and measurement menu to explore their behavior and their characteristics. By allowing him to explore these tools and menus, and encouraging him to talk about what he was seeing and doing, I was able to gain insight into his understanding of the xy plane and the language he used to describe the mathematics. For example, when he had moved a point into the third quadrant he said "I'm in a negative area here; still in the positive area going upwards."

When asked to move the cursor to a location where both coordinates of a coordinate point were positive, Marlon moved the cursor to the right from (0,0) along the x-axis. When prompted again, he moved the cursor to (0,9). I asked him to place the cursor where both coordinates would be negative and he placed it at (-14,0). Because he was not moving his cursor off of the axes, I asked him to explore what would happen if he took the point off of the axes. He eventually placed a point in the 3rd quadrant and said "Okay so I have them all in the negative area." I asked him to move the point across the x-axis, and he noted that x was negative and y was positive and indicated the corresponding portions of the axes, saying "This is {*indicating the negative portion of the x-axis*} my x {*indicating the positive portion of the y-axis*} and y.... Negative

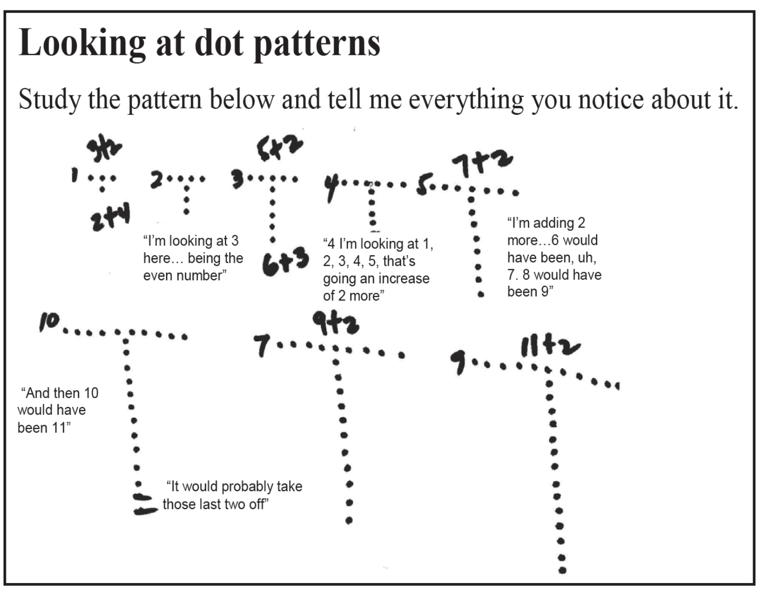

Figure 3: **Marlon's work in analyzing the pattern given in the handout "Looking at dot patterns" is supplemented here by a record of some of the statements he made as he was working.**

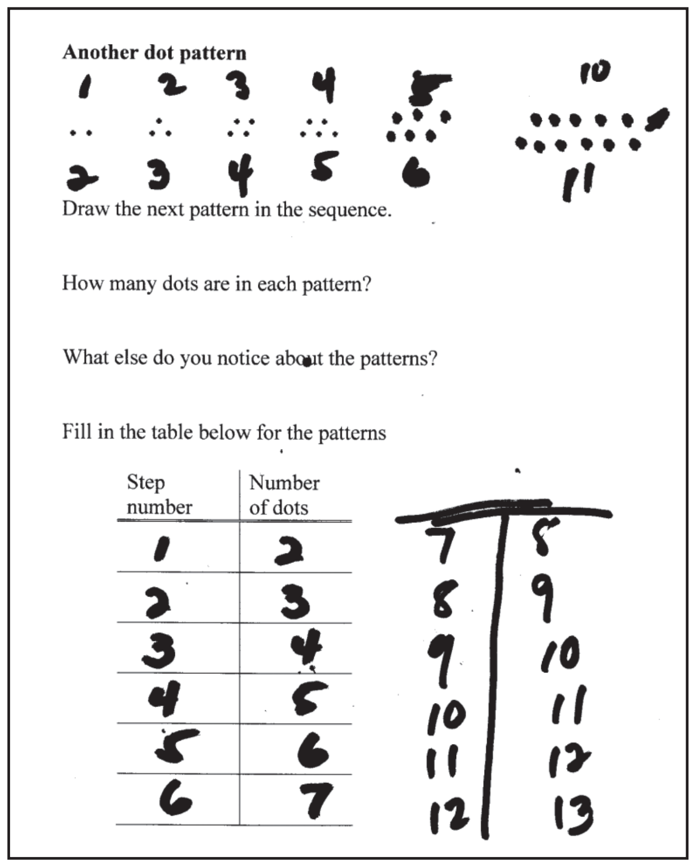

Figure 4. **A portion of Marlon's work on "Another dot pattern."**

going this way {*putting the cursor on negative portion of x-axis*} and positive going this way {*stroking cursor upwardly in the 2nd quadrant*}." He had not understood the meaning of my request to find a point so that "both coordinates" had a particular quality, but was able to describe qualities of points and their relationship to the axes.

In session three, following a review of his previous explorations of the xy plane, I turned his attention back to "Another dot pattern." He looked at it again and remembered the patterns he had noticed. Pointing to the first entry in the table of values, he noted that:

> [H]ere it's still going numerical[ly] going downward {*gesturing down the step number column*}. Skip a number {*gesturing across from step number 1 to its associated number of dots, which was 2*} still going downward {*gesturing down the number of dots column*}.

Because I wanted him to graph this data on the xy plane, I used guiding questions to lead him to the idea of associating the table with x and y coordinates. I noted that there was a left and a right coordinate. After observing the left and right coordinates in the table, he said "This same sequence you would do like [an] x and y {*gesturing across the first row of the table*}." I then told him to open a new sketch and plot those points, which he did using the plot points menu. His plot is shown in Figure 5.

He noted that they were in a straight line, and that, "It gives you your angle of that particular grid. I mean of that ... particular plot of the numbers." When I asked him what he meant, he said "I would say actually its going on a 45 degree angle." He then retracted this statement, and noted that it would only be considered a 45 degree angle if the line intersecting the points had intersected the origin. As he examined the points he had plotted and clarified his thoughts, he said:

> Okay, here {*tracing with the cursor along the positive x and then the positive y axis*} right now I'm dealing with I would say a 90 degree angle. 45 would actually be right here {*tracing along the path where the line y = 1 would be, from the origin up and to the right*}. So it's right off of a 45 degree angle.

His technological gestures allowed me to see how he was thinking and why he did not think that the line connecting the points was at a 45 degree angle. Because this was not an intervention, and I wanted the focus to be on the function itself, I did not correct his misconception but chose to refocus his attention on the numerical data. He had created a pattern on paper predicting the number of dots in the 10th pattern. I asked him if he could make a prediction about whether or not his pictorial estimate of the number of dots in step 10 was correct or not. At this point, his table only included entries up to step number 6. Rather than considering the graphical representation shown via the technology, he made his predications using the table of values and added the addition to the table seen to the right of the original table in Figure 4.

In session four, after some refamiliarization, Marlon again graphed the points associated with "Another dot pattern" using technology and made predictions about where additional points might be located. I prompted him to consider the graphical representation by asking where he thought the next point in the table would be located. He understood that the number of dots was one more than the step number, and so when asked to predict how many dots the 20th pattern would have, he said that it would have 21 dots, describing the pattern as being "in numerical order." When asked where that point would be, he traced along the x-axis and then up into the first quadrant. His mouse movements strayed, and he actually ended up at the point (19, 20), but his method demonstrated understanding of the nature of the location of points on the xy plane.

Using Algebraic Representations

Following this discussion, I asked him questions to determine whether or not he could represent "Another dot pattern" algebraically. When I had asked him during session 2 how many dots would

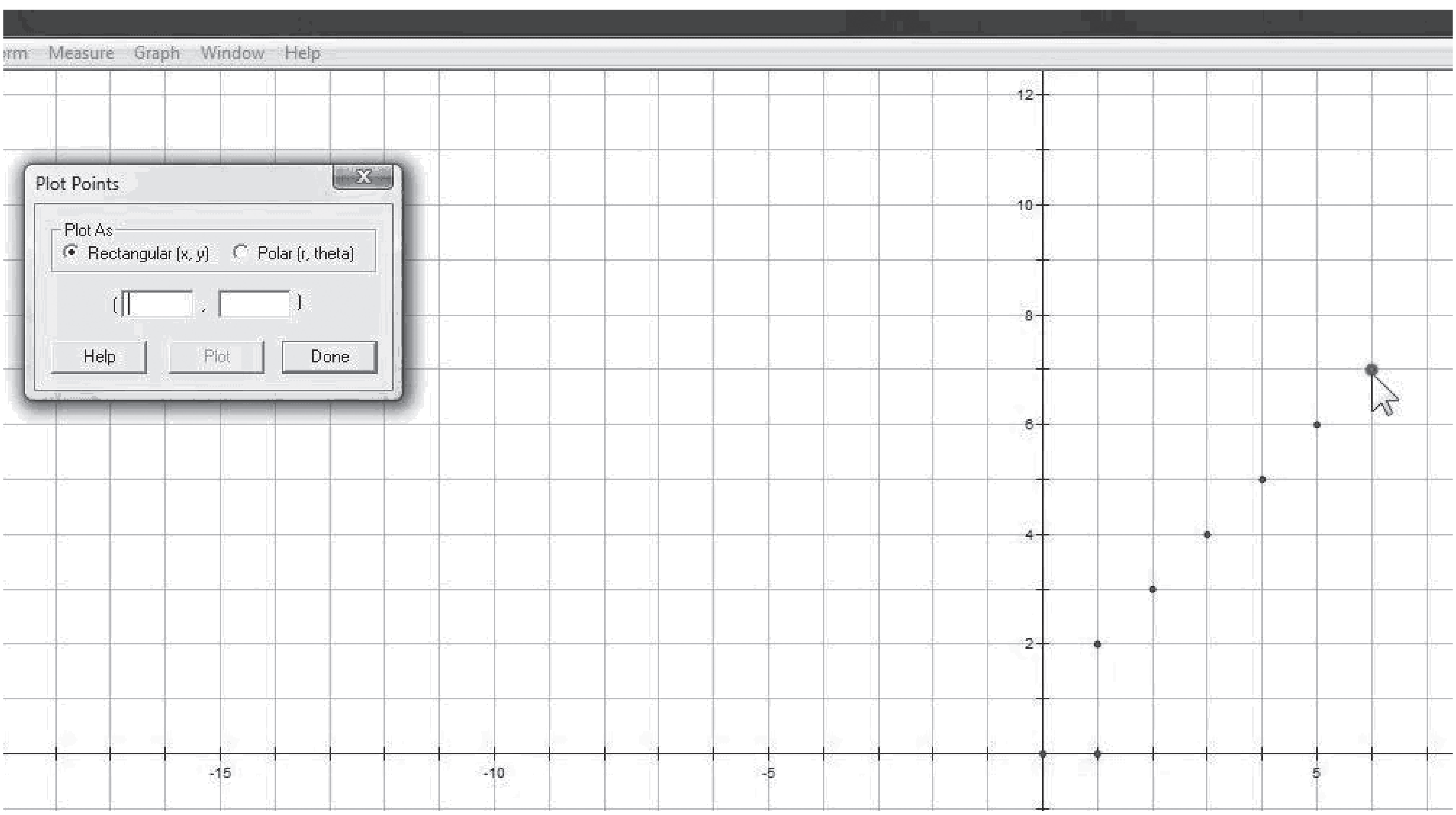

Figure 5: **Points Marlon plotted from his table of values for "Another dot pattern."**

be in the eighth step, he said, "I would just, in this case ... I would just add one ... if it was 8 it'd be 9." In session 4, I asked him what he had in mind as he noted that step 10 has to have 11 dots. He said, "all I'm doing is actually adding a 1 to that and 10 it's going to give me 11." This was a correct description of the mathematics in the pattern. When I then asked him to express the same idea using n as the step number, however, rather than giving the number of dots as n + 1, he reasoned as follows.

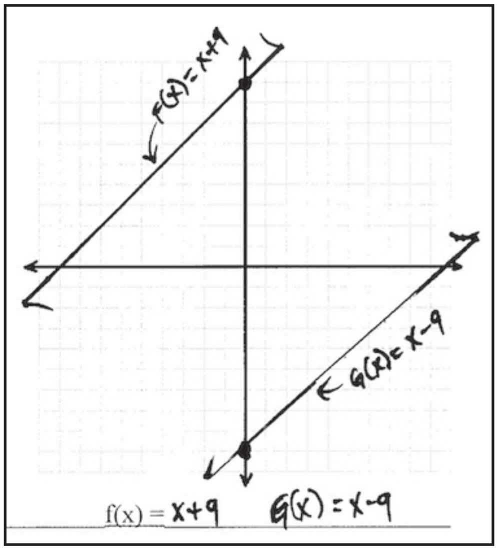

***Figure 6:* Marlon's sketch of the functions *f(x)* = *x* + 9 and *f(x)* = *x* – 9, based upon technological representations.**

> I'm just looking at n representing a certain number, a pattern and o being the next letter in the alphabet. So it's the same thing as far as the numerical pattern. I would assume that the letter is going to be in ... alphabetical order.

His reasoning was logical and made sense to him, but showed that he lacked understanding of the use of variables in mathematics. He could describe the pattern, but he could not represent it algebraically.

Creating a Function

The teaching experiment was designed in part to observe how much Marlon could learn through his own investigations. Rather than attempting to clear up his misconception through direct instruction, I wanted to see what he might learn about algebraic representations through the use of the software. I introduced him to the functions menu of the software in the hope that some exploration on his part would reveal that *f(x)* = *x* + 1 and that he could then make the conceptual connections between his understanding of the pattern and the algebraic representation. After some introductory explanation, I encouraged him to try creating some functions that used *x* as a variable. He used the technology to graph *f(x)* = *x* + 9 and noted that "it crosses over the y intersect at 9." After sketching that graph on paper, he used the technology to graph *g(x)* = *x* – 9. Figure 6 shows the sketch he made of these two functions based upon the technological images.

His conversation and gestures as he examined these two functions with the technology showed his thinking as he compared the graphical and algebraic representations.

> I notice, okay {*running the cursor down the graph of f(x)*} okay. I notice here's my -9 here {*indicating (0,-9)*} and this is my positive {*indicating (-9,0)*} okay. My 9 here, okay, as you were {*moves the cursor to (0,9)*} Here's my positive number right here. Here's my negative number right here. {*moving the cursor to (0, -9) then veering a little toward the right as he goes down*} And they're considered what {*gesturing with mouse along the graph of f(x)*} parallel to each other {*running the cursor down the graphs of both f(x) and g(x)*}.

He referred to *f(x)* = *x* + 9 as his "positive 9" and *g(x)* = *x* – 9 as his "negative 9." He was associating the *y*-intercept with the algebraic representations of the functions. I was hopeful at this point and asked him to create a function that would pass through his data points. After some discussion, the following took place. Note that because *f(x)* and *g(x)* had already been defined, the software named the new function *h(x)*.

> MARLON: I'm actually – is it value? No. What I'm actually doing is trying to put, this is *h* now and I want to get *x* + 0, {*cursor is at the origin*} Oh no no that's going to, that's going to be wrong. Um, {*cursor is at (1,2)*} ... 2 plus 2. ... If I want this line to go straight through here {*moving cursor from (0,0) to (1,2) to (0,1) to (0,0)*} I notice that {*to (0,1) to*

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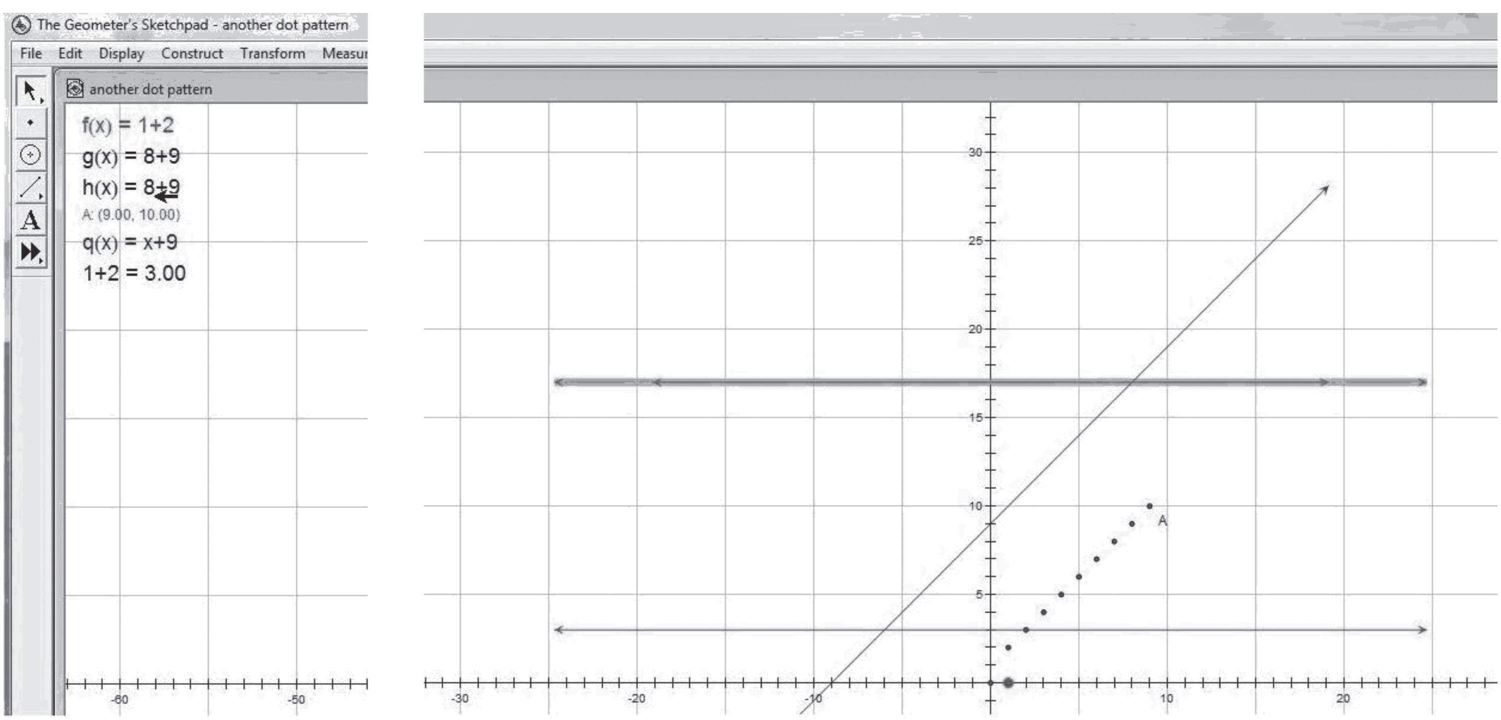

***Figure 7:* Some of Marlon's explorations as he sought to find a function that would pass through the graphed points.**

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> *(0,2)*} this is {*to (0,0) to (1,0)*} this is actually one, two {*moving cursor from (1,0) to (1,2)*} so if I go to 1 {*he changes the function to h(x) = 1*} plus 2 {*now he has entered h(x) = 1+2*}.

Consider the knowledge that Marlon had demonstrated. He knew that his pattern involved adding one to the step number to get the number of dots. He knew that the *x*-coordinate of his data points represented the step number and the *y*-coordinate represented the number of dots. He had observed that functions of the form $f(x) = x + b$ were passing through the y-axis at the value "*b*." He had observed that graphs of that form were parallel. The data points were graphed in a line parallel to those graphs. In spite of all of this, when asked to create a function that would pass through his data points, he reverted to a misconception of the representations involved. He attempted to create a function that would pass through a point (*a, b*) by defining that function using the expression "$a + b$."

Some of Marlon's work in this investigation is shown in Figure 7 (p. 8). Repeated attempts to find the function can be seen in the upper left hand corner. The function $f(x) = 1 + 2$ was intended to pass through the point (1,2). The functions $g(x) = 8 + 9$ and $h(x) = 8 + 9$ were intended to pass through the point (8, 9), done twice because at first *g(x)* was out of view and he did not think anything had happened.

I wanted to draw his attention back to what he had been observing, and so I reminded him that the parallel functions had used the variable *x*. After further examination of the graph he then graphed $h(x) = x + 1$, and saw that it passed through the data points. Marlon noted, indicating the point (1, 2) that "it could have been +2" and then said that "+2, could have +3, 4 and so on and it would still [give] me a parallel line." To follow up on what he was saying and reinforce the idea of the *y*-intercept and the parallel nature of the family of functions, I encouraged him to try some additional functions of the form $f(x) = x + b$. His misconceptions continued to interfere with his work. The new function he created did not pass through his data points, and so, rather than observing that the new function was parallel, he felt that he had done something incorrectly.

> MARLON: I plot a new function and then I'm going to say ... x + 2.... {*he has plotted* $q(x) = x + 2$} Okay, Oop. Okay I did it wrong. Alright. I did it wrong {*moves his cursor to (0,1)*}. I should have said x + ... I should have said 1 + 2. {*he moves the cursor from (0,0) to (1,0) to (1,2)*} ...
>
> It's ... cutting across the 2 {*referring to the point (0, 2)*} instead of actually cutting over one plus 2 {*indicating (1,2)*} it's actually cutting here at (0,2)... Because here's my *x* and here's my *y* {*going from (0,0) to (0,2)*}. So if I actually should have did, if [I] actually ... did it right it should have been 1 + 2 and it would have been right there {*indicating (1,2)*}.

He had seen that $f(x) = x + 9$ crossed the *y*-axis at 9, he had seen that *g(x)* crossed the y-axis at -9 and he had seen that $f(x) = x + 1$ passed through his data points and crossed the *y*-axis at 1. Nevertheless, he still insisted that 1 + 2 should have been the defining expression for a function passing through (1, 2) and the rest of his data points. He also did not anticipate the placement of $q(x) = x + 2$. His misconceptions continued to interfere with his understanding, with the technological representations he had seen, and the connections he had made. He later stated that "*x* plus 3 would have brought me here {*indicating (2,3)*}. The talk-aloud dialog highlighted the confusion Marlon was experiencing about standard algebraic representations and the manner in which his misconceptions interfered with his learning.

> *The issue is not his ability to think logically.*

Discussion

Marlon's lack of valid internal representations of standard mathematical notation interfered with his ability to build understanding through the use of technology. His journey through the study began with evidence of his lack of algebraic validity. His misapplication of the FOIL method of multiplying binomials was evidence that his internal representation of polynomial multiplication lacked sufficient validity and endurance. Although he understood and reasoned about functional patterns, he was unable to transfer this knowledge to standard representations. His lack of a conceptual understanding of the standard use of variables interfered with his ability to represent the number of dots in "Another dot pattern" algebraically. Given that *n* represented the step number, he represented the number of dots using the letter *o*, rather than $n + 1$. Marlon did demonstrate diligence and perseverance, qualities vital to the mathematical success of adult mathematicians. The quality of proactive sense making and perseverance in problem solving Marlon exhibited might be thought of as "mathematical industry." The weaknesses in Marlon's internal representations impeded the progress his mathematical industry might have allowed him.

Persistent misconceptions about the nature of the algebraic representation of a function inhibited his ability to extrapolate from the legitimate connection he had made between the graph of $f(x) = x + b$ and (0, *b*). He did not find $f(x) = x + 1$ without additional prompting to include x as part of the expression. Even after he saw the correct representation, it was not enduring. During later attempts to pass a function through (1, 2) and (2, 3), he reverted back to using $f(x) = 1 + 2$ and $f(x) = 2 + 3$ to serve that purpose. Although he had indicated to me that he had seen functional notation before, understanding of that notation did not endure. His internal representation of coordinate points as a horizontal movement *plus* a vertical movement moved in to falsely fill that conceptual gap and interfered with his ability to build upon what he was experiencing.

The data collected highlights his conceptual strengths and conceptual challenges as well as the way in which those strengths and challenges interacted. Marlon's work analyzing the dot patterns and making genuine connections between $f(x) = x + b$ and (0, *b*) demonstrate genuine intellectual strength. They are in keeping with the first mathematical standard of practice described in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2011), "Make sense of problems and persevere in solving them" (p. 6). Perseverance describes Marlon's earnestness and continued efforts to struggle with the same ideas. He made some sense of what he was seeing in the patterns. He solved the problem of the connection between the graphs of $f(x) = x + 9$ and $g(x) = x - 9$ by continuing to examine the graphs and noting where they crossed the *y*-axis.

Marlon's mathematical industry is evident in the earnest effort he made to make sense of "Looking at dot patterns." He dealt with confusion over one portion of the pattern by focusing on another portion of it that he could understand. He noticed details about the pattern and created a genuinely logical way of thinking about the even numbered patterns. Although his suggestion to use the letter *o* to represent the number of dots in "Another dot pattern" shows a lack of understanding of algebraic notation, it does show the ability to think logically and make sense of things. The issue is not his ability to think logically, the issue is his ability to think logically using standard discipline valued algebraic representations.

Marlon tried different approaches when something didn't work as he expected. Those efforts to progress were hampered by invasive misconceptions regarding standard representations and their purposes. The growth in understanding he might otherwise have experienced through the use of technology was stunted by the lack of validity of his internal representations. A conception of $f(x) = a + b$ as a way to pass a graph through the point (*a, b*) is not valid in that it does not accurately represent what the standard notation $f(x) = a + b$ is intended to represent. Concepts were not sufficiently connected to standard representations in a way that would promote continued learning.

Such a situation might be described as "conceptual barrenness," in that any conceptual understanding of the representation that may be present is insufficient to support mathematical growth. A lack of validity in Marlon's internal representations thwarted the growth that Marlon's mathematical industry might otherwise have produced, placing him within a discouraging mathematical dilemma.

Marlon was employing mathematical industry in a conceptually barren setting. He was able to understand the pattern. He was able to state a sensible way of describing this pattern. Yet despite persistent efforts over several sessions, he was not able to transfer all of this genuine understanding to a discipline valued standard representation of the function, $h(x) = x + 1$. He needed facilitation to find that representation. Even after seeing it and the graphs $f(x) = x + 9$ and $g(x) = x - 9$, he became confused about what happened when he graphed $q(x) = x + 2$. When he considered creating a graph to pass through the point (2, 3) he referred to $x + 3$. He wondered why $q(x) = x + 2$ was not passing through (1, 2) and described the graph as "hitting zero plus two," referring to the coordinate point (0, 2) as a sum. Pervasive misconceptions contaminated the representations he was seeing. Such contamination slowed his progress and would have stopped it without facilitation. Marlon did possess worthwhile mathematical industry and conceptual understanding of the nature of the pattern, but it was very difficult for him to overcome his lack of conceptual understanding of the representational setting in which he was being called upon to work.

Implications for Research and Practice

Marlon's case demonstrates a dilemma that many adult students may be facing: the employment of considerable mathematical industry in a conceptually barren setting that is discouraging to the point of defeat in many cases. An increased emphasis on conceptual understanding in the formation of mathematics education policy could signify an important turning point in our nation's efforts to improve the educational experiences of students like Marlon. Conceptual understanding had been characterized in recent years in many ways. Among those characterizations are ideas related to mathematical representations, such as the ability of a student to connect multiple representations of the same idea (Alagic & Palenz, 2006), to visualize ideas (Abramovich & Ehrlich, 2007), and to verbalize ideas (Alsup & Sprigler, 2003). The ability to connect disparate ideas and transfer knowledge to a different setting are also vital (Beitzel, Stally, & DuBois, 2011), as is understanding that situates knowledge within a web of related ideas (Van de Walle, 2007). Conceptual understanding also includes the ability to reason flexibly about problems that are posed and make connections to what is already known (Engelbrecht, Harding, & Potgieter, 2005). It is evidenced by "the ability to cope with higher levels of abstraction" (Panasuk, 2010, p. 236), the ability to notice and consider analogous situations, the ability to represent related ideas coherently, the ability to justify reasoning about ideas, and the ability to apply ideas appropriately (Common Core State Standards Initiative, 2011).

Research Connections

As standards emphasizing conceptual understanding are increasingly incorporated into our K-12 educational system, stakeholders in postsecondary education should take note of the revitalized expectations they must have for their students and the pedagogy that should accompany those expectations. As curriculum and pedagogy are prepared for developmental and college mathematics courses, attention must be paid to the level of conceptual understanding of the mathematical representations used in those courses students must have in order to interact with those representations with beneficial industry and enthusiasm. In addition, resources must be developed, researched, and field tested, to meet the needs of students faced with Marlon's dilemma. Such research and development might focus on the following questions: How can the conceptual disconnect students experience when examining standard representations be accurately assessed? What curriculums and classroom methods can best target the dilemma created by a lack of student understanding of standard mathematical representations? How can those curriculums and classroom methods value and utilize the mathematical industry students with different levels of understanding possess?

Marlon was employing mathematical industry in a conceptually barren setting.

Strategies

In order to minimize as much as possible the effects of the discouraging dilemma many students face teachers can engage in daily formative assessment practices to help identify students who are trying to build upon representations of which they have little conceptual understanding. Identifying such "disconnected" representations can allow teachers to locate and direct students to resources that might help those students rebuild conceptual pathways. Online libraries of resources for rebuilding conceptual pathways can be developed to provide teachers with an efficient way to locate the best tools. Teachers may also wish to provide additional help sessions for groups of students whom they have observed experiencing similar disconnects. Reflective journaling about the nature of the difficulty their students are experiencing can establish a data source to distinguish various disconnects; organized information about the best resources available to meet those needs will help ensure greater success for future students with similar difficulties.

Specific strategies for assisting students to build a conceptual understanding have been suggested by research. Among those strategies are the use of language to precede the use of variable, analogies to help students understand algebraic situations, and tasks that bring students to higher levels of variable usage.

Students' difficulties may be influenced by a lack of understanding of variables, thinking of them as just "letters" without understanding their role (Dias, 2000; Saul, 2001). They may not understand that other symbols may be used to represent quantities, or that some letters used in mathematics are not variables, such as the "*f*" in the expression $f(x)$. One strategy that may help them make the transition to using variables and other mathematical shorthand properly is to use English words to describe the mathematical notation verbally until they get tired of writing things out and are motivated to use the shorthand: For example, use "the image of 4 by the function *g*"(Dias, 2000, p. 194) as language preceding the use of the notation $g(4)$.

Students may be working at a lower level of algebraic understanding than teachers realize (Saul, 2001). Students may be solving linear equations by trial and error, especially in simpler problems such as $x + 5 = 8$. If asked instead to solve $x + 112.57 = 739.45$, students will be more likely to apply inverse operations as a tool; they will be forced to consider the binary operations involved and the concept of "doing and undoing" that are essential to algebraic understanding (Driscoll, 1999; Saul, 2001). At the highest level of algebraic understanding students are able to use variables to represent more complex algebraic expressions. Teachers can help students progress to this level by asking them to think of the more complex expression as "playing the role" of the variable (Saul, 2001). They can also replace variables with other types of icons or parentheses that better match their understanding of the algebraic variable to replace variables. By using such techniques, teachers can help students have a better understanding of the role of variables.

In addition teachers can ask students to connect abstract ideas to analogous situations that will allow students to build a grounded understanding of those ideas (Koedinger & Nathan, 2004). For example, when teaching students about the mathematical properties of functional relationship, as opposed to a mathematical relation that is not a functional relationship, the analogy of a mail carrier delivering the mail can be used. The set of

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letters to be delivered is the independent variable and the set of mailboxes to which it is delivered is the dependent variable (Sand, 1996). Each letter must be placed in one mailbox, no single letter can be placed in more than one mailbox, but one mailbox may receive more than one letter. A mapping representation of this situation can be created, and this analogy can be extended to multiple representations and used as an underpinning for discussions of functions.

Technology projects can provide students with opportunities to build conceptual understanding of mathematical representations provided they are at the appropriate level and build on the students' current understandings, as Marlon's work with Geometry Sketchpad illustrates. Under the pressure of time, teachers may not be able to facilitate every learning experience for every student at this pace, but they can carefully consider what knowledge they wish students to build and choose targeted activities accordingly. They can narrow a technological investigation to one key parameter or idea upon which other ideas rest. For example, teachers can have students enter data into mathematical software similar to Geometer's Sketchpad that can be closely modeled by functions having the form $f(x) = mx$ and then ask students to find the function of that form that most closely models their data. As they investigate with technology, students can observe how changes in m affect the graph, draw conclusions about the effect of that parameter on the graph, and make observations about the relationship of that parameter to the data modeled. The more students can connect other representations with algebraic representations, the more solid their understanding of algebraic representations will be.

As the students become confident in the use of technology through one or more teacher led technological investigations, they will be more likely to use technology independently for their own learning needs, and the teacher can assign increasingly more independent investigations for students to conduct outside of class. One laboratory session is enough to build such a technology sequence into a one-semester course. Having students write brief reports that include copied images from their technological work allows the instructor to gain insight into their mathematical thinking and provides the opportunity to include verbal representations in conjunction with standard mathematical representations.

Concluding Thoughts

Issues faced by adult developmental mathematics students are symptomatic of problems found throughout mathematics education. Many adult students come back to school dreaming of obtaining an education that will change their lives. They enter with the ability to quickly learn about and engage in valued mathematical standards of practice, valiantly persevering in their efforts to make sense of things. For many, however, attempts to progress in mathematics using those qualities are thwarted by the lack of validity of their internal mathematical representations and conceptual disconnections regarding standard representations. From the beginning of the semester, they are asked to employ their mathematical industry in a land of meaningless representations: continuing in this fruitless situation becomes repetitious to the point of the defeat and discouragement (Bryk & Treisman, 2010). Students reflect this frustration by describing developmental courses as "an insurmountable barrier for many students, ending their aspirations for higher education" (Bryk & Treisman, p. 19).

Marlon's case study provides a glimpse into internal barriers of conceptual misunderstandings that impede progress, even for industrious students earnestly working to progress in college mathematics. In order to enhance students' success in developmental mathematics, their learning experiences must uncover and restructure internal representations that interfere with an understanding of standard mathematical representations. Research is needed into the curriculum, resources, pedagogy, and andragogy needed to help all students in this situation progress with efficiency and effectiveness. Students like Marlon who are valiantly attempting to succeed deserve nothing less.

They are asked to employ their mathematical industry in a land of meaningless representations.

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”the intellect.” Further, such an analysis requires tracing some important meanings implied by these terms and their interrelationships.

The term intellectual often means requiring the intellect or having or showing a high degree of intelligence. The term intellect implies the ability to reason or understand or to perceive relationships, differences, and so forth. It refers to that part of the mind that knows or understands. It may also imply the power of thought, great mental ability, or a high degree of intelligence. The terms intelligent or intelligence imply having or showing an alert mind, bright, perceptive, informed, clever, and wise. They also generally imply the ability to learn or understand from experience, the ability to acquire and retain knowledge, and the ability to respond quickly and successfully to new situations. And they characteristically imply or presuppose use of the faculty of reason in solving problems, directing conduct successfully, and making sound judgments (Wiley Publishing, 2007).

Note that within these meanings are several important concepts whose meanings are essential to understanding intellectual standards: to reason, to know or comprehend, and to make sound judgments. “To reason” entails the power to think rationally and logically and to draw inferences. “To understand” is the faculty by which one understands, often together with the resulting comprehension. It entails superior power of discernment or enlightened intelligence. “To make sound judgments” is the ability to assess situations or circumstances logically or accurately and draw reasonable conclusions. “To know or comprehend” means to have a clear perception or understanding of, to be sure of. It entails clear and certain mental apprehension (Wiley Publishing, 2007).

The term intellectual, when integrated with related terms, thus entails the use of sound reasoning and judgment in the pursuit of knowledge. It typically implies the superior powers of the intellect as well as the ability to use one’s mind to make intelligent decisions, to use the faculty of reason in solving problems, and directing conduct successfully. Finally, it suggests clear perception and the logical drawing of inferences.

The Concept of Intellectual Standards

Taking into account the previous meanings and analysis, we conceptualize intellectual standards in the following way:

> the standards necessary for making sound judgments or for reasoning well, for forming knowledge (as opposed to unsound beliefs), for intelligent understanding, and for thinking rationally and logically.

In short, we use the term intellectual standards to mean standards that further good judgment and rational understanding. They are essential for the mind’s on-going awareness and assessment of the strengths and weaknesses in personal thinking and in the thinking of others. Whether focused on the inner structure of thought or its global qualities, intellectual standards are essential to functioning as reasonable, fairminded persons. However, most people rarely seem to reflect upon the standards they use to determine what to accept and what to reject. Consequently, and because the fulfillment of intellectual standards is not natural to the mind, people tend to use default standards, ones that are often highly egocentric and sociocentric. Conversely, fairminded critical thinkers recognize the primary role of meeting intellectual standards in living a fulfilling, rational life. They therefore routinely work to meet these standards. They typically recognize when they, or others, are failing to meet them.

Closing

In the next column we will detail some constellations of intellectual standards, thereby illuminating the interconnectedness of these standards as well as some fine distinctions among them. We will also differentiate between *micro intellectuals standards* and *macro intellectual standards*, and briefly discuss the common human problem of vested interest as a barrier to the adherence of intellectual standards. These theoretical distinctions are important, in order to help students learn to reason with skill within the disciplines.

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