Pathways to Professional Growth: Investigating Upper Primary School Teachers' Perspectives on Learning to Teach Algebra

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Abstract: This paper discusses upper primary school teachers' perspectives on changes to their knowledge and practice through participation in a design-based research project. It analyses their experiences using Clarke and Hollingsworth's (2002) empirically-founded model for professional growth to understand more about the mechanisms for change that might support teachers in teaching a challenging aspect of mathematics – algebra. The ten teachers referred to observations of teaching in action, and modification of their beliefs about algebra, themselves as learners, about particular students, and about teaching mathematics. They shared differing perspectives on interacting with colleagues. Some teachers described discomfort when confronted with their lack of knowledge or with their students' questions during lessons. The findings demonstrate pathways that appeared to be commonly experienced by many of the teachers and also those that highlight the individualistic nature of teacher change mechanisms. Implications for the design of professional learning in mathematics for in-service and pre-service teachers are discussed.

Introduction

The word 'algebra' can evoke unpleasant memories of a difficult abstract topic in secondary school mathematics (Greenes, Cavanagh, Dacey, Findell, & Small, 2001). The recent introduction of an Australian Curriculum has brought algebra to the attention of primary teachers because it explicitly included the content strand "Number and Algebra" starting from the early years of schooling (Australian Curriculum Assessment and Reporting Authority [ACARA], 2009). Research on algebra has highlighted the value of connecting arithmetic and algebra, and the importance of students developing algebraic thinking early on rather than postponing this until the secondary years of schooling (Cai & Moyer, 2008; Carraher, Schliemann, Brizuela, & Earnest, 2006; Radford & Pierce, 2006). A meta-analysis of algebra research studies emphasised the effectiveness of focussing on developing students' *conceptual* understanding of algebra (Rakes, Valentine, McGatha, & Ronau, 2010). Yet how do teachers learn how to "teach a more powerful and general mathematics for understanding" (Blanton & Kaput, 2008, p. 361) when they are likely to have been schooled in narrow procedural approaches to algebra and symbol manipulation techniques? The provision of teacher professional learning for teaching algebra at primary levels of schooling is needed, both for beginning and experienced teachers (Lins & Kaput, 2004). In particular, their awareness of students' difficulties in learning algebra is of increasing importance (Saul, 2008). Algebra teaching and learning has been highlighted as "a major policy concern around the world" (Hodgen, Küchemann, & Brown, 2010).

The purpose of this study was to investigate upper primary school teachers' perceptions and experience of professional learning for teaching an important area of

mathematics that has traditionally been viewed as challenging to teach and to learn. Clarke & Hollingsworth's (2002) empirically-founded model for professional growth was used to understand more about the mechanisms for change that might support teachers in their endeavour to teach mathematics conceptually, to implement the new Australian Curriculum for algebra, and to prepare students effectively for learning algebra at secondary levels of schooling. A large-scale research and professional learning project provided the opportunity to investigate potential pathways for teachers' development with a sub-project that utilised a design-based research methodology over one year. Titled *Contemporary Teaching and Learning Mathematics* (CTLM), the project was conducted by the Mathematics Teaching and Learning Research Centre at the Australian Catholic University for five years (2008-2012) and funded by the Catholic Education Office, Melbourne. Teachers from 82 Catholic primary schools in Victoria each participated for a two-year period. The study described here was a sub-project of CTLM that focused on the professional learning of upper primary teachers in a specific domain of mathematics (algebra).

In the research reported, the following question was addressed: How do teachers' descriptions of their experiences while participating in a professional learning program on teaching algebra correspond to possible pathways for professional growth as conceptualised by Clarke and Hollingsworth's (2002) model?

This article is based on findings from the in-depth case study of 10 practising upper primary teachers who participated throughout one school year. The following section provides details on the context for the project by reviewing research on the types of knowledge needed for teaching algebra that were the focus of the study and by providing an overview of the literature on teacher professional learning – in general and in relation to teaching algebra.

Context and Background

Teaching algebra right from the early years has emerged as a central theme in current mathematics education (Greenes, et al., 2001) and varying views have been expressed on what algebra actually is, and what defines algebraic thinking (Kaput, 2008; Kieran, 2004). There is consensus, however, that *generalisation* is foundational, the cornerstone of mathematical structure (Kruteskii, 1976). Functional thinking in algebra has been defined as a type of "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalisations of that relationship across instances" (Smith, 2008, p. 143). Functions are used to model many real-world applications and functional thinking is important for learning in mathematics courses in the later years of schooling, particularly those on Calculus, for which an understanding of functions is foundational. Calculus underlies innovation and economic success across many science and engineering domains, and expertise is needed in this area of mathematics (e.g., Mullis, Martin, Gonzalez, & Chrostowski, 2004). The following sub-section briefly reviews the literature on the types of knowledge upper primary teachers need in order to develop their students' functional thinking through pattern generalisation.

Research on the Different Types of Knowledge Needed for Teaching Upper Primary School Algebra

Having conducted an extensive literature review on algebra, Kieran (2007) concluded that existing research has barely begun to explore the dimensions of knowledge that teachers

need for teaching algebra, particularly related to the development of students' algebraic thinking, their approaches, misconceptions and difficulties. Yet she asserted that the prolific research on the algebra *learner* can be utilised for research on the algebra *teacher*. The following two sub-sections seek to connect these two bodies of research by reviewing the literature on learning algebra and applying it to the types of knowledge considered necessary for teaching upper primary levels of mathematics. Hill, Ball, and Schilling's (2008) conceptualisation of different types of content knowledge and pedagogical content knowledge is used to frame four particular types of knowledge that were considered in the design of the professional learning program. This provides a basis for examining teachers' descriptions of their experiences of professional learning and analysing their perceptions of changes to their knowledge.

Shulman (1986) defined another type of knowledge beyond the two types of teacher knowledge conceptualised as *content (subject matter) knowledge* and *pedagogical knowledge* – *pedagogical content knowledge* (PCK) – that includes "the ways of representing and formulating the subject that make it comprehensible to others" and "knowledge of strategies most likely to be fruitful in reorganising the understanding of learners" (p. 9). The research lexicon on teaching and teacher education is familiar with this term. In the domain of mathematics, Hill et al. (2008) built on these types of knowledge and developed a framework that includes three types of content knowledge and three types of pedagogical content knowledge:

CONTENT (SUBJECT MATTER) KNOWLEDGE
- *Common Content Knowledge (CCK)*
- **Specialised Content Knowledge (SCK)**
- *Knowledge at the mathematical horizon*

PEDAGOGICAL CONTENT KNOWLEDGE
- **Knowledge of Content and Students (KCS)**
- **Knowledge of Content and Teaching (KCT)**
- **Knowledge of Curriculum(KC)**

The four types of knowledge highlighted in bold are those which this study incorporated in the design of the professional learning project: one type of content knowledge and three types of pedagogical content knowledge.

Content Knowledge

Common content knowledge (CCK) relates to the mathematical knowledge used in everyday life by adults and "is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics" (Hill et al., 2008, p. 377). *Specialised content knowledge* (SCK) enables teachers to "accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill et al., 2008, p. 378). It is "entirely mathematical, but it is not mathematical work done by many non-teaching adults" (Hill, Sleep, Lewis, & Ball, 2007, p. 133). Both CCK and SCK, however, do not imply knowledge of students or of teaching. In this study, SCK has been used to describe knowledge about pattern generalisation that is relevant to teaching upper primary students.

The recently introduced *Australian Curriculum: Mathematics* referred to the expectations of upper primary students being able to describe, continue, and create patterns and sequences, and to describe the rule that creates a sequence (ACARA, 2009). There are two ways of describing these rules which generalise quantifiable aspects of the pattern or sequence. Stacey (1989) referred to "near generalisation" which involves finding the next item using step-by-step drawing or counting, and "far generalisation" which involves finding

a general rule for any item (p. 150). Confrey and Smith (1994) described them respectively as co-*variation* and *correspondence*. A co-variation approach describes the relationship between successive items in a pattern – also known as *recursive generalisation* or a *local rule* (Mason, 1996). A correspondence approach perceives the relationship between two quantities or variables (the item/term position number in the pattern/sequence and a quantifiable aspect of the item/term itself – also known as *explicit generalisation* or a *direct* or *closed* or *relational rule*). Figure 1 provides an example of co-variation and correspondence approaches respectively in generalising a geometric growing pattern.
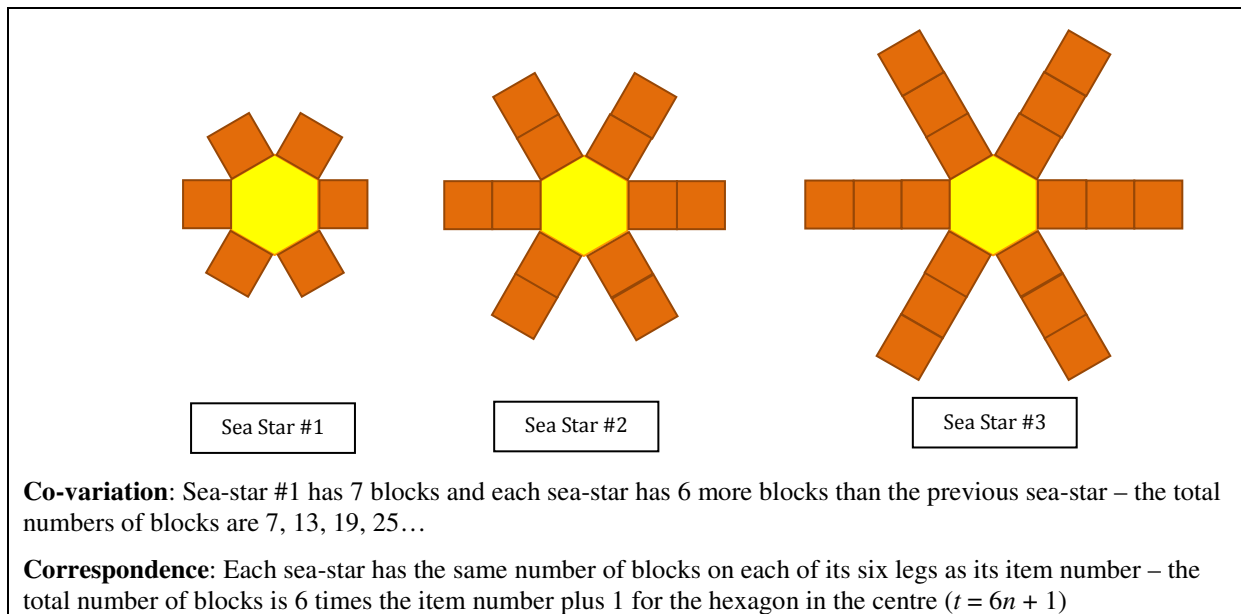

Sea Star #1

Sea Star #2

Sea Star #3

Co-variation: Sea-star #1 has 7 blocks and each sea-star has 6 more blocks than the previous sea-star – the total numbers of blocks are 7, 13, 19, 25…

Correspondence: Each sea-star has the same number of blocks on each of its six legs as its item number – the total number of blocks is 6 times the item number plus 1 for the hexagon in the centre ($t = 6n + 1$)

Figure 1: Two approaches to understanding functional relationships in a growing pattern

Figure 2 represents the same growing pattern in a table of ordered pairs.

Item position number	Item (e.g., total number of blocks)
1	7
2	13
3	19
4	25
5	31

Co-variation: "*When the item position number increases by 1, the item increases by 6.*"

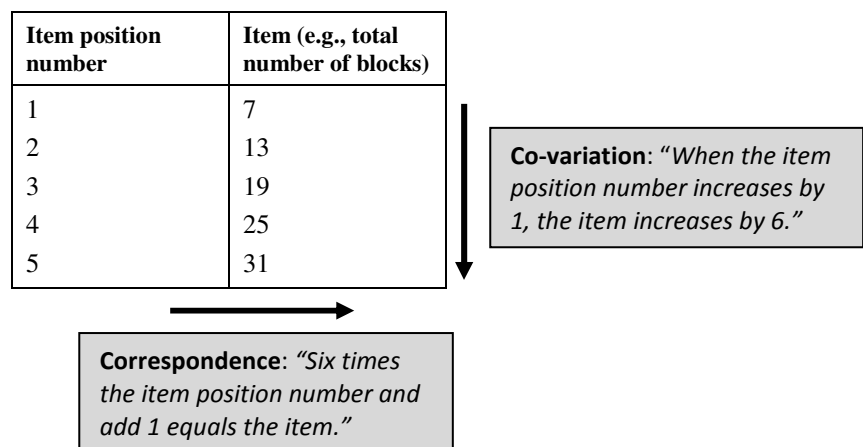**Correspondence**: "*Six times the item position number and add 1 equals the item.*"

Figure 2: Two approaches to generalising functional relationships using a table of ordered pairs (adapted from Smith, 2008, p. 147)

Radford and Pierce (2006) emphasised that algebraic thinking involves more than simply noticing a commonality between items in a growing pattern (which is possible with recursive approaches) and requires grasping that it applies to *all* possible items and being able to *express* it directly (explicit generalisation).

Pedagogical Content Knowledge

Pedagogical content knowledge is divided into three categories in Hill et al.'s (2008) model. The first type, *knowledge of content and students* (KCS) is defined as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (p. 375). Teachers with this knowledge attend to how students typically learn a concept, and to common mistakes and misconceptions. It implies an understanding of students' thinking and what makes the learning of particular concepts easy or difficult. In terms of pattern generalisation, earlier studies found that students had difficulty moving from co-variation approaches for continuing a pattern to correspondence approaches and creating a rule for a function (e.g., Confrey & Smith, 1994; English & Warren, 1998; MacGregor & Stacey, 1995; Stacey, 1989). Kaput (2008) emphasised the importance of students seeing the variable on which the pattern depends (e.g., in Figure 1, the item numbers underneath each sea star). Carraher et al. (2006) showed that students may even work correctly with a table of values where the two variables are in fact listed (e.g. Figure 2) but may have difficulty noticing the correspondence between them because they rely on merely extending each number pattern vertically.

The "knowledge of teaching moves" (p. 378) is conceptualised as a second type termed *knowledge of content and teaching* (KCT). KCT includes knowledge about how to choose representations and examples, how to *build* on students' thinking and how to address student errors effectively. The research literature details an extensive number of teaching approaches to developing students' functional thinking. Many of these incorporate the use of multiple representations of a functional relationship – diagrams of a growing pattern with item numbers, verbal and worded descriptions, symbolic expressions and equations, tables of values, and graphs (e.g., Confrey & Smith, 1994, Kaput, 1999; MacGregor & Stacey, 1995). A number of studies advocated the use of concrete materials in constructing growing patterns so that students are able to notice the changes between items and the structure within an item (Markworth, 2010; Moss, Beatty, Barkin, & Shillolo, 2008; Warren & Cooper, 2008). Friel and Markworth (2009) provided examples of several types of geometric patterns of increasing levels of complexity which give students multiple opportunities to experience the process of noticing the structure of the items – answering the question, "What is it that all these instances have in common?" (Kaput, 1999, p. 146) – and then creating a rule for the functional relationship.

The third type of PCK is conceptualised as *knowledge of curriculum* (KC) and matches Shulman's (1986) *curricular knowledge*. Ball, Hill and Bass (2005) additionally emphasised the importance of teachers not only knowing the content of curriculum but judging how to utilise it to present, emphasise, sequence and instruct. In this study exposure to a variety of curriculum documentation – state, national, and international – was incorporated in the design of the teachers' professional learning project to support their development of this type of PCK.

The next sub-section considers varying perspectives on teacher professional learning and how it is conceptualised and researched.

Approaches to Framing Teacher Professional Learning

Research continues to address the complex and controversial issue of effective professional growth and how it is conceived and measured. As with general theories of learning, how teacher learning is viewed and defined will affect how professional development programs are both implemented and evaluated. *Cognitive* perspectives on learning pay attention to teachers' individual development of a variety of types of knowledge

(Hill et al., 2008; Shulman, 1986) as a response to their involvement in professional learning. Lave's (1996) participatory or situated perspective considered learning as "an aspect of participation in socially situated practices" (p. 150). She highlighted the value of research focussing on "ways of participating and ways in which participants and practices change" rather than simply on tools and techniques (p. 157). Sfard (1998) conceptualised these two paradigms as *acquisition* and *participation*. The acquisition metaphor views learning as the process of gaining ownership of knowledge / concepts / ideas / meaning / facts by reception, transmission, internalisation, or construction. The participation metaphor replaces the idea of 'knowledge' as a commodity with 'knowing' as an action. Learning involves practices, doing rather than having.

Studies that examine teacher professional learning have tended to frame their research in terms of either the participation or the acquisition paradigm for learning yet often actually encompass ideas from both metaphors. Some have given precedence to the interactions between teachers for their learning whereas others have focussed on individual teachers' development. For example, Putnam and Borko (2000) framed their research using a situated perspective of learning but included the idea of individuals appropriating concepts and theories as their own. They described three approaches that were considered to support teacher learning. These included: researchers working alongside teachers in their classrooms, teachers bringing reports of their experiences in the classroom to workshops, and workshops that focus on teachers' learning of subject matter. Kazemi and Franke (2004) also framed their research on teacher learning using a situated theory of learning, investigating shifts in ten teachers' participation in monthly meetings over a year where they brought self-selected samples of student work to examine together. Although they paid attention to the *collective* transformation of participation of the group, they referred to the activity of analysing students' work as effective for developing teachers' *knowledge* of how their students were thinking and progressing (KCS). The researchers stated that teachers improved their own ability to think mathematically by learning to interpret student strategies they did not initially comprehend.

Some studies of teacher learning framed their research more in terms of an acquisition paradigm and sought to examine changes in teachers' knowledge or understanding, yet used contexts for learning in which teachers interact with each other. For example, Zwiep and Benken (2013) researched upper elementary and middle school teachers' learning of mathematics and science through content-rich learning workshops. They focussed on changes in teachers' content knowledge, their perceptions of mathematics itself, and of how students learn mathematics. They found that challenging teachers' existing content knowledge created "uncomfortable moments that led to change and growth" (p. 319). They suggested that the "content can be a critical vehicle through which change can be made in teachers' understanding and perceptions of mathematics" (p. 320).

Sfard (1998) argued that in trying to avoid acquisitionist concepts, the participation metaphor does not account convincingly for a learner's previous experience and how this is carried from one situation to another. She suggested that relinquishing either metaphor would result in problematic extremes. Instead each could be utilised for local sense-making, rather than seeking a paradigm to cover the entire field. Garet, Porter, Desimone, Birman, and Yoon (2001) considered both perspectives on what constitutes professional learning. They surveyed 1027 mathematics and science teachers to identify "three core features of professional development activities that have significant, positive effects on teachers' self-reported increases in knowledge and skills and changes in classroom practice" (p. 916). They described these features as: a focus on content knowledge, opportunities for active learning with collective participation of groups of teachers from the same school, and the coherence of programs with other activities. Desimone (2009) added the importance of *duration* of

professional development programs, noting that both the total hours spent on the activity and the span of time over which the activity is spread, contribute to the effects of a program on teacher's learning. Guskey and Yoon (2009) analysed 1300 empirical studies on teacher professional development. Those studies that were deemed to improve student learning outcomes were described as including (in addition to the above features): outside experts who presented ideas directly to the teachers (not via a school's own teaching coach) and supported their implementation through sustained follow-up after the main professional development activities.

The design of this study incorporated the above features by considering individual teachers' professional development in the context of participation with colleagues in existing teaching teams from their own school, and involving cycles of alternating meetings and lessons during one school year. The teachers' own perspectives on learning how to teach algebra were considered important. The study aimed to consider aspects of the professional learning program which the teachers described that might relate to the participation and acquisition paradigms and contribute to perceived changes in their knowledge or practice. The next sub-section reviews literature that considers research on professional learning, specifically for teaching algebra.

Research on Teachers' Professional Learning of Algebra

Despite explicit attention being paid to the learning of algebra in the earlier years of schooling, Carraher and Schliemann (2007) described research on the teaching of early algebra as in its infancy. Kieran (2007, p. 744) argued that in research on algebra teaching, "little attention has been paid thus far to the study and development of teachers' pedagogical content knowledge." A handful of studies that related to teachers' professional learning focussed on their development of knowledge. Warren (2006) researched teachers' development of both content knowledge and pedagogical content knowledge of algebra for the early years of schooling. She described the development of a professional learning framework that utilised sociocultural perspectives on learning and also incorporated an initial phase of "expert input and sharing" of mathematical "knowledge in action" via a demonstration lesson (p. 537). The second phase involved a cycle of collaborative lesson planning in pairs with expert input and feedback via email, teacher implementation in classrooms, reflection in a group, and decision-making about the next sequence of lessons. This study built on Warren's (2006) framework by incorporating an initial demonstration lesson in each class by the researcher and additionally team-teaching with each teacher throughout the year and several face-to-face interactions during meetings (rather than email).

In another study, Steele, Hillen and Smith (2013) investigated the development of mathematical knowledge for teaching patterns and functions using a content-focussed course with pre-service and practising teachers (mostly at secondary levels of schooling). They found that the key features of their teaching experiment that supported teacher learning of this aspect of algebra were: focussing on a specific area of the curriculum relevant to teachers, regular re-visiting and refining of the main concept, and teachers attending to tasks first as learners and then as teachers.

The study described here was designed with a consideration of the literature reviewed and sought to incorporate elements that research has highlighted as most likely to support teacher learning of algebra, from both acquisition and participation paradigms. The following section describes these elements in detail.

Research Design

This study adopted a design-based methodology where teachers and researcher experience the project as a collective effort and where teacher learning and student learning are two joint goals (Gravemeijer & van Eerde, 2009). The three key aspects of this methodology are instructional design and planning, ongoing analysis of classroom events, and retrospective analysis (Cobb, 2000). Teachers and researcher inquire together "into the nature of learning in a complex system" with the intent of producing "useable knowledge" (Baumgartner, et al., 2003, p. 7) – principles and "explanations of innovative practice" (p. 8). Interactions between materials, teachers and learners are enacted through continuous cycles in order to produce meaningful change in contexts of practice (Baumgartner, et al., 2003). In this study, these cycles involved the teachers in collaborative planning, implementing, evaluating, and revising lessons with the researcher in their year-level teaching teams (Hiebert & Stigler, 2000).

The researcher team-taught alongside each teacher for most of the lessons to provide a supportive environment for teachers to experiment iteratively with their teaching "on the basis of conclusions they themselves draw from data from their own classrooms" (Gravemeijer & van Eerde, 2009, p. 523). The researcher also supported the teachers' classroom experimentation by providing "a set of exemplary instructional activities and materials" sourced from the literature (p. 512). An example of one of the tasks is presented in an appendix. Teachers could revise and adapt these materials for their own students. The teachers and researcher co-analysed students' participation, work samples and inferred learning to revise the learning tasks and develop subsequent tasks. Both "careful review of the data and a reflection on the process of the teaching experiment" to understand more about "what induced the changes observed" were considered important (Gravemeijer & van Eerde, 2009, p. 514). The researcher was well-equipped to accept the role of 'expert' in this study given her experience teaching mathematics across primary and secondary levels and her familiarity with the relevant research literature.

The following sub-section describes an empirically-substantiated model for professional growth that was used for designing the professional learning program for the teachers in this study and for analysing the subsequent data to understand more about the mechanisms for change that might support teachers in learning to teach a challenging but important aspect of mathematics.

A Model to Conceptualise the Process of Professional Learning

A model for professional growth was used to frame the study conceptually in terms of the processes of teacher learning and to use as an analytic tool for examining teachers' perspectives on their experiences and the researcher's observations of their participation. An early version of the cyclic model was developed by Clarke (1988) and based on Guskey's (1985) linear models, but with a stronger emphasis on outcomes salient to the teacher. It was refined by Clarke and Peter (1993) after their research with secondary mathematics teachers and was developed further by Clarke and Hollingsworth (2002), using empirical data from three studies. Its design as a dynamic model sought to incorporate multiple possible teacher change pathways.

The striking similarity between the iterative character of design-research methodologies and models for teacher learning (Gravemeijer & van Eerde, 2009) is evident in this model. It is based on the premises that teacher learning flourishes where teachers work together (Gravemeijer & van Eerde, 2009) and that teachers are "active learners shaping their professional growth through reflective participation in professional development programs

and in practice" (Clarke & Hollingsworth, 2002, p. 948). The model, pictured in Figure 3, conceptualises the process of change in a teacher's professional learning through the mediating processes of reflection (dotted lines in model) and enactment (solid lines) between four different change domains. The change environment refers to the context in which teachers work and influences their professional growth by the level of access to professional development programs, restricting or supporting different types of participation, encouragement or discouragement of classroom experimentation, and provision or otherwise of administrative support to enable teachers' application of new ideas (Clarke & Hollingsworth, 2002).
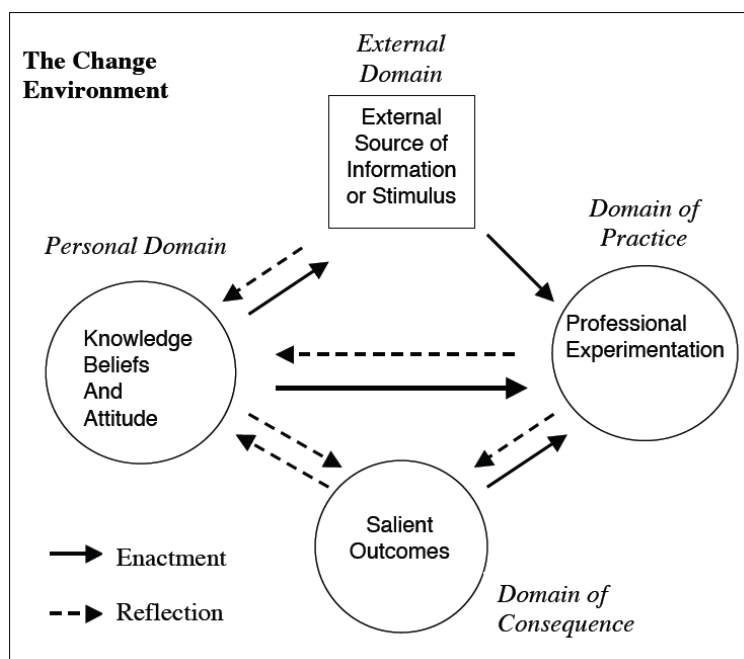

Figure 3: The interconnected model of professional growth (Clarke & Hollingsworth, 2002, p. 951)

Clarke and Hollingsworth's (2002) model was considered a valuable analytic tool for the study because it resonates with an appreciation of the complexity of teaching and of professional change leading to growth. It was empirically founded on studies of teachers' professional learning in mathematics and accommodates both pathways to change that teachers might commonly experience and those that are perhaps more idiosyncratic in nature because of teachers' individual response to different aspects. In this way, the model supports the anticipation and the encouragement of multiple avenues for change which seem to mirror realistically the possible mechanisms by which teacher learning might occur. Additionally the model does not require choosing between acquisitive and participative theories of learning since it can handle both interpretations of learning as the development of knowledge and of practice, which is consistent with the researcher's stance that such perspectives do not need to be dichotomous. The use of the model allowed the analysis to focus on the teachers' individual development of knowledge, their interactions with others for learning, and perceived changes in their practice.

This study's design incorporated the following change domains of the model in order to investigate the teachers' perspectives on their experiences of the professional learning program and to explore possible pathways to the development of knowledge and practice in teaching algebra:

- *External Domain*: Provision of sources of information, stimulus, and support in the form of: demonstration lessons by the researcher; advocated practices in teaching this aspect of algebra; professional reading, exemplary instructional activities and materials as documented in the research literature; discussion of the relevant content in various curriculum documentation; and iterative facilitated discussions with other teacher participants to reflect on their students' learning, revise lessons, and collaborate on lesson planning.
- *Domain of Practice:* Enacting of new teacher knowledge through iterative classroom experimentation in the form of team-teaching of lessons, post-reflection, and analysis of student work samples.
- *Personal Domain:* Throughout their participation, teachers were able to reflect on their changing knowledge, beliefs and attitudes, and were asked to explore these more formally in later individual interviews and group interviews. The researcher was also able to observe perceived changes in teachers' engagement, personal responses, and beliefs during lessons and meetings.
- *Domain of Consequence*: In their final interviews, teachers were asked to describe the salient outcomes for themselves and for their students after their involvement in professional learning. Changes in their knowledge for teaching algebra were also investigated in a final written survey and they were asked about their future enactment of professional experimentation in this area of mathematics and in general.

The design of the professional program for teachers sought to offer the teachers a variety of opportunities for learning consistent with possible individual inclinations and in keeping with the model's conceptualisation of multiple pathways for professional growth. Clarke and Hollingsworth (2002) emphasised that different teachers may interpret an experience in different ways and that it is an individual teacher's *interpreted* change, rather than only observable change, that is crucial to subsequent change in their own knowledge, beliefs and practice. The individual and group interviews conducted with teachers at the end of their participation focussed on teachers' own interpretations and perceptions of their experiences of professional learning and changes in their Personal Domain. These were then related to any changes in their engagement, personal responses or beliefs as perceived by the researcher via observation notes and analysis of meeting audio-recordings.

The next sub-section explains how the collection of preliminary survey data was utilised in the design of the study.

Teachers' own Suggestions for Professional Learning

A survey of 105 upper primary teachers was conducted prior to the study to examine what practising teachers actually knew about teaching functional thinking, since there was found to be little information on this in the literature. The findings from the survey on teachers' mathematical knowledge for teaching algebra and the implications for their professional learning (Wilkie, 2014) informed the design of the case study, in particular the content and types of experiences provided to the teachers during their participation. The survey also included an item that asked teachers to suggest the most helpful types of professional learning for algebra. Their responses were coded using NVivo qualitative analysis software. The emergent categories, in order of decreasing frequency, are presented in Table 1. Several teachers listed more than one type of support, for example, "Improved knowledge through PD and discussion with peers after hands on with students." This response was categorised as "Expert input – training", "Collaboration with colleagues" and "Experimentation."

Category	Percentage of teachers	Illustrative examples
Ideas and resources	38.1%	"Lots of ideas. We love hands on relevant teaching ideas that link to the curriculum and can be assessed" "A variety of lesson activities and extensions activities for those who find the growing patterns and rule identification come easily"
Expert input - Training	29.5%	"I need more training. I find patterns difficult to find and need to have an answer prior to my lessons" "More professional input"
Collaboration with colleagues	15.2%	"Working as a team/level to plan activities" "Talk to same level teachers/staff"
Expert input - Demonstration lessons	13.3%	"See an example of it being taught" "Demo lessons would be helpful"
Professional reading	8.6%	"Some professional readings on what it is we need to achieve" "Time to read and absorb it which I will do in time"
Experimentation	8.6%	"The chance to do some lessons and check how they go and what my understanding of content is" "Time to experiment"
Ambiguous response	1.0%	"Looking at patterns and how they look e.g. caterpillar pattern great way of students identifying rule"
No response	14.3%	

Table 1: Teachers' suggestions for types of professional learning support ($n = 105$)

It can be seen that the teachers' suggestions related mostly to the *External Domain* in the previously described Clarke-Hollingsworth model – external sources of information or stimulus, such as ideas and resources, expert input (both training and demonstration lessons), and professional reading. In this study, resources that were provided to the teachers included professional reading, an assessment rubric, task ideas, suggested lesson plans, student task handouts, pictures, photographs, suggested assessment tasks, lists of hands-on equipment needed, suggested solutions to tasks, and researcher-assessed rubric scores for their students' completion of an initial assessment task. Collaboration with colleagues can also be considered as a form of external stimulus for professional learning since teachers can receive information and stimulus from each other. This interaction relates to the previously described participatory paradigm for learning. Interestingly, in a survey of all of the CTLM teachers (not just the upper primary teachers) on their experiences of professional learning over a two-year period, the most helpful aspect indicated was the *planning sessions with other teachers* (Clarke, D., et al., 2011). These sessions incorporated both expert input and collaboration with colleagues, and were conducted in teachers' own contexts – their school environment and their own mathematics programs. In the algebra study described in this article, the researcher attended all project meetings onsite at schools with each of the three teaching teams throughout the year to provide expert input and to take an explicitly participatory role in the teachers' collaboration.

Less than 10% of teachers referred to the *Domain of Practice* and their interest in professional experimentation. It is likely that the wording of the question may have affected

this response and that if teachers were asked about their *own* plans for professional learning, rather than for suggestions per se, a higher proportion may have referred to this change domain.

Professional Learning Design

Based on a review of the literature on developing functional thinking and teacher learning, the results of the previously mentioned teacher survey, and those teachers' own suggestions for professional learning, a program was designed to incorporate the development of teacher content knowledge and pedagogical content knowledge, and all of the categories suggested by teachers themselves, with a focus on participative opportunities for learning. An outline of the design is presented in an appendix.

Data Collection and Analysis

This study was designed to explore teachers' professional learning over time through several interactions in lessons and meetings during the school year and to utilise several sources of data (Creswell, 2007). It incorporated a "descriptive and interpretive" approach to data collection and analysis (O'Toole & Beckett, 2010, p. 43) that acknowledges the shaping role of the researchers and the need for "ongoing reflexive attention" (Yates, 2003, p. 224). Initial data included the previously mentioned survey of 105 upper primary teachers. At that time, the principals and school mathematics leaders of the schools participating in CTLM were informed of the project and subsequently indicated their upper primary teachers' likely interest. Several schools responded and two of these were selected, with consideration of the researcher's university teaching commitments and the number of participating teachers and classes. Data from the subsequent collective case study of 10 teachers for a one-year period form the basis for the discussion in this article. A sequence of five lessons in each teacher's class, with pre- and post-meetings in teaching teams and attended by the researcher, was timetabled. These meetings were audio-recorded and included data on teachers' discussion of: students' mathematical activity and work samples; classroom norms and mathematics practices; exploration of concepts using instructional materials; evaluation of the previous lesson, and planning for the next lesson. Learning tasks were also designed to solicit informative written data on students' mathematical thinking and interpretation which were analysed and discussed in teaching team meetings (Cobb, 2000). A researcher's journal was kept to document observations of teacher engagement, changing practices, and reflections from lessons and team meetings. The various data collected provided for the analysis of the teachers' interactions with each other and the researcher for learning (participatory paradigm) and on their individual development of different types of knowledge (acquisition paradigm). The video-recording of lessons was not considered financially viable for this study but may have provided additional data on the teachers' changing practice throughout the year. Figure 4 provides an overview of the data collected during teachers' participation in the research.

Data collection	Research methods	Sources of data
INITIAL	Survey	– Questionnaires (*n* = 105)
MAIN	In-depth collective case study Cycles for 5 lessons: **Demonstration lesson** ↓ **Planning with researcher (expert) & colleagues** ↓ **Lesson with students** ↓ **Debrief with researcher & colleagues** *(Teacher's own experimentation)*	For 10 participants in 3 teaching teams: Yrs 5/6 (*3 teachers*) from one school, and Yr 5 (*3 teachers*) and Yr 6 (*4 teachers*) from another school: – Audio-recordings of meetings (planning & debrief) – Student work samples from 10 classes for 5 lessons Photographs from lessons Researcher's journal
FINAL	Individual interview	– Audio-recordings of 10 interviews
	Individual post-survey	– Questionnaires (*n* = 10)
	Group interviews in teaching teams	– Audio-recordings of 3 groups

Figure 4: An overview of the data collection process for the research

The issue of "voice" in research is discussed extensively in the literature (e.g., Clough & Nutbrown, 2007; Creswell, 2007; Mertens, 2005). It was considered important in this study to seek teachers' perspectives – their activities, experiences and perceptions about learning to teach algebra. An individual interview with each of the teachers towards the end of the school year explored their experiences of professional learning and aspects that they perceived as contributing to their knowledge about the effective teaching of algebra and to perceived changes in their practice. The teachers also participated in a group interview with members of their year level teaching team (three different teams across the two schools) to reflect on the experiences of their students as well as on challenges, surprises, and suggestions regarding the development of functional thinking. This was to elicit further insight into the teachers' development of knowledge of content and students (KCS) and of content and teaching (KCT) in particular. Sample questions from the semi-structured interview schedules are provided in an appendix.

A final survey of the 10 teachers (using the same questionnaire as the initial survey with 105 teachers) investigated changes to their content knowledge, pedagogical content knowledge, practices and attitudes after their participation in the case study. The content of the questionnaire items had not been re-visited with these teachers during the year. Findings about changes in the teachers' knowledge for teaching algebra are to be reported elsewhere.

Data (audio and textual) analysis from the collective case study was undertaken cyclically throughout the year to enable emerging ideas to re-shape perspectives, improve instrumentation, and allow for additional data gathering (Miles & Huberman, 1994). Themes were developed from line-by-line inductive coding of interview transcripts, pattern searching,

grouping of codes into conceptual sets, and triangulation with the audio-recordings of meetings and observation memos from the researcher's journal. The use of QSR NVivo qualitative analysis software supported initial coding, the refinement of coding and themes, and the adaptation of themes at different levels of abstraction (Creswell, 2007). Peer review of data analysis, coding and theme development was undertaken by the co-author, as director of the overall CTLM project, to increase the robustness of findings (Lincoln & Guba, 1985). Table 2 presents a list of the codes created from the analysis of the teachers' individual interviews and the three group interviews.

Code	Number of teachers	Number of references
Contributed to professional learning		
Comparing notes with colleagues	9	17
Experiencing change in perceptions of algebra	5	6
Experiencing impetus to improve own content knowledge (CK)	5	10
Receiving input or resources from expert	9	19
Noticing different student-grouping strategies	4	8
Noticing the design of tasks for students	10	34
Observing expert teaching	8	15
Observing sharing time in lessons	8	11
Re-framing knowledge of how students learn algebra by observing		
Noticing student engagement or enjoyment	10	22
Noticing students' variety of strategies	3	9
Observing their students' growth	8	21
Surprised by particular students' responses		
Capable students who struggled	3	5
Students who exceeded expectations	8	12
Issue or concern		
An aspect of the project's implementation	8	29
Concern about their own CK or PCK	8	19
Concern about their students		
Struggling students	4	5
Students' lack of experience or knowledge	2	2
Students' negative perceptions of algebra	4	8
Own perceptions of algebra		
Time as a challenge		
Outcomes of participation	9	27
Recommendations		
Supporting student learning	10	41
Supporting teacher learning	7	14

Table 2: Coding hierarchy created inductively from individual and group interviews of the ten teachers

This article discusses emergent themes from the analysis that focus specifically on teachers' experiences of participation in the professional learning program, to identify how these relate to changes in particular domains and to highlight possible pathways to change in their knowledge and practice for teaching algebra. The teachers' interpretations of their experienced processes of change and the outcomes that they considered salient to their own context are explored. They were assigned pseudonyms for the purpose of reporting on the study's findings.

Discussion of Findings

Previous research has highlighted the perceived value of a socio-cultural approach to teacher professional learning that takes place within teachers' own contexts and involves inquiry-based collaboration with other teachers. This study sought to explore these aspects of professional development with upper primary school teachers learning to teach a challenging aspect of algebra. It aimed to analyse their perspectives on their experiences of a design-based research project, to note which aspects they attended to in their reflections, and to explore how these relate to possible pathways for changes to their knowledge or practice. Although the data collected from the project provided valuable information on the teaching and learning of algebra, the purpose of this paper is to focus explicitly on the teacher in terms of their perceived salient outcomes and professional growth. The following discussion focuses on the teachers' own reflections about their experiences using data from their individual and group interviews. It is structured around three emergent themes that draw attention to both patterns of similarity in teachers' descriptions of how changes in a particular domain related to change in others and also to individualistic processes. These are discussed with reference to Clarke and Hollingsworth's (2002) previously presented model. The three themes are: *Modifying beliefs by observing their students*, *Developing confidence through comparing notes with colleagues*, and *Dealing with discomfort*.

Modifying Beliefs by Observing their Students

Throughout the year, it was noticeable in meetings that the teachers frequently recounted to each other their observations of their students' actions during the algebra lessons, particularly those students' responses that surprised them. In interviews, all of the teachers highlighted that watching their students engaging with the tasks attracted their attention. "That really helped me, seeing you, how you got the kids thinking" (Fiona). "It was good to see, and just having the time to watch what you do and watch how the kids – I was able to take notes and see what kids were doing" (Gemma). These comments about observing their students were found to relate to a number of common foci.

One focus which drew all of the teachers' attention was the positive engagement of their own classes during lessons, which seemed to be a source of surprise. They each referred to noticing their class' or a particular student's positive affect or engagement:

> Heather: But just seeing her eyes, and maybe seeing her excitement that she got it. That was good to see, and another boy Jake.

> Tom: Probably their 'aha' moments, for particularly some of the kids where you could just sort of see the light switch on and that sort of made sense to them.

A few teachers indicated that they had been initially apprehensive about their students' likely response to the tasks or even to algebra itself. At times it was hard to differentiate between teachers' own feelings about algebra and their perceptions of their students' attitudes. One teacher explained that "the unknown can be quite scary, and the word 'algebra' doesn't bring nice thoughts, really" (Fiona). Another teacher said "I think sometimes even as adults we get a bit scared, we're just like 'Uhhmm'" (Paula, g[1]). Another teacher explained the initial reactions of other teachers in the school to their participation in the project: "In the staffroom people are like 'Oh G-! Thank G- I'm not in Year 5/6!' because they were quite daunted by what we were doing" (Gemma, g). For several teachers, it seemed that noticing their students' interest and engagement over the course of the five lessons

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[1] Response during group interview

contributed to a change in their own beliefs about algebra:

> Gavin: I wasn't sure how they would find it, and I was a bit nervous because I thought this could end up being too challenging for them, but it wasn't. It was challenging, but it wasn't to the point where they became disillusioned and lost confidence. They were excited and they were happy about you coming back.

> Sarah: I think when we all saw what you were doing, we were far more relieved than – we didn't really know what to expect. But then, we were hands-on, you were there… and the kids were totally engaged (g).

One teacher said of algebra that "it's not as scary as we think it is" (Fiona). Others said that "it doesn't have to be as mysterious as what it appears to be" (Gavin) and that "it's not as threatening" (Molly). Ball (1996) described one consequence of professional learning as "revising deeply held notions about learning and knowledge" (p. 2). In this study several of the teachers indicated that their perhaps negative perceptions about algebra were revised over time as they watched their students engage in learning experiences with a positive affect or attitude.

This theme highlighted a change sequence which appeared common to all of the teachers' experiences: a change in their external domain via a source of stimulus in the form of their own students' responses to being taught algebra by someone else, leading to a change in their personal domain which was a modification of their beliefs about and attitude towards algebra. It is worth mentioning that this was a process involving multiple lessons, not a once-off observation. It is speculated that this helped strengthen the association teachers made between algebra learning and positive student affect and was less likely to be dismissed as a once-off coincidence. The salient outcome for several teachers appeared to be an increased interest in algebra and how it could be taught; as one teacher reflected, seeing her students "empowered" and succeeding in their learning "really spurred [her] teaching on" (Gemma). Several teachers raised their intent to continue their own classroom experimentation with teaching algebra after the end of the project. This process is captured in the Clarke-Hollingsworth model via an arrow of enactment from salient outcomes up to professional experimentation, demonstrating the processual and cyclical nature of change.

A second focus common to a number of teachers was their learning about how children develop functional thinking through observing their own class in action; this relates to their development of Knowledge of Content and Students (KCS). One teacher said that "to see that developmental process over the whole time was really enlightening I think" (Paula). They "[knew their] own children" and having someone "teach [*their*] children" (Trisha, g) helped them observe their students' responses to the tasks. A pattern that emerged from the teachers' reflections was the need to reconsider their "assumptions about students" (Ball, 1996, p. 2) after observing unexpected student responses in the lessons. Eight of the teachers described their surprise about the response of some of their students to a task they thought would be too hard:

> Paula: Initially I went, 'Oh my goodness, my kids are never going to be able to do this.' That's just how I felt and I said a couple of things to the others, 'I'm not sure how they're going to go with this' or whatever. But I guess sometimes they can surprise you.

> Molly: The ones that were usually, that usually struggled in maths were actually the ones to pick up on some of the strategies. That was a real surprise.

> Yvette: I think the highlight would be and it got me a little bit excited, was my low kids that sometimes struggle through some maths concepts no matter how you present it (you're like 'oh no') they were able to look at the patterns, and they could come up with a formula which I thought was fantastic. A child that I thought 'oh maybe they won't do as well,' just surprised me.

Some of the teachers speculated on the reasons for students exceeding their expectations. One teacher related it to the use of visual representations in the tasks which supported her "arty students" (Heather). This relates to research by Rivera and Becker (2006) who found that students who were able to use the *figural* clues in a geometric growing pattern rather than rely on numeric clues were more likely to be able to create an explicit generalisation. Lannin, Barker, and Townsend (2006) also found that students who moved too quickly to numeric strategies had difficulty making connections across different generalisation tasks. Another teacher also referred to "visual" patterns as benefiting her students who typically struggled in mathematics and additionally commented on the novelty of the learning tasks, that they were different to usual (Molly). Another teacher thought that having a real-life context for each task and providing time to explore was important for students who usually struggled – they would "forget they had to be this person who couldn't do it" (Sarah).

Interestingly, three teachers also referred to their surprise that certain students they deemed as highly *competent* in other areas of mathematics seemed to *struggle* with functional thinking. They did not expect that it was not the capable students who were able to generalise explicitly but other students deemed less capable mathematically. One teacher said that "a couple of children that do well in maths generally – overall, they're my clever ones in most areas – didn't do so well in these lessons" (Fiona). There was the sense that teachers related this issue to an unfamiliar learning process for these usually capable students – "a whole new way of thinking" (Fiona). A different approach to generalisation – figural and structural perhaps rather than only numerical – seemed to appeal to students *other* than those whom teachers would have described as capable in mathematics.

A third focus which was common to several teachers related to changes to their knowledge of content and teaching (KCT). They described how they noticed the use of questioning (asked by the researcher during lessons) to help students notice features of the pattern structures, to recognise quantifiable aspects of patterns as the variables, and to the correspondence between variables and the structures.

> Fiona: When you were talking to children, I was listening to see how you were getting them to explain or articulate what they were saying, so that I could model that as well.

All of the teachers commented in their interviews on particular aspects of the lessons that they perceived as being effective for teaching functional thinking (KCT). Frequent comments related to the use of hands-on materials (pattern blocks, tiles, and counters) to create growing patterns rather than working with "just number sequences" (Fiona). One teacher said, "I suppose it was more interesting than I expected... Those concrete blocks were really great for the kids to see" (Heather, g). Several teachers referred to the chance to "play", to make, to "explore", and to connect the patterns with the different variables. Another said "they're seeing it, they're doing it, they're understanding it" (Gemma). One teacher contrasted the use of hands-on material with his experience of learning algebra at school:

> Gavin: What helped was the fact that there was a lot of hands-on, as well. It helped them to conceptualise… When I was at school, algebra, it was just all up there, and I really struggled.

Several teachers referred to the real-life context of the tasks, such as using a story or role-play to provide additional connections to students' prior knowledge:

> Tom: That was probably the thing, just sort of to see how you related to a real-world problem and see how you sort of talked to the kids.

> Sarah: Watching you come in with a real-world problem, and then showing us a hands-on concrete illustration of what that would look like, was far easier to do than I would have thought. (g)

Others commented on the increasing difficulty of the patterns themselves as the lessons progressed throughout the year.

Eight teachers drew attention to their experiences of "sharing time" at the end of each lesson. The researcher (and eventually some of the teachers) observed students throughout the lesson and subsequently selected particular students or pairs of students to describe and demonstrate their ways of visualising the pattern and their approach to explicit generalisation to the rest of the class. Several teachers noticed that some students were able to connect their ways of thinking to other students' solutions, to "seeing the penny drop at times" for students while watching another student explain their strategies (Gavin). For the researcher, the students' considerable interest in each other's solutions during sharing time was noticeable. One class even wanted to stay in and finish during recess! "You run out of time because they all want to share!" (Molly, g). Quite a few teachers mentioned that they were going to try and use sharing time more in their own mathematics teaching practice. Lave (1996) highlighted the benefits to students in being able to tutor others, to engage with each other and depend on each other for learning. In this study, students tutoring the class during sharing time additionally benefited the teachers in improving their knowledge of algebra and teaching algebra (SCK and KCT), refining their knowledge of their students (KCS), and modifying their beliefs. An outcome salient for many of the teachers but unexpected (by the researcher) was their interest in experimenting with the teaching strategy of sharing time in their own teaching of other areas of mathematics.

Using the Clarke-Hollingsworth model, it can be seen that the use of a design-based research methodology enabled a cyclical process of teachers experiencing the same external stimulus (the observation of their students being taught by someone else), leading to their increased professional experimentation (supported by team-teaching) and change in their attitudes, beliefs and knowledge about teaching and learning. Philipp (2007) stated that teachers' beliefs and attitudes can be changed by seeing practices that are effective. For the professional learning of algebra, this study demonstrated that one possible mechanism that seems to resonate with different teachers and has the potential for changing knowledge and practice is the incorporation of *cycles* of changes in teachers' external domain and in their domain of practice, rather than a perhaps simplistic linear program.

Developing Confidence through Comparing Notes with Colleagues

In this second theme, the external sources of information or stimulus to which teachers attended in their reflections were various aspects of their involvement with other teachers in the project. Unlike the previous theme, however, some of their experiences of change in this domain appeared to be an *impingement* on their professional learning for a few of the teachers. A participatory metaphor for learning focuses on involvement in a community of practice, with access to ongoing activity, other members, information, resources and participation opportunities (Lave & Wenger, 1991). For a handful of teachers in this study, it appeared that some aspects of their participatory experience led to salient outcomes that were not supportive of their professional growth. This relates to Clarke and Hollingsworth's (2002) notion of the change environment influencing a teacher's professional learning, yet in this study the issue was less to do with the previously described levels and types of support in the school context, and more to do with pre-existing interpersonal issues between and among members of one of the teaching teams.

> Sarah: We've worked at improving our group dynamics this year… but we're all quite different… I think we're quite professional, but it will take us another couple of years to, I think, be really comfortable.

The concern about displaying a lack of knowledge in front of others was raised by these teachers. "Sometimes you feel like that you have to say something cool" (Tom). Interestingly, a few teachers appeared to experience this peer-related anxiety as an added incentive to work on improving their knowledge. One teacher said having to meet with others "forced [her] to look at [her] work" and "kept [her] on task" (Heather). Another teacher found that comparing notes with colleagues increased his sense of being different: "You think, 'Oh, I'm the only one that's thinking that'" (Gavin). There was the sense that teachers' day-to-day relationships with each other outside of the context of the algebra project decreased their level of involvement in the project meetings. This was noticed by the researcher over time in meetings where a sense of tension seemed to impede the teachers' focus in the meetings and their willingness to share. This issue was raised independently by the school mathematics leader who described similar experiences with the same team in other meetings. Perhaps involvement in a community of practice outside of their school environment, such as an online community or an external professional development course, may have resulted in different outcomes for these teachers (but are outside the scope of this study). This again highlights that teacher professional growth is very much related to how an individual perceives and interprets change in their different domains and what may be salient to them varies according to their own perspective.

Despite the idiosyncratic and negatively-perceived experiences of a few teachers in their participation with others, nine out of the 10 teachers did describe at least some aspect as being associated with a beneficial salient outcome. In the Clarke-Hollingsworth model the collaboration with colleagues and the researcher can be viewed as an external source of information or stimulus, with their sharing of classroom experiences, discussion of student work samples, and discussion of future lessons. The aspect of involvement most frequently highlighted by the teachers was examining their students' works samples together:

> Molly: So I struggled with that, not knowing what I was looking for until we nutted it out in our planning session and the three of us just fed off each other. And then that was quite easy after that.

Another teacher said that "talking about and using students' work was really good and confirming." She also found it useful at the same time to "de-brief [her]self about [her] own understanding of what [she] had learnt as well" (Paula). One teacher found that comparing students' work helpful for clarifying the difference between co-variation and correspondence approaches to generalisation and that it was "particularly helpful to look together and say 'That's an example of that, that's an example of that. What do you think?'" (Trisha). Another teacher found that moderation and comparing her students' work with other classes helped her set realistic expectations of her own class. A few teachers found it "affirming" to compare their interpretation of student work with the researcher's, and then to discuss these as a team. "It was good to see that we were on a par so we understood. Like what we were looking for was what you were, you had looked for" (Gemma). It also clarified their own understanding of what students' responses meant (KCS):

> Gemma: You could say, you know, 'I've got a kid that did this. I don't understand.' So we'd talk about it, or you'd pick a student out and say, 'Oh, this person did this. Let's explore what was going on there.' And I mean it's great to do that.

For those teachers whose students' responses were surprising, either by exceeding their expectations or by struggling unexpectedly, being able to share their surprise with other teachers appeared to help them to explore together reasons why their students were responding unexpectedly, to solve the 'puzzle' collaboratively and to re-adjust their expectations. As with Kazemi and Franke's (2004) research on promoting collective inquiry with student work, this study also found that the use of student work on common tasks that

came from teachers' own classrooms enabled the teachers to build common ground, develop shared meaning and increase their ability to articulate this with mathematical language. Sharing each others' student work samples from actual lessons they had participated in, encouraged teachers to shift from a "general pedagogy to one that is particularly connected to their own students" (p. 204).

Several teachers highlighted the value of knowing from others "a bit more about what went on, because each classroom didn't seem that it happened exactly the same [sic]" (Fiona). One teacher said she learnt "just as much from the debriefing, from everyone else and what their experiences were" as from involvement in the lesson itself – "doing it as well" (Paula). Some teachers found that meeting together "broke the isolation of being a teacher down, because it was collegial" (Gemma) and provided "validation" (Tom). One teacher said, "I was able to see whether I was on the right track or not. If they did something similar or their kids responded in a similar manner then that helped me" (Yvette). Some teachers discussed their differing interpretations of similar experiences. For example, one teacher commented positively on the use of longer-than-expected wait time by the researcher when questioning the class whereas another teacher expressed concern that such lengthy wait time meant that her students had been "shy" and too scared to give a wrong answer (Molly).

The reflections of the teachers on their participation in meetings and their informal interactions with colleagues seemed to relate to their perceived benefit of being able to *compare* with others because of common ground – their classroom experiences with similar tasks, their students' responses, their students' work samples. This contrasted with the previously mentioned tension between members of one of the teaching teams. Graven (2004) found significant gaps in the literature on research employing participatory perspectives on learning to explore teachers' development of mathematical confidence. She sought to analyse the conceptualisation of confidence in collective practice and viewed confidence as "both a product and process of the mathematics teachers' learning" (p. 179). The teachers' association between their collaboration during the professional learning program and their changing confidence partly resonate with Graven's conceptualisation of confidence as "part of an individual teacher's ways of learning through experiencing, doing, being and belonging" (p.179), but there are other complexities illustrated by the inter-personal tensions in one of the groups.

Dealing with Discomfort

Experiencing discomfort or difficulty in the process of learning can be seen not as an impediment but an impetus for deeper, transformative learning, particularly if the learner can commit to engaging in a dynamic process of grappling with their assumptions and difficulties (Nelson & Harper, 2006). For the teachers who participated in this professional learning program, their reflections during interviews highlighted their experience of discomfort in two aspects of their involvement: completing the initial questionnaire on their prior knowledge for teaching algebra (for a copy of the full questionnaire please refer to the appendix in Wilkie, 2014), and certain moments in class during the algebra lessons. For some of the teachers, their perceptions of not knowing enough initially about upper primary algebra, and later not 'being the expert' in their classes for this area of mathematics were a source of discomfort. These two experiences can be viewed as external sources of information or stimulus, but the second occurred in teachers' domain of practice because of the cyclical nature of a design-based methodology which incorporates experimentation alongside external input. These sources of discomfort and the change mechanisms that teachers associated with them are discussed in turn.

Eight teachers referred to concern about their knowledge of algebra (both CK and PCK). Some of their reflections indicated that the experience of completing an initial survey on their prior knowledge highlighted for them their need for professional learning about algebra (the survey had been completed before they had received the invitation to participate in the project). One teacher said she was "aware of her lack of knowledge" and that she "struggled with that test and it wasn't fitting or couldn't click into it" (Trisha). Other teachers related their lack of knowledge to their lack of experience in learning it in the past:

> Gavin: It's not something that I think – I know I certainly in the past haven't done much on anyway… This is an area that is not covered so much.

> Fiona: Beforehand I would say I knew very little about algebra… it's an area we don't know very well at all.

Although a source of discomfort, there was the sense of increased motivation to address their lack of knowledge by participating in a professional learning opportunity: "So hey, if you're going to come in and teach us to teach it, then that's got to be a bonus" (Fiona). Two of the teachers were quite new to the teaching profession and seemed less uncomfortable with their self-perceived lack of knowledge: "Being a new teacher, I'm happy to learn off anybody" (Paula). In this case, a external source of information negatively-perceived by some teachers nevertheless was associated with reflection leading to a change in their personal domain (beliefs and attitude) and to enacting further willingness to experience change in their external domain.

The second source of discomfort was experienced by some teachers in class during interactions with their students. One teacher highlighted the discomfort she experienced when a student shared an unanticipated alternative solution to a task that caught her unawares in class (during her first attempt at leading sharing time). She described preparing for the task: "I came up with the answer and I truly thought that was going to be the only way of getting it" (Fiona). She said that the experience "threw a spanner in the works" initially but she dealt with it by asking the student to explain their alternative solution. She later reflected:

> Fiona: 'Well, this group came up with this.' And a few of them going, 'Huh?' and others saying, 'We came up with this way.' 'Oh'. They're both right and they're both differently written, but they're both right. So, yeah, I thought that was quite – that was also a surprise, but a challenge trying to be on top of it.

The challenge of being aware that there are different ways of visualising a pattern and therefore different formats for expressing a functional rule was also highlighted by other teachers. The sense of not really grasping these possibilities beforehand generated a sense of anxiety about being in front of students during lessons and not looking like an expert:

> Heather: If I have a worksheet, and I haven't looked at it, and I don't know the answer, just the loss of face I have is just tremendous, and it just deflates me so much.

She said that receiving tasks beforehand, going through them, and discussing the different solutions were "really helpful". Another teacher agreed that going through "the possible outcomes or answers or solutions" was an important aspect of preparation for a lesson. Wanting to be able to assess his students' work after each lesson, although not a quick or easy process, was an impetus for grappling with the mathematics: "It meant that you had to look at it because there may be different ways, there's not just one answer, and it opens up the possibilities and that takes a little bit longer to assess" (Gavin). Ball (1996) emphasised the importance of a teacher's development of knowledge for "interpreting students' unexpected statements and solutions" since guiding a class discussion "can be treacherous when the teacher is unsure of the terrain being explored" (p. 2).

Another teacher described a sense of discomfort when walking around the class looking at students' answers and being asked by them if they were correct. He commented, "Your kids come at things different ways" (Tom). For him, struggling to interpret and judge the correctness of his students' various solutions in front of them appeared to focus his later attention on the students' different solutions at sharing time (researcher-led in this case for all of the lessons) so that he would be able to respond appropriately to student questions in the future. He commented that when the researcher had "specifically picked out" students, it was helpful for highlighting the variety of solutions to the rest of the class. It also appeared to increase his KCS and KCT.

There seemed to be a noticeable and not unsurprising association between a teacher's level of understanding on how to generalise a particular growing pattern for themselves and their subsequent level of engagement with students on the same growing pattern in a later lesson. If they had grappled with and understood the pattern for themselves, they were more likely to take an active role in the classroom rather than an observational role. For a challenging area of mathematics such as algebra, the use of team-teaching during classroom experimentation sought to alleviate teachers' anxiety and provide a 'safety net' for experimentation. In this study, this strategy produced mixed success, with some teachers being increasingly more willing to experiment with their teaching, and others adhering to a mostly passive role. Two teachers preferred to focus their attention on assisting a small number of students and avoided wandering around the class and conversing with students about their thinking. For one of these teachers, her experience of discomfort in class was associated with a salient outcome of a *decreased* sense of confidence, despite the team-teaching approach:

> Trisha: I found it personally quite hard. I think my confidence probably lowered more than– because I'm aware of my lack of knowledge and that my brain just doesn't think along those lines. I struggle with it.

The other teacher related her discomfort, not to algebra as a challenging area of mathematics, but to having a larger class that year (28 students after 20 students previously) and the pressure of meeting their diverse learning needs. But she did refer to a sense of discomfort during the algebra lessons: "I don't feel I own it; I don't know where I'm heading" and "I admit I did feel a bit – not threatened, but inadequate – when I was helping out, when I was giving. I would love to just sit back and watch you" (Molly). This teacher's negative salient outcome is described by Ball (1996) who said that in professional development experiences teachers may confront their own uncertainties in understanding, and experience feelings of inadequacy and shame. This study demonstrated that the teachers' individual inclinations influenced their perceptions of the professional learning experiences and led to varying outcomes salient to them, including perhaps a *decreased* willingness to engage with learning to teach algebra.

Implications and Conclusion

An interconnected model for professional growth involving various domains of change and the potential for multiple pathways for development fitted well with design-based research methodology in studying teachers' professional learning of an area of mathematics (algebra) perceived to be difficult to learn and teach. The cyclic methodology, involving several repeated opportunities for teachers' learning, could be viewed as stimulating and re-visiting change in the different domains of the Clarke-Hollingsworth model and by providing for the development of different types of knowledge that teachers need as conceptualised by Hill et al. (2008). The modelling of the professional learning experiences of the teachers

handled the sense of complexity of individual teachers' pathways to change and that such change takes time and often repeated cycling of interactions between changes in the four different domains. This created longer 'chains' of interwoven change sequences – some common to many teachers and some unique to one – rather than a simplistic and perhaps unrealistic 'one-size-fits-all' linear process.

A one-dimensional conceptualisation of professional development did not resonate with the evidence from the study of idiosyncratic pathways of individuals, even though the study did highlight some noticeable patterns that were common to several and sometimes all of the teachers. "Individuals (teachers) value and consequently attend to different things (they consider different things salient)" (Clarke & Hollingsworth, 2002, p. 954). This seems particularly true for mathematics professional learning which requires the development of several types of mathematical knowledge for teaching, not just the mathematics itself – the content (or subject matter) knowledge alone.

A perspective on learning that embraces both acquisitive and participative paradigms was found to provide access to various ways of exploring the nature of change in the teachers' professional development. The teachers themselves encompassed aspects of both conceptualisations in their reflections – their descriptions of learning as the development of their knowledge, as participation in their collaborative activity with others and involving changes to their beliefs and teaching practice.

The involvement of the researcher in the teachers' domain of practice throughout the year, as a type of external stimulus (through demonstration teaching in classes) and to provide supportive team teaching, nevertheless produced mixed results and was not perceived as beneficial by one of the teachers. She reported a subsequent *decrease* in confidence. This outcome also highlighted the complex nature of professional learning and that what may be perceived by one teacher as facilitating their growth may be perceived by another as impeding it. There did appear, however, to be a noticeably positive change pathway common to several teachers that related to their perceived value of being able to observe their students being taught by someone else and to attend to particular aspects of their responses in lessons.

The opportunities for the teachers to be able to discuss their classroom experiences during meetings together were facilitated by common tasks for their students. Their collaborative debriefs, student work analysis, and comparison of different perspectives on student responses appeared to lead to increases in teachers' knowledge, both content knowledge and pedagogical content knowledge. But this situated experience for professional learning seemed problematic for one team of teachers with pre-existing inter-personal tension that impinged on their ability to relate comfortably in meetings.

This study adds empirically to the current research literature by demonstrating the importance of multi-faceted professional learning opportunities that consider the individual's needs, interests, disposition, and the likely variety of different pathways to growth. It has implications for the mathematics education of pre-service teachers by highlighting the potential value of providing opportunities for them for pre-service classroom experimentation that align with and connect to their experience of external sources of information and stimulus, such as university-based study. Providing opportunities for pre-service teachers to collect student mathematics work from their own lessons during classroom experimentation, perhaps during placement, and examine their responses collaboratively with support from a teacher educator is worth exploring. This is likely to increase the likelihood of growth via change mechanisms affecting multiple domains and resulting in meaningful and long-lasting salient outcomes for their future teaching practice. The idea of classroom experimentation is raised frequently in the literature about teacher learning, but it can also be related to Dewey's (1904) notion of a *laboratory* approach to training pre-service teachers during school placements (additional to the usual apprenticeship concept) – opportunities for classroom

experimentation, for taking initiative, adjusting one's teaching, and reflecting critically. He suggested that this approach would enable subject-matter (content) knowledge and educational principles (pedagogical content knowledge) to become real and vital to them.

The study has built further on research efforts to understand the contextual factors in teachers' change environments that may promote or hinder professional learning. Several factors that appeared to promote the teachers' learning related to the nature of design-based research in re-visiting and refining teachers' knowledge through an ongoing process over time. Having an external source of information or stimulus within the teachers' domain of practice (the researcher's teaching of their classes) also seemed to support most but not necessarily all of the teachers' learning. For algebra, which is considered a challenging area of mathematics, exploring ways to address the issue of some teachers' ongoing negative affect would be worthwhile. As one teacher in the study said, algebra "challenges a teacher's thinking, as well as a child's thinking" (Fiona). Ball (1996) asserted that we need to learn more "about what helps teachers learn to manage dilemmas wisely, with a combination of confidence and humility" (p. 8). In this algebra study, a strategy that seemed to alleviate teachers' anxiety about their perceived expertise in the classroom was collaborating with colleagues beforehand on generalising a particular pattern to be used in an upcoming lesson and to seek as many different solutions for the same task as possible. One other previously discussed negative factor was the pre-existing strained inter-personal relationships between the teachers in one of the teaching teams which interfered with their willingness to engage with each other collaboratively. These issues are useful for informing the design of future professional learning opportunities for teachers.

The Clarke-Hollingsworth model distinguishes between the notions of *change* and *growth* by highlighting that professional *growth* is evident where changes in different domains are associated with salient outcomes that are maintained *over time*. It would be worthwhile in future research efforts on teachers' professional learning of algebra to consider ways to examine the long-term effects of teachers' participation on their subsequent classroom experimentation, knowledge, and practice. Research could focus on how certain change sequences, particularly those that resonated with several of the teachers, are related to lasting teacher growth as observed in their ongoing practice, through continued enaction and reflection. Although the experience of participating in professional development on algebra seems to have "demystified it" a little more for some of the teachers (Gavin), there is much more to grapple with in understanding and addressing the professional learning needs of teachers effectively in this important area of mathematics To build on the work described here, the author is engaged in ongoing research with lower secondary students and teachers.

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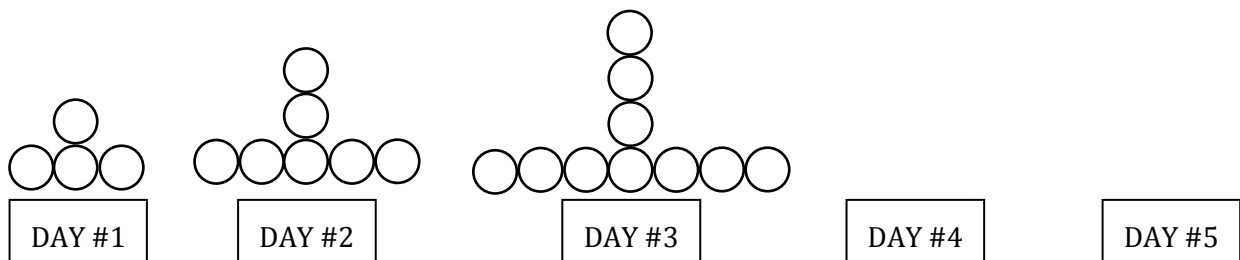
Appendix: Sample Learning Task

The 'upside-down T' plant

In my garden one day, I saw a tiny plant with 4 leaves (Day #1). The next day it had grown and had more leaves (Day #2). On the following day, it had grown even more leaves. Each day I noticed that it continued to grow in the same way.

| DAY #1 | DAY #2 | DAY #3 | DAY #4 | DAY #5 |

a) In the space above, add pictures of what the upside-down T plant will look like on each of the next 2 days (Day #4 and Day #5).

b) What do you notice about the structure of the plant and the way it grows each day? If you can, colour the leaves of the pictures above in different colours to show what you see, and explain your thinking below.

c) How many leaves will the plant have on Day #7? Explain / show how you obtained your answer.

d) How many leaves will the plant have on Day #17? Explain / show how you obtained your answer.

e) If someone gives you any day number, how do you find the number of leaves the plant will have on that day? Explain / show how you obtained your answer.

f) On what day number would the plant have 100 leaves? Explain / show how you obtained your answer.

Appendix: Outline of the Professional Learning Program

Initial training with researcher *(February)*
- Definitions for functional thinking, growing patterns, generalisation
- Discussion of national and international curriculum documentation
- The two types of generalisation – co-variation and correspondence
- Learning progression for upper primary students (see Wilkie, 2013 for more details)
- Preparation for demonstration lesson

Initial demonstration lesson *(March)*
- Observation of researcher teaching own class where students worked on individual rich assessment task (sample task in Appendix)

Team-teaching of 5-lesson sequence for developing students' functional thinking *(March – September)*
- **Lesson #1 focus**: Assessing student's prior knowledge of growing patterns and ability to continue, describe, and generalise linear functional relationship in a geometric growing pattern
- **Lesson #2 focus**: Generalising growing patterns using a correspondence approach by starting with one prototype item from the pattern (3 growing patterns at different levels of difficulty; group discussion then pairs)
- **Lesson #3 focus**: Noticing the structure of unordered items of a growing pattern to encourage different ways of visualising their structure and generalising using a correspondence approach (2 different tasks for students to choose from; working in pairs)
- **Lesson #4 focus**: Assessing students' generalisation of a growing pattern, ability to recognise invalid use of proportional reasoning, and to create and interpret scatterplot of the linear relationship
- **Lesson #5 focus**: Generalising more difficult and non-linear growing patterns (as appropriate) and understanding different ways of visualising the structure leading to different expressions of the functional rule

Collaboration with colleagues *(February – October)*
- Meetings in teaching teams (Year 5 and Year 6 at one school and Years 5/6 at the other) between each lesson in the sequence to discuss experiences, share student work samples, compare student responses, receive further input from researcher, evaluate lesson, and prepare for next lesson

Experimentation *(March – December)*
- Teaching extra lessons during year as decided by teachers
- Opportunities to assess student work using learning progression as a rubric and then comparing with researcher assessments

Ideas and resources *(February – October)*
- Handouts of professional reading, learning progression for functional thinking, examples of growing patterns, and explanatory notes on generalisation of different patterns
- Suggested lesson plan outlines, student task handout drafts, and lists of hands-on equipment for lessons
- Researcher scores for initial assessment task for each student

Appendix: Sample Questions from the Semi-structured Interview Schedules with Teachers Individually and in Groups

Sample questions for individual teachers

What was the highlight for you in observing your students' development of algebraic thinking throughout this year?

  What was the greatest challenge for your students?

  Did anything surprise you in relation to their learning?

How did you find assessing your students' algebraic thinking (during lessons, looking at their written work afterwards, discussing with colleagues)?

What is the single greatest thing you have learned this year about teaching algebraic thinking?

Has participation in the project contributed to your professional learning in the area of algebra?

  If yes, what aspects of the project in particular were helpful for you?

What was the most helpful thing about the debriefing and planning meetings with other teachers?

  What was the least helpful thing?

Has your level of confidence in teaching algebraic thinking changed at all?

  Has your level of confidence in assessing algebraic thinking changed at all?

Is there anything you might consider doing differently in your teaching of algebra as a result of participation in this project?

Is there anything you might consider doing differently in your mathematics teaching practice *generally* as a result of participation in this project?

What advice would you give to another teacher who is about to teach Yr 5/6 algebra for the first time?

Are there any other issues, suggestions or information you would like to mention?

Sample questions for group interviews

Were any aspects of algebraic thinking challenging for your class to learn throughout this year?

  Were any aspects easier than you expected?

Were there any aspects of algebraic thinking that you felt were challenging to teach?

  Easier than expected?

What aspects of the algebraic thinking lessons worked well for your class this year?

What changes might you as a team consider making for next year / time when teaching algebraic thinking?

Do you have plans to teach additional lessons on algebraic thinking?

  If so, what do you have in mind?

If the CEO (Catholic Education Office) were to develop a professional development program for teachers specifically on teaching algebra, what advice or suggestions would you give to them?

Are there any other issues, suggestions or information you would like to mention?