Gibney, J. (2014). Provoking mathematical thinking: Experiences of doing realistic mathematics tasks with adult numeracy teachers. *Adults Learning Mathematics: An International Journal, 9*(2), 97-115.

Provoking mathematical thinking: Experiences of doing realistic mathematics tasks with adult numeracy teachers

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Abstract

This action research project looks at what happened when a small group of adult numeracy teachers with widely different experiences of learning and teaching mathematics explored their own informal numeracy practices and undertook a series of collaborative mathematical tasks. Evidence from qualitative data collected during the enquiry suggests that *realistic* tasks can provoke a range of mathematical thinking and learning responses which allow us to identify ways in which *procedural* and *conceptual* thinking is being used, and to track learning journeys through different stages of problem-solving. Although more experienced numeracy teachers could move between and within their '*real worlds*' and '*maths worlds*' with intent and ease, others had less integrated experiences, often valuing perceived mathematical powers over their own intuitive powers, with mixed success.

Key words: mathematical thinking, action research, adult numeracy teachers, realistic, realisable, mathematisation, collaborative classroom, intra- and extra-mathematical.

Introduction

Historically, within the UK, adult numeracy teaching is a field that many people move into sideways, often from teaching other disciplines. The requirement for practitioners to have a set level of personal mathematics skills was introduced only relatively recently and it is not untypical to find teachers of numeracy who lack confidence in their own mathematical ability (Cara et al., 2010). Personal mathematics development is therefore an important component within many pre- and in-service adult numeracy teacher education programmes. Teachers are encouraged to develop their mathematical thinking throughout their training, both by participating in class activities and pursuing private study. As a tutor and course leader on such a programme, I have observed that when it comes to building a personal mathematics portfolio, many teachers exhibit fairly mechanistic and unreflective ways of working. This is true not only in terms of the approaches they adopt, but also the sorts of independent tasks they choose to undertake - often a surprisingly narrow diet of content-driven and competence-based exercises. The purpose of this research project was to explore how to better support adult numeracy teachers to develop and extend their own mathematical thinking. The rationale for this extends beyond the perceived need for adult numeracy teachers to 'upskill' and is based on the underlying assumption that developing teachers' confidence, awareness and insight into their own mathematical thinking, will better equip them to develop and extend the mathematical thinking of their learners.

Method of enquiry

The classroom, tutor and teachers

The twelve teachers in the group participating in this enquiry were aged 25-55, from socially and ethnically diverse backgrounds and included two teachers whose first language was not English. They were all undertaking professional development in adult numeracy teaching and consistently demonstrated high levels of motivation and engagement although their personal experiences of mathematics, both as teachers and learners varied tremendously.

Within the group, we had negotiated a shared sense of adult numeracy as involving more than basic mathematical skills, or the application of mathematics in everyday life but rather numeracy as a way of negotiating the world through mathematics, *"not less than maths but more"* (Johnston & Tout, cited in Coben, 2004, p3). In the course of working together during the year, we had tried to develop a co-operative and conjecturing classroom - a milieu that explicitly challenged deficit models of adult numeracy. This ethos was influenced by the idea of *funds of knowledge* describing the informal knowledge, skills and experiences that adult learners can draw on but may not be evidenced by formal qualifications (Moll et al, 1992; Baker, 2005), a concept that can be broadened to include interpersonal and metacognitive skills.

Responses to initial classroom probes into their mathematical thinking suggested that few of the teachers moved flexibly between different representational modes. The most mathematically experienced wanted to adopt a symbolic or algebraic response whenever possible, with few trying out more practical approaches. The least experienced saw this use of 'formal' mathematical methods as their ultimate goal, placing less value on other approaches. This apparent lack of variety on the teachers' own mathematical journeys was often in contrast to the active learning and multi-sensory approaches they were developing to support mathematical thinking with their own learners. The initial focus of the enquiry was to explore how to provoke adult numeracy teachers to think and act less mechanistically as 'doers' of mathematics themselves.

Methodology

The enquiry adopted an action research approach based on the idea "*that a practitioner is involved in analysing a situation, planning an alternative action, carrying out that action, and then evaluating the effects of what they have done"* (Mason, 2002, p172). The research was broken down into three smaller cycles or phases of enquiry and reflection. These were undertaken over an eight week period in the final semester of the course.

All participants within the group were involved in research design tasks for about an hour a week in class with some out-of-class time required for auditing, self-reflection and write-ups. Although an essentially social constructivist perspective informed the research focus and the design of classroom interventions, the research methodology itself was mixed. Data collection from tutor field-notes and audio-recordings of semi-structured group discussions focussed on teachers' interpretations and evaluations of tasks undertaken in and out of the classroom, suggestive of an ethnographic approach. Other sets of data, however, were generated from audio-recordings of pair discussions, stimulated recall interviews, written work and tutor observations which aimed to capture responses to paired and individual tasks. Though more typical of positivist methodologies, these provided rich qualitative data which allowed me as a practitioner researcher to experience more fully what happened as teachers engaged in tasks.

Mason (2002, p52) suggests that in researching one's own practice, it is useful to differentiate between giving a *brief-but-vivid* "account-of what was seen, heard, experienced" and analysing, explaining or "accounting-for" incidents. *Accounts-of* will be used to illustrate salient incidents and experiences, along with excerpts from edited transcripts of audio-recordings and examples of teacher responses to tasks. Data analysis will be through a mixture of event sampling using and adapting pre-specified categories from wider theoretical and empirical research, and *accounting-for* recurring phenomena using key constructs and frameworks which are reported within each of the three action research cycles.

This paper will now outline key findings from cycle 1 of the enquiry before going on to focus in particular on significant moments arising from data generated within naturally occurring peer-peer discourse between two pairs of teachers during the second and third cycle of the enquiry.

Cycle 1 Awareness raising

Gattegno (1988, p167) highlights the importance of teachers sensitising themselves to their own behaviours, emotions, and awarenesses:

> Teachers need to make themselves vulnerable to the awareness of awareness, and to mathematization, rather than to the historical content of mathematics. They need to give themselves an opportunity to experience their own creativity and when they are in contact with it, to turn to their students to give them the opportunity as well.

In considering what sorts of mathematical activities to use within this action research, I wanted tasks that would support teachers to take the initiative and become more fully engaged in their own mathematical thinking. Schoenfeld (1994) developed a broad and age-independent description of what learning to think mathematically means:

1. Developing a mathematical point of view – valuing the process of mathematisation and abstraction and having the predilection to apply them.
2. Developing competence with the tools of the trade and using these in the service of the goal of understanding structure – mathematical sense-making.

But what did mathematical thinking and mathematisation look like 'outside formal mathematics classrooms'? Research has demonstrated that adults have access to many informal numeracy practices (Street, 1984; Nunes, Schliemann & Carraher, 1993a; Baker & Rhodes, 2007). The idea that teachers need to become aware of learners' innate or natural powers to think mathematically (Mason and Johnston-Wilder, 2006) is echoed in a number of recent research reports (Swan, 2006; Swan and Swain, 2007). Indeed, much official discourse now actively encourages adult numeracy teachers to *"build on the knowledge learners already have"* (Swain et al, 2007, p. 7).

The belief in the importance of teachers' recognising their own *funds of knowledge* and exploring innate *mathematical sense-making* powers themselves, provided the initial impetus for considering *everyday* contexts and numeracy in the task design. By exploring what we as adult numeracy practitioners noticed about our own numeracy practices, would any shared characteristics, prior knowledge or behaviours related to mathematical thinking emerge to inform the design of tasks for subsequent action research cycles, for both experienced and less experienced participants?

Task design 1

During the first week of the enquiry, teachers and tutors made diary notes about what they identified as their numeracy practices over the course of a 'work-day' and a 'non-work day'. These were mostly handwritten on two large A3 diagrams resembling a clock face. A further record sheet was completed during the second week. This required us to identify and classify mathematical behaviours we noticed according to what Bishop (1988) identified as six universally occurring activities: counting, locating, measuring, designing, playing and explaining.

Analysis

Each week, findings were shared with peer partners. Subsequent whole group discussion were animated, as numerous and at times conflicting *accounts-of* and *accounts-for* were generated:

> ***Accounts 1***
>
> The supermarket does all the price comparisons – I just read the labels.
>
> We're on a really tight budget so I'm working out stuff with money all the time.
>
> I get the kids to help with the adding up when we're in the supermarket.
>
> I was quite shocked – I do more maths out of work than when I'm teaching.
>
> It took ages to park this morning. Usually there are a few places left, but today it was practically deserted.
>
> I didn't realise how much time I spend in the car at the moment – there's journey times, buying petrol, using maps and Google directions, speeds and signs, even working out the best lane to be in where all the road works are.
>
> I'm totally addicted to Sudoku at the moment – my son and I try to see who can finish first.

Teacher and tutor *accounting* suggested that as well as becoming more sensitised to our own numeracy practices, we engaged in a diverse range of socially and culturally situated mathematical behaviours. Although some of us identified possible mathematical topics and themes related to particular situations or times, others discussed what they actually did. Many omitted or ignored things they did not consider mathematical but *"just common sense"*. This illustrates how difficult it is to design learning tasks tailored to each individual's particular experiences.

Data from this part of the enquiry did however suggest some common characteristics of the group's everyday numeracy practices which tended to involve purposeful activities which were often collaborative e.g. family activities involving playing, cooking, shopping or constructing. These were often linked to particular roles and could be dependent on and shaped by particular tools or realia e.g. maps, self-service checkouts, petrol pumps, Sat Navs. This is in line with findings from similar studies into everyday numeracy practices (Lave, 1988; Harris, 2000; FitzSimons, 2005). For example, in reviewing a range of empirical research some 20 years ago, Resnick (1987) noted that much activity outside classrooms is socially shared. She contrasted examples of shared knowledge and understanding, tool manipulation, contextualised reasoning and situation specific competencies from everyday numeracy practices with the sorts of individual knowledge and skills, abstraction, symbolic manipulation and generalised learning more likely to be experienced in many formal mathematics classrooms.

Implications from Cycle 1

This initial analysis suggested that the teachers' informal numeracy practices could be drawn on more effectively by providing tasks which afforded:

- Opportunities for them to work together on problems.
- Access and use of cognitive tools.

- Direct engagement with objects and situations rather than purely symbolic thinking.
- Use of situation-specific competencies (adapted from Resnick, 1987).

However, the overall goal was to further develop and extend these teachers' mathematical thinking; to build on existing knowledge *and* ensure those with little or less successful experience of learning maths were empowered to operate successfully within formal mathematics classrooms too. To this end, the framework above merely presented possible points of departure.

In terms of identifying an actual topic base for the mathematical tasks to be used in the next cycles of the research, I was particularly struck by the relatively infrequent use made of 'standard' measures or indeed measuring devices during awareness raising activities in Cycle 1. Discussions with teachers revealed resonated experiences and generated additional complex, contingent and subjective strategies for measuring and estimating everyday phenomena:

Accounts 2

In the morning I know when the bath's getting full … I can hear how long I've got to drink my coffee

I can estimate how much it'll cost by how full the trolley is.

I know how much squash to add by the colour – not dilution ratios!

I measure how crowded a place is by how far I have to go to get to an uncrowded place.

Buying petrol has nothing to do with gallons or litres…

Don't need an alarm clock… my dogs tell us when it's time to get up.

Cycle 2 Plausible estimates

Subsequent research and review of potential mathematical thinking tasks which could be adapted in accordance with the research focus and findings to date, uncovered a number of suitable open-ended tasks based on estimation and measure. A set of classroom assessment tasks (CATs) which had already been field-tested were chosen for cycle 2 of the research. These involved *"Making plausible estimates"* based on Fermi-type problems (Ridgeway and Swan, 2010).

Task design 2

Figure 1 details the task objectives presented to the whole group:

The aim of this task is to provide the opportunity for you to work with your partner to:

- make sensible assumptions
- develop a chain of reasoning
- choose suitable units
- communicate your assumptions and reasoning effectively to peers

Extension: Identify upper and lower bounds i.e. what range of values would you give in order to be pretty certain that you have included the true value being estimated?

Figure 1. Plausible Estimations

Tahta (1981) makes a useful distinction between *inner* and *outer* tasks which helps here to distinguish between the explicit *outer* task of finding a plausible estimate and the intended *inner* task which would allow both teachers and tutor to gain experience of what mathematical thinking and communicating might look, behave and feel like.

By building on a range of theoretical and empirical research, Goos et al (2004, p100) identify five assumptions they argue are crucial to creating a culture and '*community of mathematical inquiry*':

1. Mathematical thinking is an act of sense-making, and rests on the processes of specialising and generalising, conjecturing and justifying.

2. The processes on mathematical inquiry are accompanied by habits of individual reflection and self-monitoring.

3. Mathematical thinking develops through student scaffolding of the processes of enquiry.

4. Mathematical thinking can be generated and tested by students through participation in equal-status peer partnerships.

5. Interweaving of familiar and formal knowledge helps students to adopt conventions of mathematical communication.

Mindful of the desire to value and develop teachers' informal and formal mathematical experiences, I found the first of these assumptions resonated strongly with the notion of accessing learners' *innate powers* and the last two strongly influenced my decisions to conduct the plausible estimation sessions with particular peer partners, and to require teachers to present and justify their findings to the whole group. The focus of analysis within this cycle of the research also moved onto data generated by two pairs of teachers within the group who fulfilled certain contrasting characteristics related to previous experience of teaching and learning mathematics.

Teachers M and N were confident in using higher level mathematical skills, had studied mathematics at university level and each had at least five years' experience of teaching mathematics to adult learners mainly within further education settings. They were given the 'mummies' task in Figure 2.

An unravelled roll of paper is 33 metres or 100 feet long.

Will one roll be enough to wrap a person up?

Figure 2. Mummies

Teachers R and S were less confident in their mathematics skills and knowledge, had no formal mathematics qualifications beyond a foundation level and had quite recently become involved in teaching adults numeracy within their respective work-based training organisations. They worked on the 'briefcase of pennies' task in Figure 3.

Suppose you filled a briefcase with one penny coins.

How much money would you have?

Figure 3. The briefcase of pennies

Before considering in more detail what unfolded as these teachers engaged with their plausible estimation tasks over the next two week period, it is important to outline further theoretical frameworks which significantly impacted on both the conduct and analysis of data from this second cycle of enquiry.

Realistic maths and mathematisation

The idea of relevance and realism within mathematics teaching is complex and contested. Many *authentic* mathematics and *real* problem solving approaches advocate settings and situations which try to motivate and engage learners by using topics relevant to their immediate concerns. However Swain et al (2005) argue that is the quality of an individual's engagement with a problem that makes math *meaningful* rather than its utility or everydayness. Others, like Cooper and Dunne (2004) highlight the hidden rules younger learners must negotiate when tackling *contextualised* word problems and how these can adversely impact on learners from different cultural or social backgrounds.

Realistic mathematics is a term which better describes the sorts of tasks adopted within this enquiry and relates to an approach developed by Freudenthal (1991) which accentuates the actual activity of *doing* mathematics and advocates the power of learners to make things real for themselves by using their imagination. Such *realistic* tasks require learners to mathematise subject matter from *real* or *realisable* situations and reinvent mathematical insights, knowledge and procedures in the course of *"their own mathematical activity rather than from the traditional view of presenting mathematics to them as a ready-made system with general applicability"* (Gravemeijer cited in Barnes, 2004, p5). These situations can include contextual problems or mathematically authentic contexts for learners where they experience the problem presented as relevant and real. Such a process- rather than content-driven approach was consistent with the research focus of developing more relational and creative mathematical thinking and built on earlier project findings. The 'plausible estimation' tasks are examples of such *realistic* tasks.

Horizontal and vertical mathematisation

Additionally, in responding to the challenge that both learners and teachers often experience in trying to distinguish between concepts and procedures in mathematical thinking, Treffers (1987) developed the idea of *horizontal* and *vertical mathematisation* within this *realistic* maths framework. According to Freudenthal (1991, p41) horizontal mathematisation "leads from the world of *life* to the world of *symbols*" which Barnes (2004) suggests happens when learners use their informal strategies to describe and solve a contextual problem. On the other hand, *vertical mathematisation* occurs when the learners' informal strategies lead them to find a suitable algorithm or to solve the problem using mathematical language. For Freudenthal (ibid), this is where "symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly". For example, in the case of the 'plausible estimation' tasks, the process of establishing the important information required and using an informal strategy such as trial and improvement to arrive at an estimate would be *horizontal mathematisation*. Translating the problem into mathematical language through using symbols and later progressing to selecting an algorithm such as an equation could be considered *vertical mathematisation*, as it involves working with the problem on different levels. This framework will be used within the analysis of data generated during teacher work on the 'plausible estimation' tasks.

Analysis

When first presented with these Fermi-type problems, teachers responded with surprise and some uncertainty:

Extract 1

R: Make it pound coins and I might have a go.

S: How are we going to do this then?

R: Haven't a clue.

This intrigue right at the start of the tasks worked effectively to harness teachers' emotions. Teachers M and N worked flexibly between mainly *iconic* or visual and *symbolic* modes of representation. They modelled a human being as a cylinder, deciding this would be the most effective way of minimising surface area rather than use a collection of smaller cylinders; agreed symbolic formulae for this surface area; assigned variables and established a direct proportional relationship between foot length and height.

R and S initially worked in an *iconic* mode with mental images of their own bags and briefcases, before looking around the room for more immediate concrete items which approximated their mental image of the briefcase on the task sheet. They stayed in this *enactive* mode, using a ruler to measure an actual briefcase and the diameter and depth of a 1p coin. Cooper & Dunne (2004) might argue that this is an example of teachers not understanding conventions for estimating and rounding. However, taking time within this mode allowed R and S to feel confident in their estimates before moving on to iconic and symbolic approaches. M and N, in contrast showed throughout the task that they knew and understood 'the rules' of the mathematics classroom:

Extracts 2

M: We've worked to the nearest whole number throughout so we should use 2m for the height.

And later:

N: The task just wants us to decide yes or no, so we've done it. We don't need to work out how much more paper we'd need.

Ironically, their numerical estimates were the weakest chain in their initial argument as they got caught up in the mathematical conventions and opted for a height of 2m rather than their original 1.8m. The extension task which M and N completed required them to establish upper and lower bounds and brought the idea of *dimensions-of-possible-variation* (Marton and Booth, 1997) into play. By considering the *range-of-permissible-change* (op cit) for the heights of adults and children, constraints on their first model were established and a new model created which would incorporate a baby's particular body shape and size. This extension task had extended the teachers' mathematical thinking by requiring them to look for *invariance in the midst of change* (op cit). Overall, M and N demonstrated sophistication in their relational understanding and worked together to integrate relevant real-life experiences as they engaged flexibly between *horizontal* and *vertical mathematisation* of the problem (Treffers, 1987). The ease with which I was able to identify and analyse M and N's responses to this task using the frameworks provided is also significant as it may provide an *account of* what I readily recognise and implicitly value as 'mathematical thinking'.

R and S had a significantly different journey through the task. S began to *mathematise vertically* using an algorithmic procedure involving volume. She wanted to work out the volume of the case and then to divide by the volume of a single coin and remembered that V= l x w x h *"maybe"*. R, on the other hand, began with *horizontal mathematisation* - drawing lines of pennies as he built up a sense of volume through layering. His approach required him to find out how many coins were needed to layer out the briefcase and he struggled to make sense of the approach adopted by S:

Extract 3

R: I'm just using a practical approach that I understand but S's solution is more mathematical.

R's implicit value judgment here about some sorts of mathematical thinking being more valuable than others resonates powerfully. Barnes (2004, p59) argues that using a formula does not necessarily imply better conceptual understanding and warns of the *"danger of focusing too much on vertical mathematisation"*. In fact, when R and S discovered that their initial results did not match, it was by using R's *horizontal mathematisation* that they were able to establish that an error must have been made with S's conversion rates. Indeed, R had a very solid conceptual understanding of volume whereas S had adopted *vertical mathematisation* mechanically but without Freudenthal's (op cit) other two requirements: comprehension and reflection. This prevented her from spotting the common misconception made in converting between units of volume. It was by reverting back to R's layering approach that both were able to work out that 1 cm^3 = 10mm x 10mm x 10mm and come to an agreed plausible estimate. Having to articulate for the whole class their chains of reasoning, initial assumptions and ways of validating results helped both R and S strengthen their understanding of the general algorithms they had adopted, although more time to consider upper and lower bounds may have consolidated this further.

Although R and S were less confident in terms of their *intra-mathematical* skills, it is important to note that they brought a range of social, communication and meta-cognitive skills and experiences to the task process which allowed them to discuss, peer check and when necessary seek help from peers and tutor. They were tenacious, supportive of each other and prepared to take their time, progressing with small steps along repeated cycles of what Mason, Burton and Stacey (1985, p156) describe as *"the helix of manipulating – getting a sense of – articulating"* when thinking mathematically.

Both sets of teachers were provoked by the tasks to fall back on their own experiences and access a range of personal 'everyday' or *extra-mathematical* knowledge in diverse ways. M and N drew confidently on their experiences of child birth to establish estimates for the width of the head and the length of a new born baby. However, when asked how they might check their findings, neither M nor N wanted to try out their solutions. The intrinsic motivation and interest was in the *intra-mathematical* process – the accuracy of their final estimation all but redundant. M and N had quickly moved from the *real* world to their own *mathematical* world and intended to stay there. In contrast, when asked how they would check their estimate, R and S went straight back to *extra-mathematical* knowledge of their *real* world:

Extract 4

S: You'd not use volume at all – you'd empty the case and weigh them.

R: Just like they do in the bank.

This willingness (or not, in the case of teachers M and N) to re-engage with the *real* world scenario in order to evaluate not only the plausibility of the estimates found but the validity of the *mathematisation* process itself may have significant implications for the teachers' own professional practice. Arguably, failing to reinterpret and validate mathematical results within *real* situations can result in leaving *unrealistic* modelling unexposed. For some learners this sort of uncritical mathematical thinking does nothing to close 'the gap' between *real* and *maths* worlds.

When asked whether the 'Pennies in a Briefcase' task had been useful, R and S responded:

Extract 5

S: Yes, it got us thinking – we had to use lots of different sort of maths.

R: We'd forgotten lots. I think I understand units of volume better now.

Although these 'Plausible Estimation' tasks required only a basic knowledge of geometry, numeric skills and units of measure, the teachers did engage in more relational and connected thinking. Misconceptions related to conversions of units, use of appropriate formulae and rounding errors were identified through self and peer monitoring and teachers seemed to develop a more intrinsic feeling for the plausibility of their estimates. The value of developing conceptual and procedural knowledge in tandem seemed clear to all participants, and some teachers were also able to reflect more confidently and critically on their chains of reasoning.

Coding framework for plausible estimates

A more analytical comparison of the mathematical thinking and specific problem solving strategies the two pairs of teachers adopted in moving from *real* worlds to their *maths* worlds and (sometimes) vice-versa is difficult, not least because they were undertaking two different tasks. However, by adapting a framework devised by Arleback (2009), I was able to encode data from recordings of peer-peer discussions during the 'Pennies in a briefcase' and 'Mummies' tasks:

1. Reading: reading the task and getting an initial understanding of the task

2. Making model: simplifying and structuring the task and mathematising

3. Estimating: making estimates of a quantitative nature

4. Calculating: doing maths - performing calculations, solving equations, drawing diagrams

5. Validating: interpreting, verifying and validating results: calculations and the model itself

6. Summarising: summarising the findings and results in writing or orally

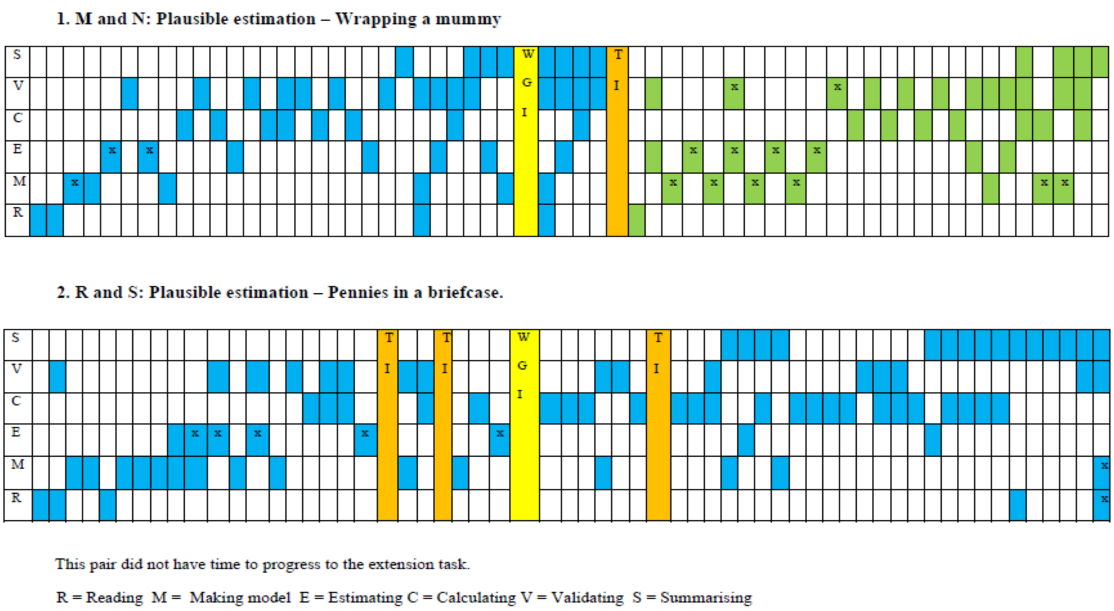


Figure 4. Mathematical behaviours during 'plausible estimations'

Figure 4 aims to capture a macroscopic and fairly dynamic picture of how teachers were *heard* to move between different 'behaviours' during the audio-recordings of the first 30 minutes of paired work on these tasks. Coded activities are identified within blocks, representing approximately 30 second time intervals. A whole group tutor intervention (WGI) took place after 15 minutes, and tutor interventions (TI) for particular pairs are also identified. X indicates where teachers have explicitly used *extra-mathematical* knowledge and experiences in diverse ways as outlined earlier.

Interestingly, although R and S had divergent calculation strategies during their tasks, the actual mathematical behaviours displayed in the diagram were similarly categorised within this framework as was the modelling stage which did not differentiate between horizontal and vertical mathematisation.

Implications from Cycle 2

Although the main value of these diagrams to me as a practitioner comes from the actual process and challenge of coding and categorising the peer-peer discussions, they do also provide some triangulation of earlier observations on how and when *extra-mathematical* knowledge is used, some new insights into the timescale of comparative progress through the tasks by both pairs, the frequency with which the teachers validated results and the time taken to summarise findings in preparation for articulation to the whole group. Arleback (2009) noted similar phenomena with his learners and observed that validation of results involved checking calculations, estimations *and* the initial model. However although both pairs of teachers here used articulation to summarise *and* peer validate their calculations, results and decision making processes throughout, M and N were more reluctant to 're-enter' the messier *real* world once they had found a comfortable place of abstraction in their *maths* world.

Cycle 3 Creating measures

For the final cycle of the research, a second set of field-tested mathematical tasks were used. These aimed to prompt teachers *"to evaluate an existing measure of an intuitive concept and then create and evaluate their own measure of this concept"* (Ridgeway and Swan, 2010). A key component within this cycle would be the requirement for both pairs of teachers to test and evaluate any measures created back in the *real* world.

Task design 3

Requiring teachers to start from everyday concepts – *steep-ness, sharp-ness, awkward-ness, compact-ness, crowded-ness and square-ness* – to mathematise phenomena by creating their own measures seemed even more closely related to the experiences of awareness-raising in the first cycle of the enquiry and consistent with the sort of mathematisation and guided re-invention advocated by a *realistic* maths approach. As well as provoking mathematical thinking, I hoped these tasks would afford *meaningful* two-way connections between *real* and *maths* worlds.

Experiences during cycle two of the enquiry suggested that peer partners worked well together. This time however, I provided more scaffolding in the form of prompts in teachers' work packs, so that tasks could be sustained and worked on independently. These included regular self-monitoring and reflection opportunities, consistent with the second and third assumption identified earlier as crucial to a *community of mathematical inquiry* (Goos et al, 2004). Teachers worked on these extended tasks in class each week for an hour over a three week period. Although they had individual work packs, pairs were expected to work collaboratively to reach a point where they would be able to go out on campus to test whether their measures actually worked. A written summary of findings 'so far' with commentaries, photographs and individual reflections on the creating measures process would provide evidence for teachers' personal mathematics portfolios.

Before tasks were distributed, an introductory activity was undertaken to encourage teachers to consider themes, processes and specific features evoked by particular concepts:

- With your partner, take a few minutes to discuss what the concept of **'sharp-ness'** means to you both.
- This might include thoughts, images, experiences, associations, special words or phrases, contexts or feelings.
- Use a concept map to record your initial responses.

Figure 5. Example of introductory activity for 'creating measures' task

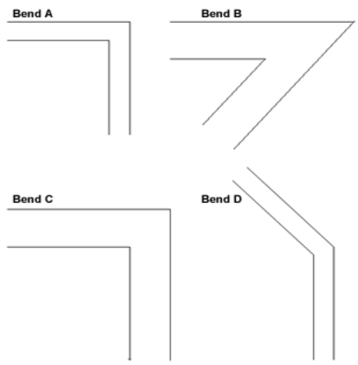
When finally presented with their actual tasks, several teachers experienced what Mason & Johnston-Wilder (2006, p96) describe as *"a contradiction of expectation"* which they argue is a useful disturbance to provoke activity:

Extract 6

M: Oh, it's nothing to do with pain or needles …

The actual 'creating sharp-ness' activity presented to teachers M and N is shown here:

Without measuring anything, put the four bends in order of "sharp-ness".

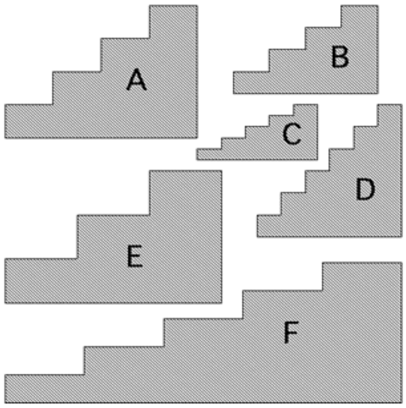


Explain your method clearly.

Figure 6. 'Sharp-ness' Activity 1 Warm-up

This first activity specifically invited teachers to engage with *iconic* modes of representation. By inviting them to 'look first, and act later', I hoped that the teachers would use their own mental imagery and innate sense-making powers to identify similarities and differences between images, to specialise and generalise, order and classify and begin to become aware of some of the properties of the bends, or in the case of teachers R and S, the staircases which they might be able to explore later:

Without measuring anything, put the staircases in order of "steep-ness".



Explain your method clearly.

Figure 7. ‘Steep-ness’ Activity 1 Warm-up

Although these two dimensional images were less life-like than those used in the earlier ‘plausible estimation’ tasks, they were not conventionally mathematised to one-dimensional lines. Another feature of the classroom at the start of this third cycle of enquiry was the availability of mathematical equipment – tools for measuring, different sorts of paper including square, graph and dotty, calculators, counters, centicubes, etc. Indeed, all tasks required teachers to undertake some hands-on measurement, ensuring that everyone got involved at an *enactive* level, quite literally manipulating, constructing and measuring particular properties of their task concepts. Nunes et al (1993b) identify the significance of such *measuring* tools in supporting mathematical reasoning in younger learners and increasingly adult learners are being re-introduced to the power of multi-sensory approaches to mathematical sense-making. These tasks required that my teachers did the same.

Figure 8 shows how the learning objectives for the ‘steep-ness’ task were introduced to teachers R and S:

Objectives

This problem gives you the chance to:

- criticise a given measure for the concept of "steep-ness"
- invent your own ways of measuring this concept
- examine the advantages and disadvantages of different methods.

Figure 8. ‘Steep-ness’ Task objectives

Analysis

In their initial discussions on the staircases, R and S identified a range of factors influencing their perceptions of steepness: personal preferences about heights and depths of steps, fitness and stamina, carrying shopping bags, going up or down, taking single or multiple steps, individual heights and builds, disabilities, indoor or outdoor steps, surfaces, ‘length’ of staircases. Their considerations were very much rooted in the social context of the staircase journeys – who, why, when, where, how often. Rather than a straightforward exercise in finding gradients, R and S were tackling a much more complex modeling task within the real-world scenario they had created.

M and N on the other hand again moved almost immediately to abstract mathematisation, exploring how they could use trigonometry to create a measure of ‘sharpness’, focusing solely on angles and width with no consideration of other contextual factors. When prompted, they were able to generate other variables: roads, lanes, vehicles, weather, surface, speed, visibility, gradient, etc. but the relevance of these only really became apparent to them when they went outside to test their new measure in the messier real world. Figure 9 provides a brief *account-of* their measure for ‘sharpness’:

Measures of Sharpness

M: $$r = \frac{x}{1 - \sin\theta}$$

N: $$\frac{r}{x} = \left(1 + \frac{\tan\theta}{\tan\left(\frac{90-\theta}{2}\right)}\right)$$

They prove: $$\frac{1}{1 - \sin\theta} \equiv \left[1 + \frac{\tan\theta}{\tan\left(\frac{90-\theta}{2}\right)}\right]$$

when $\theta \neq 90°$

And test their measures outside with real world bends.

Figure 9. ‘Measures of sharpness’ invented by teachers M and N

This may also convey some the unconscious assumptions and value judgements that I, as someone more comfortable within the abstract maths world of algebra myself, make about what mathematical thinking looks like. It certainly *accounts-for* some of my confidence that such realisable tasks can provide effective points of departure for diverse groups of teachers to engage in doing, thinking and communicating mathematically and to recognise what this engagement entails.

Although data from this third cycle of the enquiry provided many other textured examples of ways in which the ‘creating measures’ tasks provoked teachers to engage in mathematical thinking, I will finish by focusing on one further incident that was particularly significant and indeed disturbing:

> ***Extract***
>
> S: Before today I thought I could look at a slope and know how steep it was. But when you do the measurements, you realise it’s different. I’ll never decide about steepness by eye again.

What had happened for this teacher to conclude that her intuitive understanding and experience of steepness in the *real* world was wrong? Data from the audio-recording of S and R’s work and stimulated recall interviews suggest that S drew this puzzling conclusion as a result of some very ‘logical’ deduction:

> ***Account 8***
>
> R and S measure the height and slope of staircases on campus.
>
> Back in the classroom, they use Pythagoras to calculate length.

> They produce scale drawings - 'staircase triangles'.
>
> They measure the angles.
>
> The steepest staircase isn't the one they thought it would be.
>
> You can't trust your eyes to measure steepness.
>
> Do it by measuring in future!

Ironically, R and S had no need to use Pythagoras at all but had been so excited in *"finally understanding how to do it"* that they built it into what was otherwise a reasonable algorithmic approach to measuring steepness, believing their calculations would be more accurate if they only had to use two real-life measurements. However, rather than consider that they might have made a calculation error, S instantly gave up her own internal sense of what a reasonable result should look like, trusting to the "*power of mathematics*" and in particular, the power of formulae, over-riding the evidence of her own eyes. R who was much less critically engaged in the process, was happy to concur with S and seemed unconcerned that evidence from calculations totally contradicted his initial observations 'by eye'.

This episode suggests that for S, the world of formal maths although exciting was still very external to her own internal world. It also suggests something about how she valued different sorts of knowledge – with formal mathematical powers at the top of the hierarchy and her own at the bottom. It took time, considerable peer checking and more experiences of measuring and testing staircases around the campus before S's mathematical and personal worlds began to reintegrate. R and S may have recovered from this incident but it continues to resonate strongly with previous personal experiences.

If learners override *extra-mathematical* understanding, how can they develop their ability to judge whether their answers are sensible and how often do they leave classes not knowing any more how to do something that made sense to them at the start of the lesson? In the case of R and S, the incident actually provided a sort of dissonance that generated another very fruitful point of departure. However, in a short time-restrained session where curriculum and assessment demands may prevent teachers and learners taking the time to move within this *horizontal* phase of mathematisation to deal constructively with misconceptions and bridge gaps between real and maths worlds, how damaging might this sort of mathematical experience be to learners' self-confidence and self-concept?

While researching this phenomenon further, I found an article in which Meissner (2006) suggests that we have a number of internalised representations or micro-worlds which inform our subjective domains of experience. He identifies a reflective and subjective domain of experience (SDE) and argues that although both are important for flexible thinking one can often be more dominant over another, particularly when a new problem or conflict arises:

> The individual prefers to ignore the conflict rather than modify the SDE or adopt another SDE. In mathematics education it is quite natural that an 'analytical-logical' behaviour remains dominant and that conflicting, common-sense experiences or spontaneous ideas get ignored. (Meissner, 2006, p3)

This is an interesting theoretical construct with which to try to understand why R seemed relatively unperturbed by cognitive dissonance, while S was so easily enticed to relinquish her own *common sense experiences.*

At this stage of the enquiry then, my initial disappointment that carefully selected and adapted mathematical tasks had resulted in some teachers dismissing rather than valuing their own intuitive mathematical powers, was tempered by the fact that engagement with these same tasks had generated phenomena that provided insight into another interesting and valuable point of departure related to how we move between and within our formal and informal, real and mathematical worlds.

Summary discussion

The initial focus of this action research project was to improve practice in supporting adult numeracy teachers develop and extend their own mathematical thinking. At each stage of this inductive process, as a participant observer I have collected, reflected on and evaluated data related to teachers' responses to a series of research design tasks. In particular, using audio-recordings to reflect on classroom discourse during collaborative work on mathematical tasks and in oral presentations to peers generated evidence of rich, cyclical and non-linear problem solving and mathematical thinking processes. It was a real privilege to listen to teachers interacting together with energy, trust, humour, perseverance, intelligence and humanity.

During this enquiry, I hoped to gain insights into a group of adult numeracy teachers' mathematical thinking but learned a great deal more about my own assumptions, beliefs, and expectations. In focusing on the quality of my own interventions and interactions with teachers, I need to recognise that I can be just as *mechanistic* and *instrumental* in supporting work on mathematical tasks as they can be in solving them. I also recognise, value and am more likely to favour mathematical thinking and behaviours which mirror my own formal mathematical experiences and interests and need to be fully conscious of this if I am to further develop my own inclusive practices in supporting teachers to develop mathematical thinking.

Teachers and tutors come to formal mathematics classrooms with *funds of knowledge*, which include diverse and contingent informal numeracy practices which are culturally and socially situated. These often go unrecognised, are not valued or are held subconsciously. Raising awareness of these through systematic reflection can provide valuable insights into hidden personal and interpersonal resources and propensities which can be harnessed or challenged to support teachers' own mathematical thinking and, hopefully, their professional practice.

More *enactive* and *iconic* approaches can open up or close down possible lines of inquiry in unexpected ways. Similarly, tasks which specifically require teachers to take more time in *manipulating* and *getting-a-sense of* the mathematical structures of a problem, though often more time-consuming, are less likely to result in teachers adopting mechanistic or instrumental approaches.

Conclusion

What unfolded during this small-scale practitioner enquiry suggests that doing *realistic* mathematics tasks within a *community of inquiry* can provoke a range of mathematical thinking and learner responses. These allow us to identify ways in which procedural and conceptual thinking can be used within *horizontal* and *vertical mathematisation,* and how learner journeys can be tracked through different stages of problem solving. Such tasks can also provide *meaningful* starting points to teachers with varying levels of prior mathematical experience. However, teacher and tutor beliefs and assumptions about what constitutes mathematical behaviour can support or constrain the intent and ease of movement within and between their *real* and *mathematical* worlds, and vice versa. While teachers with more experience of mathematics could do this flexibly, despite preferences and predispositions to reside in more formal mathematical mental environments, others with less confidence or less well developed *intra-mathematical* knowledge and skills dismissed their own innate *sense-making* and *extra-mathematical* knowledge too readily, with mixed success.

Recommendations for future practice

Adult numeracy and mathematics teacher education courses need to support students to engage regularly in a variety of sustained, open-ended and *realistic* mathematical tasks, with further extended tasks signposted for independent study.

If teachers are to develop greater awareness of what mathematical thinking looks, feels and sounds like, more self and group reflection and evaluation tasks need to take place with explicit reflections on *inner*, *outer* and *meta*- tasks encouraged within personal maths portfolios and group discussions.

New mobile technologies are being used increasingly and naturalistically within sessions: listening to, watching and analysing targeted audio- and video-recordings of engagement in their own mathematical thinking tasks will support teachers to develop *awareness of awareness* further.

The key literature, frameworks and constructs which informed the context and conduct of this enquiry along with the specific mathematical tasks used could be shared and contribute to reading lists used on other adult numeracy teacher education courses.

Throughout this paper, there has been an underlying assumption that developing teachers' confidence, awareness and insight into their own mathematical thinking, will better equip them to develop and extend the mathematical thinking of their learners. Adult numeracy teacher educators need to identify and value further opportunities for students to explicitly evidence and reflect on how they are using their own experiences of thinking and acting mathematically to inform their practice with learners.

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