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EXPLORATIONS IN ELEMENTARY MATHEMATICAL MODELING

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Abstract: In this paper we will present the methodology and pedagogy of Elementary Mathematical Modeling as a one-semester course in the liberal arts core. We will focus on the elementary models in finance and business. The main mathematical tools in this course are the difference equations and matrix algebra. We also integrate computer technology and cooperative learning into this inquiry-based learning course where students work in small groups on carefully designed activities and utilize available software to support problem solving and understanding of real life situations. We emphasize the use of graphical and numerical techniques, rather than theoretical techniques, to investigate and analyze the behavior of the solutions of the difference equations.

As an illustration of our approach, we will show a nontraditional and efficient way of introducing models from finance and economics. We will also present an interesting model of supply and demand with a lag time, which is called the cobweb theorem in economics. We introduce a sample of a research project on a technique of removing chaotic behavior from a chaotic system.

Key words: mathematical modeling, difference equations, cooperative learning, inquiry-based learning, cobweb theorem in economics.

1. Introduction

We introduce some models from finance and business represented by a single first order difference equation as a part of a course in mathematical modeling aimed at freshmen undergraduate students. The course is designed as a one-semester course in the liberal arts (general education) core, where students are required to take one or two mathematics courses.

The main mathematical tools are difference equation, matrix algebra and Markov chains. The course does not require training in calculus.

The use of technology in the form of computer algebra system (CAS) such as DERIVE and/or spreadsheet such as Excel, and cooperative learning [3] are integral part of the course. Students work in small groups on carefully designed activities that guide them in exploring and discovering the concepts on their own [4]. It is important to note that most of the concepts are intuitively introduced.

In this paper we show how the students use difference equations to model annuities, sinking funds, mortgages and amortization. They use a computer algebra system, graphing calculator, or spreadsheet to iterate the difference equations to find the solutions and investigate the behavior of the graphs. We will also present an interesting model of supply and demand with a lag time, which is called the cobweb theorem in economics. We introduce a sample of a research project on a technique of removing chaotic behavior from a chaotic system.

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2. Some Models from Finance and Business

2.1 Annuities. The students work in small groups of 4 on activities that are designed to allow them to model real life financial situations with difference equations and to explore related concepts on their own.

Activity 2.1 It is known that, for certain income brackets, money deposited in an individual retirement Account (IRA) and its interest are tax-deferred. Karen Smith sets up an IRA at the beginning of the year and deposited $3,000. Karen will deposit $2,000 at the end of each year, and the interest rate is 6% compounded annually.

A) Model this situation by a difference equation.

B) What will be the future value of the account at Karen's retirement age of 60 if she starts the IRA at the age of 25?

Discussion: The students in groups work on modeling this situation and then use DERIVE or spreadsheet to iterate the difference equation to answer part B. After a few minutes the instructor asks the representative of one group to represent their approach and then allow discussion among groups. Here is a typical students approach. Let y_n be the balance in the account after n years. This situation can be modeled by the difference equation:

$$y_n = y_{n-1} + 0.06y_{n-1} + 2,000 = 1.06y_{n-1} + 2,000 \qquad (2.1)$$

where $y_0 = 3,000$. We are looking for y_{35}, which can be obtained by iterating the difference equation (2.1). Most of the groups use DERIVE's function ITERATE to find y_{35} by authoring and approximating the following expression:

ITERATE (1.06y + 2000, y, 3000, 35)

The output is 245,927 which is y_{35}. Note that the in order to obtain the ordered pairs (n, y_n), $n = 0$, 1, 2, …, 35, and graph y_n vs. n, author, approximate, and plot the DERIVE expression:

ITERATES ([n + 1, 1.06y + 2000], [n, y], [0, 3000], 35)

Activity 2.2 Assume you open an account that pays 4% compounded annually and deposit $2,000. You will deposit 2% more into you account than you deposited in the previous year.

A) Model this situation by a difference equation.

B) What is the total amount in the account after 20 years?

Discussion: The students in groups model this situation but the iteration of the difference equation is not a straight forward. The following is an approach by a group of students to model the situation. Let y_n represent the total amount in the account after n years and let d_n be the deposit at the end of the n th year. We have

$$d_{n+1} = d_n + 0.02d_n$$

$$d_{n+1} = 1.02d_n, \quad d_0 = 2000 \qquad (2.2)$$

$$y_{n+1} = y_n + 0.04y_n + d_{n+1}$$

$$y_{n+1} = 1.04y_n + d_{n+1}, \quad y_0 = 2000. \qquad (2.3)$$

However, most of the students can not iterate the difference equation (2.3) since its right-hand side contains two dependent variables y_n and d_{n+1}. The groups and instructor's discussion suggests that equation (2.3) can be iterated if its right-hand side is transformed into a function of the dependent variable y_n and the independent variable n. Students from previous sessions know that the difference equation $x_n = ax_{n-1}$, where a is a constant, is equivalent to $x_n = a^n x_0$. Consequently, the difference equation $d_n = 1.02d_{n-1}$ is replaced with $d_n = (1.02)^n d_0 = 2000(1.02)^n$. Thus, the current situation is modeled by the difference equation:

$$y_{n+1} = 1.04y_n + 2000(1.02)^{n+1}, \quad y_0 = 2000 \qquad (2.4)$$

Now DERIVE's function ITERATE can be used to find the ordered pair $(20, y_{20})$ by authoring and approximating the expression:

ITERATE ([n + 1, 1.04y + 2000(1.02)^(n + 1)], [n, y], [0, 2000], 20)

The output is [20, 76310.10].

After these activities the instructor introduces the concept of annuity and the associated terminology such as ordinary annuity, the term and future value of an annuity.

2.2 Mortgages and Amortization. While working on the following activity the students explore modeling loans and determine the periodic payments on a loan as well as other related questions.

Activity 2.3 You want to buy a new car that costs $16,000. The dealer offers $4,000 for a trade-in with a down payment of $2,000. You borrow $10,000 from a bank for 5 years at an annual rate of interest 12% compounded monthly. The payments are made at the end of each month. What will the monthly payment be?

Discussion: Students model the given situation by a difference equation such as:

$$y_n = y_{n-1} + 0.01y_{n-1} - p = 1.01y_{n-1} - p, \qquad (2.5)$$

where y_n represents the balance of the loan after n months and p is the monthly payment, where the interest rate per month $= 0.12/12 = 0.01$. In this situation $y_0 = 10,000$ and $y_{60} = 0$. Since the value of p is unknown, we cannot iterate equation (2.5). The students usually estimate a value of p. If you take a loan of $10,000 without interest and you need to repay it over 60 months, the monthly payment would be 10,000/60 = $166.67. Since there is an interest rate on the loan, the monthly payment must be more than 166.67. Students simply select an estimated value of p, substitute it in (2.5), and use DERIVE ITERATE routine to evaluate y_{60}, which should be 0 or very close to 0. The students adjust their estimation of p based on the value of y_{60}. After several trials they find that if $p = 222$, $y_{60} = 36.29$ and if $p = 223$, $y_{60} = -45.36$. This concludes that $\$222 < p < \$223$. $P \approx \$222.25$.

Usually some students would wonder if there is a formula that can be used to solve the above problem instead of guessing the monthly payment. The students are guided to develop the analytical solution of a non-homogeneous first order linear difference equation in the form

$$y_n = ay_{n-1} + b \qquad (2.6)$$

where a and b are constant real numbers and $a \neq 1$. We have $y_1 = ay_0 + b$, $y_2 = ay_1 + b = a(ay_0 + b) + b = a^2 y_0 + ab + b$. Similarly, $y_n = a^n y_0 + a^{n-1}b + a^{n-2}b + ... + b = a^n y_0 + b(1 + a + a^2 + ... + a^{n-1})$. Using the sum of a geometric sequence, we get

$$y_n = \frac{-b}{a-1} + \left(y_0 + \frac{b}{a-1}\right)a^n \qquad \text{if } a \neq 1 \qquad (2.7)$$

The instructor asks the students to use (2.7) to find the monthly payment p in the above activity. They substitute values of: $a=1.01$, $b=-p$, $y_0=10,000$, $n=60$ and $y_{60}=0$ in (2.7) and solve the equation for p, which is $222.444.

2.3 Equilibrium and Stability. The following activity is designed to intuitively introduce students to the equilibrium value of a first-order linear difference equation with constant coefficients and the stability of the equilibrium value.

Activity 2.4 For each of the following difference equations find the ordered pairs (n, y_n), $n=0,1,2,...,24$ for the given initial conditions of y_0 and describe the graphs:

A) $y_n=0.6y_{n-1}+8$, (n, y_n), $n=0,1,2,...,24$ for (i) $y_0=8$, (ii) $y_0=20$, and (iii) $y_0=28$.

B) $y_n=1.1y_{n-1}-4$, (n, y_n), $n=0,1,2,...,20$ for (i) $y_0=35$, (ii) $y_0=40$, and (iii) $y_0=45$.

C) $y_n=-0.6y_{n-1}+16$, (n, y_n), $n=0,1,2,...,24$ for (i) $y_0=6$, (ii) $y_0=10$, and (iii) $y_0=14$.

D) $y_n=-1.5y_{n-1}+5$, (n, y_n), $n=0,1,2,...,12$ for (i) $y_0=1$, (ii) $y_0=2$, and (iii) $y_0=3$.

Discussion: The students use DERIVE or a spreadsheet to explore the behavior of the numerical solutions of the given difference equations with given initial conditions. They will recognize the equilibrium value of the difference equation and determine whether the equilibrium value is stable or unstable. The instructor develops with students the definition of an equilibrium value of a difference equation and how to determine it. The definitions of stable and unstable equilibrium values are developed too. In addition, the instructor discusses with students the equilibrium values in the context of financial and business models.

Now the students are ready to do the following activity.

Activity 2.5 Make a conjecture about the stability of the equilibrium value of the difference equation

$$y_n=ay_{n-1}+b$$

in terms of the constants a and b.

Discussion: Lively discussion among the group mates and the groups will lead students to conclude that an equilibrium value is stable if $-1<a<1$, that is $|a|<1$. It is interesting that the stability of an equilibrium value does not depend on the value of b.

3. A Model of Supply and Demand

We will consider the farming industry where a crop is produced once each year and the farmer decides on the amount of the crop to be planted based on the current year price of the crop. Note that the amount of crop planted this year is harvested next year and would represent the next year supply. Let S_n and D_n be the number of units supplied and demanded in year n respectively and let P_n be the price per unit in year n. If the price of the crop is high in the nth year, the farmer anticipates that the price level will be maintained and will decide to plant a larger amount of the crop. The following year with a large harvest of the crop, the supply S_{n+1} exceeds the demand D_{n+1} and will result in dropping the price of a crop P_{n+1} in order to create more demand to equal the supply. As a result of the drop in the price P_{n+1} the farmer will decide to plant a smaller amount of the crop. Consequently, the supply S_{n+2} will be less than the demand D_{n+2} and will result in increase of the price P_{n+2} and the farmer will decide to plant a larger amount of the crop and this process repeats itself.

We will make the following assumptions:

1. The supply of the crop S_{n+1} in year $n+1$ is a linear function of the crop's price in the previous year P_n. The supply S_{n+1} increases when the price P_n increases.
2. The demand of the crop D_n in year n is a linear function of the crop's price in the same year P_n. The demand D_n decreases when the price P_n increases.
3. In any year n, the market price P_n of the crop is the price at which the demand D_n equals the supply S_n.

The students use these assumptions to construct a model, investigate it, and make conjectures. From assumption 1 we have

$$S_{n+1} = m_s P_n + c_s \tag{3.1}$$

where m_s and c_s are positive constants. The constant m_s is the slope of the line (11), which is one unit increase in price, produces an increase of m_s units in supply. m_s represents the *sensitivity of suppliers to price*. The constant c_s is the S -intercept. From assumption 2, the demand equation is

$$D_n = -m_d P_n + c_d \tag{3.2}$$

where m_d and c_d are positive constants. Since m_d is positive, the slope of the demand line is negative and means that a one unit increase in price produces m_d units decrease in demand. m_d represents the *sensitivity of consumers to price*. Assumption 3 asserts that

$$D_n = S_n \tag{3.3}$$

From (3.1), (3.2) and (3.3) we get

$$-m_d P_n + c_d = m_s P_{n-1} + c_s$$

$$P_n = aP_{n-1} + b \tag{3.4}$$

where $a = -m_s / m_d$ and $b = (c_d - c_s)/m_d$.

The price at which the supply and demand lines intersect is called in economics the *equilibrium price*, that is the intersection of S_n and D_n. Equivalently the equilibrium price, call it P_e, is the equilibrium value of difference equation (3.4). The equilibrium price is called *stable* if the prices converge to P_e, otherwise is called *unstable*.

Equation (3.4) is a first order linear difference equation with constant coefficients. The behavior of the solutions of (3.4) depends on the value of the constant a. Noting that $a < 0$, we are interested in investigating (3.4) in three cases: Case 1: $-1 < a < 0$ (i.e. $m_s < m_d$); Case 2: $a = -1$ (i.e. $m_s = m_d$); Case 3: $a < -1$ (i.e. $m_s > m_d$).

Activity 3.1: Consider the supply and demand model (3.4) in the following cases:

Case 1: $m_s = 0.25,\ m_d = 0.5,\ c_s = 2,\ \mathrm{c_d} = 8,$ and $P_0 = 10$

Case 2: $m_s = 0.5,\ m_d = 0.5,\ c_s = 1,\ \mathrm{c_d} = 8,$ and $P_0 = 8$

Case 3: $m_s = 0.6,\ m_d = 0.5,\ c_s = 1,\ \mathrm{c_d} = 7.6,$ and $P_0 = 6.5$

For each case do the following: (i) Find the equilibrium price P_e. Find a numerical solution $(n, P_n), n = 0,1,...,14$, graph it and analyze its behavior. (ii) Determine whether the equilibrium price P_e is stable. Describe the relationship between stability/instability and sensitivity of the suppliers and consumers to price changes.

Discussion: In Case 1, $a = -m_s / m_d = -0.5$, $b = (c_d - c_s) / m_d = 12$, and equation (3.4) becomes, $P_{n+1} = -0,5P_n + 12$. The equilibrium price $P_e = b/(1-a) = 8$. The graph of the ordered pairs $(n, P_n), n = 0,1,...,14$ is shown in Figure 3.1. The price oscillates around 8, that is the solution is alternately above and below the line $P = 8$. The amplitude of the oscillation decreases and eventually the price P_n approaches 8 as n becomes large. Consequently the equilibrium price is stable.

In Case 2, $a = -1, b = 12,$ and $P_e = 6.$ The graph of $(n, P_n), n = 0,1,...,14$ is shown in Figure 3.2, where the price oscillates between 8 and 4 that is between P_0 and $-P_0 + b$.

In Case 3, $a = -1.2, b = 13.2,$ and $P_e = 6.$ The graph of $(n, P_n), n = 0,1,...,14$ is shown in Figure 3.3, where the price oscillates around $P = 6.$ The amplitude of the oscillation increases without bound. The price diverges from the equilibrium price. Consequently, the market price is unstable. Note that this model will fail if the price becomes negative.

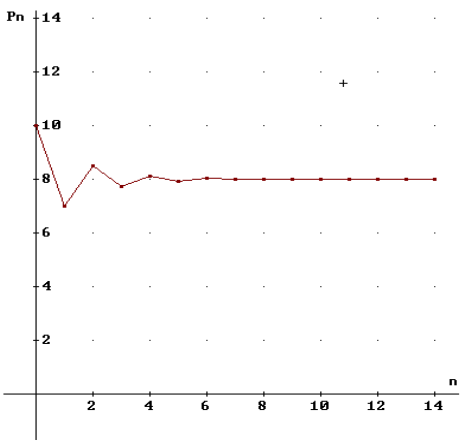

Figure 3.1

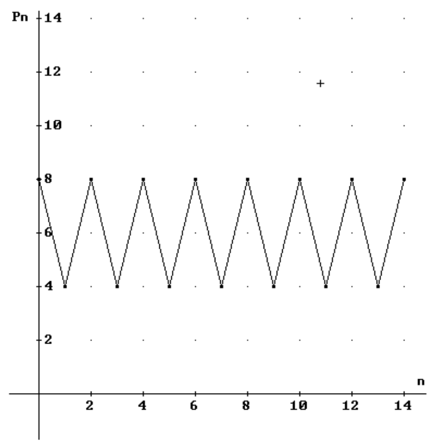

Figure 3.2

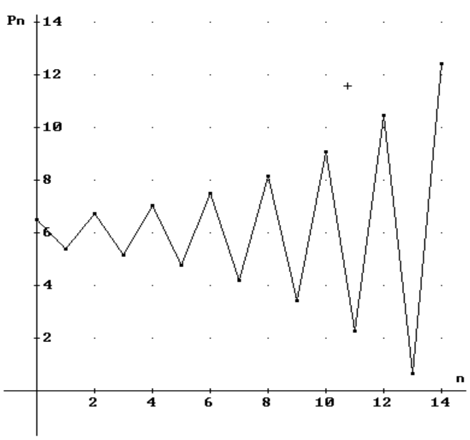

Figure 3.3

Conjecture: Suppose that the price of a product satisfies the above three assumptions in the supply and demand model. It can be concluded from the above activity that the stability of the equilibrium price, that is the market price, depends on the value of $a = -m_s / m_d$. We have three situations:

Situation 1: If $a > -1$, that is $m_s < m_d$, then we have a stable price, i.e. the market price tends to stabilize. In other words, we have a stable market if the sensitivity of suppliers to the price is less than the sensitivity of consumers to the price.

Situation 2: If $a < -1$, that is $m_s > m_d$, then we have an unstable equilibrium price. This means that we have an unstable market if the sensitivity of suppliers to price is greater than the sensitivity of consumers to the price.

Situation 3: If $a = -1$, that is $m_s = m_d$, then the market price oscillates and is a 2-cycle. The market price is unstable.

These three cases can be summarized in the following statement: If the producers are less sensitive to the price than the consumers, then the market price tends to stabilize. Otherwise, we have an unstable market price.

Practical considerations: In practice, it is difficult to calculate the constants in this model, m_s, m_d, c_s, and c_d for a particular product. In reality, we are interested in knowing whether the equilibrium price is stable or not. Recall that the above conjecture concludes that the equilibrium price is stable if $m_s < m_d$. Consequently, to determine the stability of the equilibrium price, we need to compare m_s and m_d. In other words, we need to observe the reaction of suppliers and consumers to different prices of the product to determine who the more sensitive one is.

A big concern arises if it has been determined that $m_s > m_d$ for a certain product, for example potatoes, and consequently, the market price is unstable. In order for the market to change and become stable, $m_s < m_d$ is needed. One way to increase m_d (the consumers' sensitivity to price) is for the government to act as the consumer. This can be done in one of two ways: (i) the government supports the price; (ii) the government pays the farmers to limit the potatoes they grow.

4. Logistic equations and chaos.

Many situations cannot be modeled by linear first order difference equations in the form (2.2), but can be modeled by first order nonlinear difference equations in the form (4.1),

$$x_n = x_{n-1} + a x_{n-1} - b x_{n-1}^2 \qquad (4.1)$$

where a and b are constants. An equation in the form (4.1) is called logistic difference equation. Letting $x_n = \frac{1+a}{b} y_n$, equation (4.1) is scaled to equation (4.2) which is called logistic map,

$$y_n = (1+a) y_{n-1}(1 - y_{n-1}) \qquad (4.2)$$

Students explore the behavior of the solutions of (4.2) for different values of $a, 0 < a \le 3.$ They recognize the intervals for a when solutions behave in different ways such as: the solution oscillates around the limiting value and converges to it ($1 < a \le 2$). This type of oscillation is called *damped oscillation*; the solution oscillates between two values ($2 < a \le 2.4495...$) and is called *periodic solution with period 2 (2-cycle)*; the solution repeats itself every four iterations ($2.4495... < a \le 2.56...$) and is called *periodic solution with period 4 (4-cycle)*; and the solution does not approach a fixed point or a periodic cycle and behaves erratically and is very sensitive to initial conditions ($2.56994... < a \le 3$). This type of behavior is called *chaotic*.

5. Sample research project: a technique to remove chaotic behavior from a system.

There are some methods to control chaotic systems. Güémez and Matias [2] suggested an interesting and simple method to control deterministic chaos of a system represented by a single difference equation. The method works by applying a series of proportional feedbacks (adjustments) on the system variable performed periodically with a certain period. The method, like other methods, stabilizes a given unstable orbit. However the method is simpler than other methods.

Consider the logistic map (4.2), with $\lambda = 1 + a$,

$$y_n = \lambda y_{n-1}(1 - y_{n-1}) \qquad (5.1)$$

The control algorithm consists of the application every Δn iterations of a feedback to the variable y having the form

$$y_n' = y_n(1 + \gamma) \qquad (5.2)$$

where γ is a real number that represents the strength of the feedback, and $-1 < \gamma < 1$.

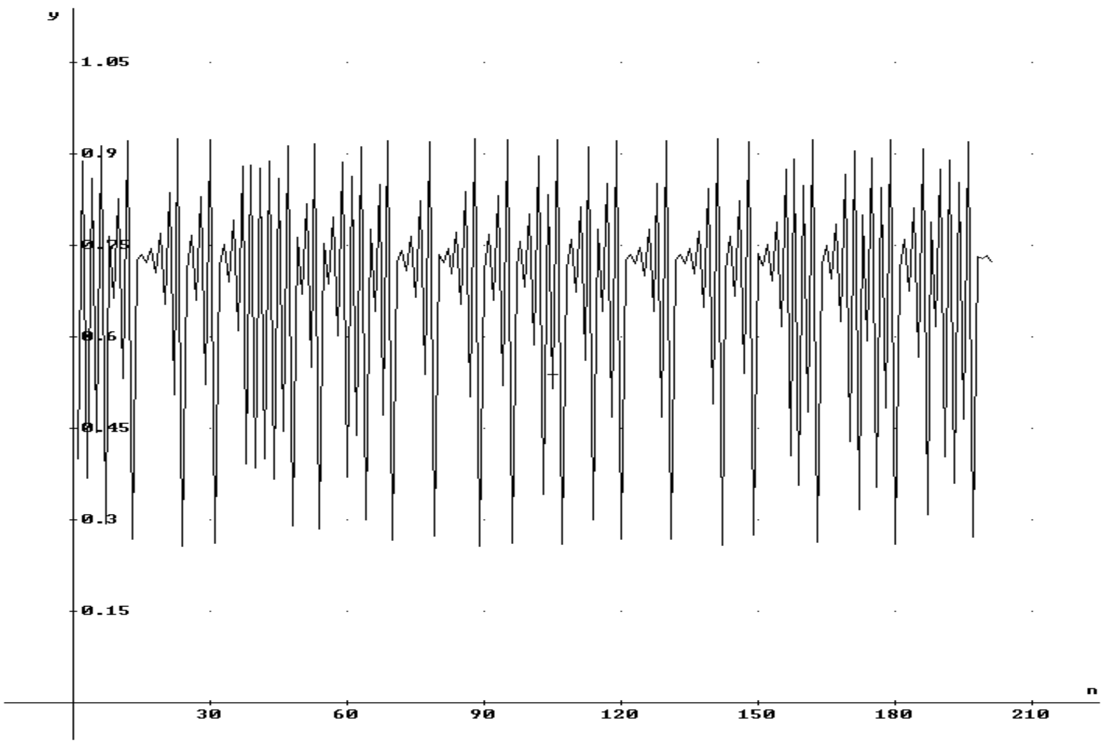

Figure 5.1 Logistic map (5.1) with $\lambda = 3.7$

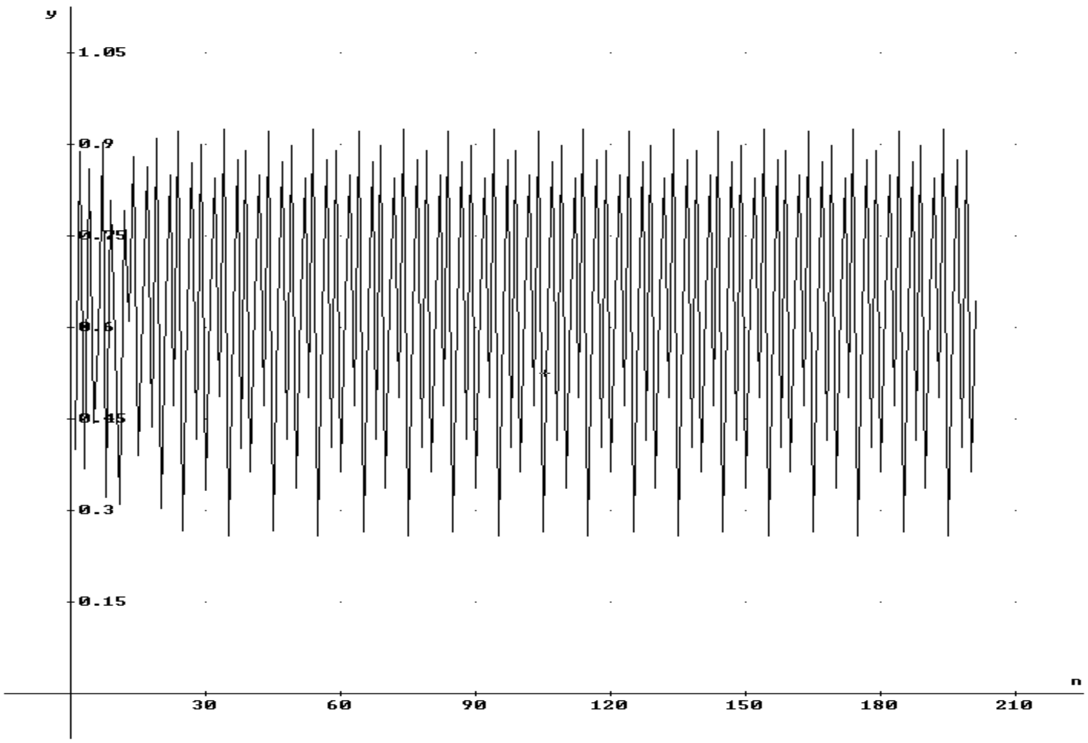

Figure 5.2 The control algorithm is applied to (5.1) with $\lambda = 3.7, \gamma = 0.05,$ and $\Delta n = 5.$

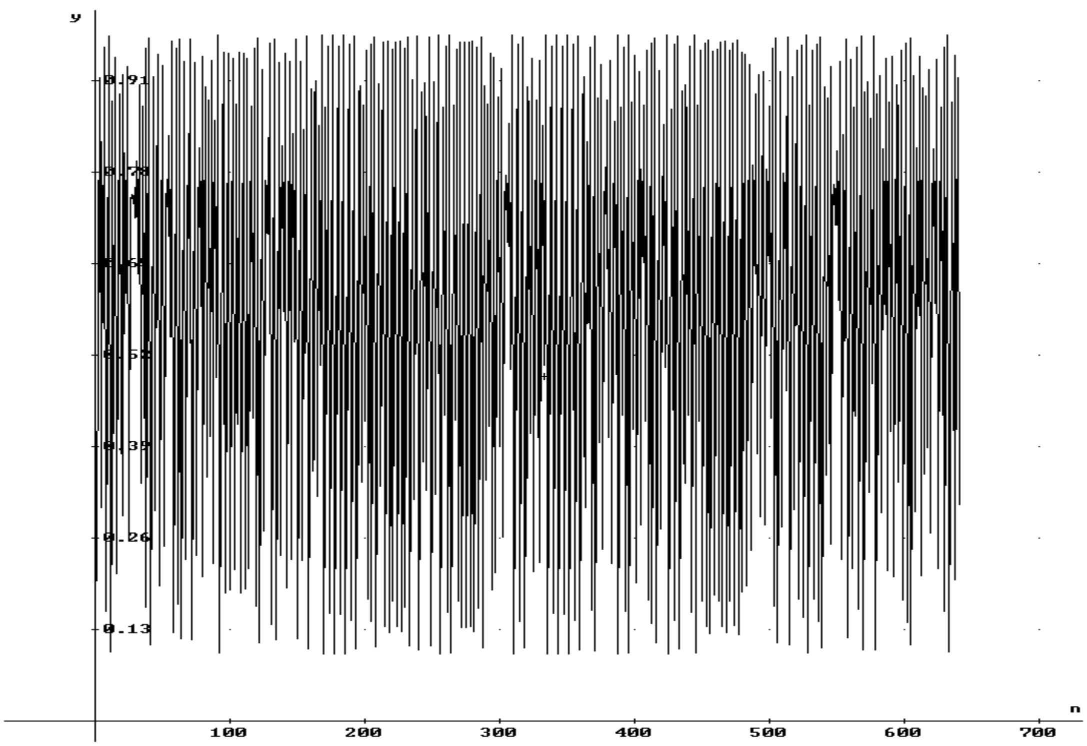

Figure 5.3 Logistic map (5.1) with $\lambda = 3.9$

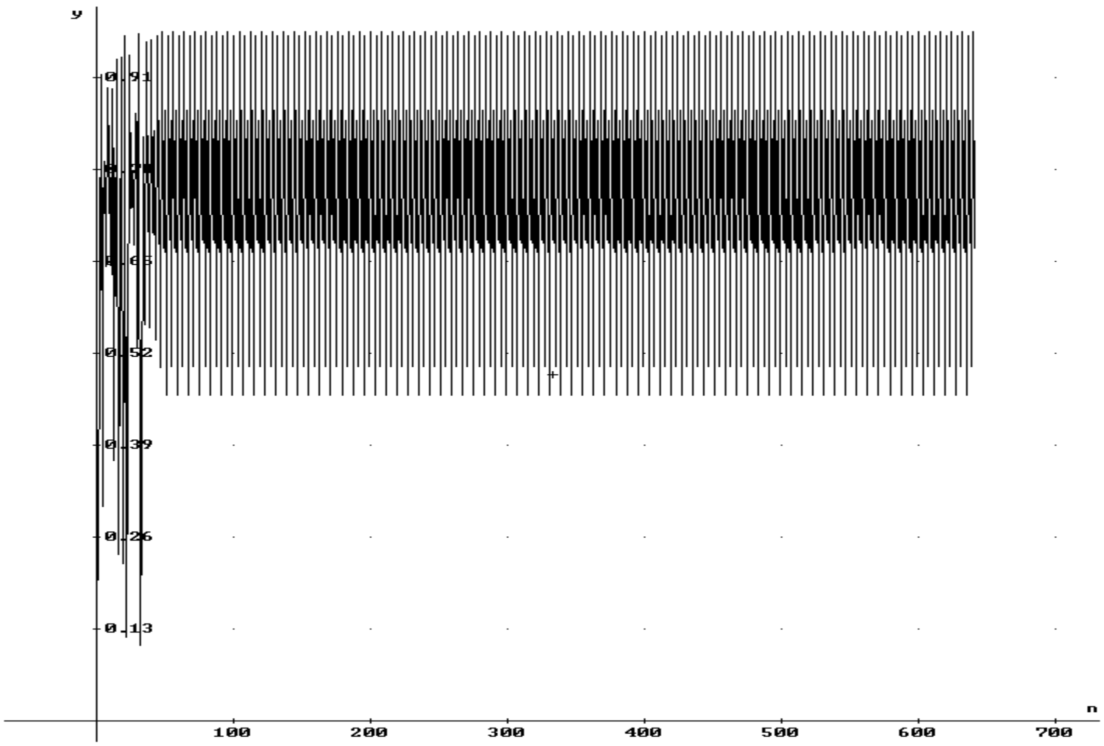

Figure 5.4 The control algorithm is applied to (5.1) with $\lambda = 3.9, \gamma = -0.2,$ and $\Delta n = 4.$

Let Δn be some positive integer representing the period of control. Thus the control is given by,

$$y_{n+1} = \lambda y_n(1 - y_n) \text{ if } n \text{ does not divide } \Delta n\text{, otherwise } y_{n+1} = \lambda y_n^{'}(1 - y_n^{'}) \qquad (5.3)$$

One way to iterate the above function (5.3), y_{n+1}, is to define it as piecewise function. We will use DERIVE function CHI. Recall that CHI (a, x, b) = 1, if a < x < b. If x < a < b or a < b < x, CHI (a, x, b) = 0. We provide two examples of applying this control algorithm to different values of $\lambda, \Delta n,$ and γ.

Example 1. Figure 5.1 shows the graph of y_n versus n for the logistic map (5.1) with $\lambda = 3.7,$ while Figure 5.2 shows the graph of (5.1) after applying the control algorithm with $\gamma = 0.05,$ and $\Delta n = 5.$ The ordered pairs $(n, y_n), n = 1, 2, ...200$ may be obtained by the following expression in DERIVE:

ITERATES ([n+1, 3.7y(1 – y) CHI (0.1, MOD (n, 5), 4.1) + 3.7y(1 + 0.05)
(1 – y(1 + 0.05)) CHI (-0.1, MOD (n, 5), 0.1)], [n, y], [1, 0.4], 200)

Example 2. Figure 5.3 shows the graph of y_n versus n for the logistic map (5.1) with $\lambda = 3.9.$ Figure 5.4 shows the graph of (5.1) after applying the control algorithm with $\lambda = 3.9,$ $\gamma = -0.2,$ and $\Delta n = 4.$ The ordered pairs $(n, y_n), n = 1, 2, ...650$ is obtained by the following DERIVE expression:

ITERATES ([n+1, 3.9y(1 – y) CHI (0.1, MOD (n, 4), 3.1) + 3.9y(1 – 0.2)
(1 – y(1 – 0.2)) CHI (-0.1, MOD (n, 4), 0.1)], [n, y], [1, 0.2], 650)

Figures 5.2 and 5.4 show that the periodicity of a system does not set immediately after applying the control algorithm. Note, that this control algorithm does not change the system parameters which is an advantage. However, the control algorithm does not work for every values of γ and Δn, which is a disadvantage.

Güémez and Matias algorithm can be applied to several real unstable systems such as the economy which is a complex system [1]. In case of economy, the proportional feedbacks in this control algorithm can be represented by injection, if γ is positive, or withdrawal, if γ is negative, of macroeconomical quantity.

The student research teams are required to review Güémez & Matias' and other related articles. They find values of γ and Δn where the control algorithm is valid and determine the algorithm's limitations. Each team researches possible application of this algorithm in a specific area and presents their findings to the rest of the teams.

Conclusion

Modeling real life situations excites students and shows them the relevancy of abstract mathematical concepts to their lives, and helps convey the elegance and wide applicability of mathematics. Difference equations represent a very sophisticated and powerful mathematical tool to model a wide range of real life discrete time situations in diverse areas. Moreover, this powerful tool does not require sophisticated mathematics background, being accessible to anyone who has successfully completed high school algebra.

Research in mathematics education shows that students learn mathematical concepts more constructively by making mental images. Easy to learn and use symbolic, numeric, and graphic software helps students in constructing these mental images in a creative and productive way. Moreover, the use of computers frees students from tedious calculations and associated with these calculations boredom, which in turn allows students to focus on translating a problem into mathematical notations, finding a solution, interpreting the numerical and the graphical information provided, and then making conjectures and writing about their findings and observations. With the use of computers, real life and realistic problems can be considered in class/lab activities.

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