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LEARNER ERRORS AND MISCONCEPTIONS IN ELEMENTARY ANALYSIS: A CASE STUDY OF A GRADE 12 CLASS IN SOUTH AFRICA

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Abstract: The paper focuses on analysing grade 12 learner[1] errors and the misconceptions in calculus at a secondary school in Limpopo Province, South Africa. As part of the analysis the paper outlines the nature of mathematics errors and misconceptions. Coding of learners' errors was done through the lens of a typological framework. The analysis showed that most of the errors and misconceptions were due to knowledge gaps in basic algebra. Also noted was that errors and misconceptions in calculus were related to learners' over-dependence on procedural knowledge which had no conceptual basis. On the other hand, learners sometimes had sound conceptual knowledge for which they had not acquired allied procedural knowledge needed to perform in particular questions. Implications of the study to the wider mathematics education community are highlighted.

Keywords: Errors, misconceptions, differential calculus

Introduction

Curriculum reform perspectives in mathematics education articulated in many research papers and policy documents aim at deepening and increasing each learner's mathematical learning and achievement (National Curriculum Statement (NCS), 1998; National Mathematics Advisory Panel (NMAP), 2008). The perspectives suggest shifts from teacher-centred to learner-centred approaches. The learner-centred approaches imply that teaching must also be directly responsive to the difficulties experienced by learners on the learning platform, such as the mathematical errors and misconceptions they experience.

In a workshop organised for mathematics teachers by the South African Department of Education, teachers were required to outline the topics that they found problematic to teach. The teachers revealed that among others calculus was one of them. One wonders why to a large extent the teaching and learning of mathematics is so difficult and ineffective. To this quest, we suspected that poor performance in mathematics is correlated to learner errors and misconceptions.

The anatomy of learner errors and misconceptions

Although errors and misconceptions are related, they are different. An error is a mistake, slip, blunder or inaccuracy and a deviation from accuracy. According to Riccomini (2005), unsystematic errors are unintended, non-recurring wrong answers which learners can readily correct by themselves. Systematic errors though, are recurrent wrong responses methodically constructed and produced

[1] In this study the terms learner and student are treated synonymously. In practice the term learner is reserved for one who studies at primary or secondary school and student for one older who studies at a higher education institution

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across space and time. Systematic errors are symptomatic of a faulty line of thinking causing them referred to as a misconception (Green, Piel & Flowers, 2008; Nesher, 1987; Riccomini, 2005). Characteristically, misconceptions are intuitively sensible to learners and can be resilient to instruction designed to correct them (Smith, DiSessa & Roschelle, 1993). Errors are visible in learners' artefacts such as written text or speech. However misconceptions are often hidden from the undiscerning observer. Sometimes misconceptions can even be hidden in correct answers (Smith, DiSessa & Roschelle, 1993), when correct answers are accidental. Educators need to *listen* carefully to determine *why* learners give answers they give so that they can correctly follow learners' reasoning.

Statement of the problem

The application of calculus in solving multitudes of theoretical and practical problems in diverse areas of human expertise represents its immense utility. The most crucial courses in higher education are calculus driven. Despite the demostrable importance of calculus in the skills development of South Africa, Luneta (2008) observed that teachers and learners in South African schools register many challenges in dealing with this topic.

If the claim that learner mathematical misconceptions are persistent, and that they knowledge acquisition is true (Smith et al, 1993; Nesher, 1987), what hope could research in mathematical errors and misconceptions in calculus give? Most teachers are unaware of mathematical misconceptions held by their learners (Riccomini, 2005). As such, they teach mathematics in line with its logical structure, quite oblivious of the need to balance the psychological standpoint from which learners ascribe their mathematical meanings (Nesher, 1987).

The purpose of this study was to investigate the errors and misconceptions learners were displaying in differential calculus, classify errors and misconceptions learners have in response to calculus questions as well as to explain how learner calculus errors link with their misconceptions.

The research question was: What is the nature of errors and misconceptions that grade 12 learners display in responding to questions in differential calculus?

Theoretical framework and literature review

Researchers propose different viewpoints of what is of essence in mathematics teaching and learning. According to Skemp (1976), the practitioners who ascribe to relational understanding of mathematics (RU) believe that the essence of mathematics is "knowing both what to do and why" (p.1). RU is about how mathematical concepts integrate with each other to form a unitary logical structure. The most radical proponents of RU assume that mathematics is pure in form and that its applied applications are irrelevant to it. Practioners who hold instrumental understanding of mathematics (IU) hold that mathematics is a tool for carrying out computations. Scant attention is given to why that tool works. With regards to calculus, IU provides a disjoint superstructure which cannot justify why calculus techniques work. We assume that errors and misconceptions in calculus occur in the space between learners' RU and IU of calculus.

Further, Hiebert and Lefevre (1986) proposed that the mathematical knowledge held by a learner could either be procedural or conceptual. They characterize conceptual knowledge as generalisable knowledge rich in connections. Procedural knowledge is regarded as competence of carrying out a mathematical task, the *know-how* of mathematics but and not the *know-why*. Procedural knowledge is usually taught through drill and practice and so can be automated to carry out specific mathematical tasks rapidly and efficiently. This speed and efficiency can be misunderstood for conceptual understanding. We also argue that calculus errors and misconceptions result if learners fail to build procedures from conceptual knowledge.

Researchers argue which knowledge ought to come first during teaching and learning of mathematics. Some for example Rittle-Johnson & Siegler (1998) believe that the knowledge that learners must learn first is of no consequence. But Orton (1983) and Vinner (1989) strongly argue that the main problem with calculus teaching is that procedural knowledge is taught at the expense of or before conceptual knowledge.

Research on calculus errors and misconceptions

Research has tried to account for problems faced by students in learning calculus, but that research is still inconclusive. Orton (1983) researched on student errors on differentiation and noted that students' procedural knowledge in routine differentiation was adequate; but that students had underdeveloped conceptual understanding of the derivative. Orton indicated that 20% of students in his study confused the derivative at a point with the ordinate, or the y- coordinate of the point of tangency. Also Porter & Masingila (2000) observed that while students differentiate and find limits among other techniques, they often are not aware, and are often surprised that there are some underlying mathematical notions signifying these techniques.

Ferrini-Mundy & Graham (1994) indicated that students' understandings of concepts which build towards fundamental calculus are mis-understood by most learners. While Porter & Masingila (2000) also reported that many students in university calculus classes possess a superficial and incomplete understanding of basic calculus concepts that most educators may not be aware of.

Tall & Vinner (1981), reported that one difficulty in understanding differential calculus was that it instituted in learners the sudden transition from the study of the discrete and the finite, to the study of the continuous and infinite. Artigue (1996) argued that some advanced calculus French students could not comprehend that 0.999 999 999 … (a limit proces) is really equal to 1 (an object). This was due to the fact that some students were confounded in viewing the limit as a process as well as an object as supported by Dubinsky, Assiala & Cottrill (1997).

Research methods

The study was undertaken through qualitative methods. Qualitative research is empathetic, striving to capture phenomena as experienced by the research participants themselves (Creswell, 2007). A qualitative methodology suited the study as it concerns in-depth descriptions of processes (Merriam, 1992) in explaining how learners reason to form errors and misconceptions in calculus. Qualitative research is suitable for this study because, it is concerned with the meanings that learners conceived or misconceived about calculus concepts. Merriam (1992) argues that achieving a deep understanding of specific phenomena and probing beneath the surface of a situation to provide a rich context for understanding the phenomena under study is the aim of qualitative research. In this study, qualitative methods help us to understand what learners mean when their productions display errors and misconceptions in calculus.

A class of 45, Grade 12 learners at a Limpopo Province rural school who had used especially developed calculus teaching materials were given a test to find out the errors and misconceptions that learners might still have on the calculus topic. The learners were taught by a part-time post-graduate mathematics education student at the University of Johannesburg. There were 26 boys and 19 girls whose mean age was 17, 6 years. Selected learners were interviewed to explain errors observed in their answers.

Data presentation and analysis

We distinguished the types of errors by coding as shown in Table 1. The categories in Table 1 are neither mutually independent nor exhaustive.

Besides we classified errors as on task (OT) or not on task errors (NOT). OT errors are those committed when engaging with calculus concepts that were being assessed. NOT errors are those that do not directly concern the concepts being tested.

In analysing responses to each question using Table 2, learners who made similar errors were grouped using a capital letter and the question number. For example, A_1 (10) represents a group A on question 1 with 10 learners making similar errors. The first column of Table 2 names the group of learners, column 2; the responses of the group or the response of a learner representing that group. The third column explains the errors made while the fourth column codes the error types from 0 to 4 according to the typology of Table 1. Errors are also coded as on task (OT) or not on task errors (NOT). On task errors were made in directly answering the question. In this study, OT errors are errors made in answering the calculus questions. NOT errors were errors made on concepts not directly linked to the question. In this study most NOT errors were on algebra.

Table 1: Categories of errors

Code	Description with examples chosen from this study
0	**Non-systematic errors.** These are slips, lapses or unintended mistakes. e.g. To show from first principles that if $f(x) = -x^2$, then $f'(x) = -2x$, some learners wrote $(x+h)$ instead of $f(x+h)$ in the difference quorient formulae but proceeded to do the proof correctly
1	**Generalisation or transfer errors.** These refer to extension of previously available strategies in new situations where they do not apply. E.g. $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}(f(x))x\frac{d}{dx}(g(x))$. A learner in the class being researched differentiated in brackets and then multiplied the derivatives. This is an error of due to procedural extrapolation.
2	**Ignorance of rule restrictions or symbolism.** Applying rules to contexts they do not apply. Failure to understand the bounds where a rule applies. E.g. If $y = \frac{\sqrt{x}-4}{\sqrt{x}}$ then $y = \frac{(x-4)^{\frac{1}{2}}}{x^{\frac{1}{2}}}$. $\frac{dy}{dx} = \frac{\frac{x}{2}-4}{\frac{x}{2}}$. Here the learner assumes that the square root sign covers all *x-4* instead of just *x*. The error occurs when numerator and denominator are differentiated separately. This error is due to equation balancing.
3	**Incomplete application of rules** E.g. To find the gradient of tangent of $y = x^3 + x^2 - 5x + 3$ at $x = 2$, A learner found the gradient (11) correctly then wrote the point as (2,11). The thinking could not proceed from there.
4	**False concepts hypothesized to form new concepts** E.g. to find the *x* and *y* intercepts of $f(x) = x^3 + x^2 - 5x + 3$, a learner wrote $x^3 + x^2 - 5x + 3 = 3x^2 + 3x - 5x = 0$ and ended there (ideas of differentiation and turning point). The learner assumed that differentiation has to do with the intercepts. Perhaps thought turning points were intercepts! This is an example of a conceptual error.

Table 2: Question by question analysis of errors on the Grade 12 Calculus test responses

Question 1. Show from first principles that if $f(x) = -x^2$, then $f'(x) = -2x$

Learners Group	Exemplar errors in the group	Interpretation	Code	OT or NOT
A_1 (12)	Some learners in this group wrote $(x+h)$ instead of $f(x+h)$ in the definition of derivative formulae but did everything else well. Others in this group wrote $\lim_{h\to 0}$ in the formula for the derivative, but later just wrote $\lim_{h\to}$, omitting the 0.	Though not penalized, it is a serious error of notation, that later hinders appreciating mathematical argument. Because of these lapses in notation, learners do not fully understand or appreciate the rigor of mathematics. Learners may take the notation lightly resulting in serious problems later. At the same time it may be just a slip. Although the learners did not write, as h tends to zero, zero was used to eventually find the limit, suggesting that it may be a slip. These errors could also be procedural due to lack of understanding mathenatical notation	O	OT
B_1 (11)	A learner in this group wrote $2(x+h)^2$ for $f(x+h)$. Then maintained the –2 to get $-2x^2-2xh-2h^2+x^2$ simplified to get $4xh - 2h^2$. Then factorised h out to cancel it; substituted 0 correctly to get a limit of $4x$. There is a misconception in that the learners do not ask themselves why their answers are different from the one given. The learners must believe that their answers are correct and the one given wrong.	Later one worked $-2x^2 + x^2$ to get 0, Suggesting that they knew from conceptual experience that although they clearly saw that the terms do not cancel to get 0, they however wrote 0. The misconception here is that the learner thinks thought she must get a zero whatever, instead of her inspecting her work to find out why her expression is not reducing to x terms that cancel. Learners thus have fixated thinking on a concept. The learners have the correct concepts but are lacking in procedure. Again, another learner simplified $-4x^2 -4xh$ to get 0. This might stem form the need to get a 0 when differentiating from first principles, despite the fact that data does not show that. This is a scenario of working from answers. Learners had errors writing $-2(x+h)^2$ for $(x+h)^2$. It appears they used the differentiation rule already of multiplying by the power and reducing the power by 1. Hence the learners just went through the motions of differentiating from first principles, such learners believe these are just motions not really necessary as the answer is already known. (This type of thinking should be clarified if such learners are interviewed). This lack of diligence makes learning of mathematics difficult for learners as they presume other explanations as superficial when they are very essential.	1 2	OT OT

Question. 2 Differentiate $y = (x^3 + 1)(x^2 - 2)$, wrt x.

Learners Group	Exemplar errors in the group	Explanations	Code	OT or NOT
A_3(7)	Correctly worked out but after getting the answer went on to write $\frac{dy}{dx} = 0$ and the expression of $\frac{dy}{dx} = 0$ in terms of x equals zero	Continued to work after getting the answer, could have been thinking of using strategies for finding turning points (TPs)	1	OT
B_3(8)	After correctly simplifying the original expression to x^5-$2x^3$+x^2-2 said equal to x^7-$2x^3$-2 = $7x$-$6x^2$-2=-$7x$-6, implying that x^5+x^2=x^7. There was no notation for differentiating although it was implied.	x^5 + x^2=x^7 used the multiplying index rule,. Differentiated two terms of a wrong expression wrongly.	3	NOT
	Differentiated x^6-$2x^3$+x^2-2 to x^8-$2x^3$+2.	Others added x^6+x^2 to get x^8	1	OT
C_3(15)	Differentiated in the brackets!	Outright misconception, failure to appreciate the product rule	2	OT

Question 3. Differentiate $y = \frac{\sqrt{x} - 4}{\sqrt{x}}$ wrt x.

Learner Group	Exemplar errors in the group	Explanation	Code	OT or NOT
A_4 (12)	Wrote $\frac{x^{\frac{1}{2}} - 4}{x^{\frac{1}{2}}} = \ldots \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}x^{-\frac{1}{2}}} = 1$, but later wrote dy/dx=0; others wrote RHS= $\frac{x - 4^{\frac{1}{2}}}{x^{\frac{1}{2}}}$. Others had $\frac{x^{\frac{1}{2}} - 4}{x^{\frac{1}{2}}}$ differentiated to $\frac{\frac{x}{2} - 4}{\frac{x}{2}}$.	Differentiate numerator and denominator separately	2 1	NOT
B_4 (2)	RHS= $\frac{(\sqrt{x})^2}{(\sqrt{x})^2} - \frac{4}{(\sqrt{x})^2} = \frac{9x - 4}{x}$,	Using difference of two squares.	1	NOT NOT

	$y=x^2-4$, $\frac{dx}{dy}=2x$			
C_4 (8)	Some had RHS= $\frac{x^{\frac{1}{2}}-4}{x^{\frac{1}{2}}}=x^{\frac{1}{2}}-4x^{\frac{1}{2}}=\frac{1}{2}x^{\frac{1}{2}}+x^{\frac{3}{2}}$ Some had an answer $\frac{4}{x}=x^{-4}$	Incomplete application of differentiating rule Error might be linked to the knowledge that $\frac{1}{x^n}=x^{-n}$, where the power changes sign if the reciprocal is removed.	1	NOT
D_4(3)	RHS= $\frac{x^{\frac{1}{2}}-4}{x^{\frac{1}{2}}}=\left(x^{\frac{1}{2}}-4\right)x^{-\frac{1}{2}}=x^{-\frac{1}{4}}-4x^{-\frac{1}{2}}$	$x^{\frac{1}{2}}-x^{-\frac{1}{2}}=x^{-\frac{1}{4}}$ misapplying exponents	2	NOT
E_4(4)	Squared the RHS to get $\frac{x-9}{x}$. Some wrote $\frac{x-4}{x}=\frac{(x-2)(x-2)}{x}=\frac{x-4x-4}{x}$ differentiated to x^2-8^2	Misapplication of squaring formula Difference of two squares, nothing to do with differentiation	2 1	NOT NOT

Question 4. Find the turning points of $f(x) = x^3 + x^2 - 5x + 3$

Learner Group	Exemplar errors in the group	Explanation	Code	OT or NOT
A_6 (3)	Solved for f'(x) =3 x^2+2x-5 to get x=1 or x=2, then wrote f(1) =-6 and f(2) = -18. So said TPs are (1,-6) and (2,-18).	The underlying thinking is correct. This was an error of performance in failing to factorise 3 x^2+2x-5.	0	NOT
B_6(9)	Factorized correctly $3x^2+2x-5=(3x+5)(x-1)=0$. Wrote $x=\frac{5}{3}$ or 1. substituted to get the points (5/3; 56/2) and (1,0)	Oversight of writing $x=\frac{5}{3}$ instead of $x=-\frac{5}{3}$	0	OT
D_6(6)	Answers (-5/3;-194/27) and (1;0).	Underlying thinking correct but performance errors in simplifying substitution.	0	OT
F_6(3)	Wrote f(x) = $3x^3$ + $2x$ -5+ 3 Divided everything by 3 ostensibly to have a coefficient of 1 on x^2.	Was thinking of a method for calculating limits	1	NOT
G_6(7)	Used intercepts points found in 10.1 to substitute to find turning points	Outright misconception	4	OT

Question 5. Find the equation of the tangent of the curve at $x = 2$

Learner groups	Exemplar errors in the group	Explanation	Code	OT or NOT
$A_8(6)$	A learner in this group found y co-ordinate (5) but could not find the gradient at x=2, but rather said the gradient was –5 and proceeded to use it correctly to find the equation of straight line	Had difficulty to conceive that the derivative could give him the gradient. Must have confusedly used the relationship between perpendicular lines to get the gradient –5 from 5	2	OT
$B_8(3)$	Found the gradient (11) correctly then wrote the point as (2,11). This thinking choked the learner who could not proceed.	Misconception	3	OT
$C_8(4)$	Could only find y co-ordinate (5). Stopped there	Incomplete working	3	OT
$D_8(8\ \ )$	Found y co-ordinate and said it was the gradient. Wrote y-1=5(x-2)		4	OT
$E_8(5)$	Could not simplify substitution to get the y value. Got –1 instead of 5. but proceeded to do everything else well	Error of performance in substitution	0	OT
$F_8(6)$	Wrong formula for gradient of line. eq Used y value		4	OT

Discussion

Drawing from the analysis in Table 2, learner achievement in calculus seems to be mainly due to knowledge gaps in algebra that forecloses substantial calculus epistemic access. This is due to the fact that algebra is the language of calculus. Difficulties are also due to learners' lack of knowledge in underlying calculus concepts, in which calculus techniques such as differentiation are instrumentally understood.

NOT errors occurred in about 40% of the cases. In the algebra realm, some learners could not add or multiply basic algebraic expressions correctly. One implied that $x^5 + x^2 = x^7$ (Learner in Group B_3). Others wrote $x^{\frac{1}{2}} - x^{-\frac{1}{2}} = x^{-\frac{1}{4}}$ (Learners Group F, Table 2). There was confusion when learners tried to use the rules for indices particularly of negative or fractional exponents viz; $\frac{1}{x^{-2}} = 2^x$ or $\frac{x^{\frac{1}{2}} - 4}{x} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}x^{-\frac{1}{2}}}$ (Learner Group A, Table 2). These procedural errors were NOT errors but are significant in that they incapacitate progress calculus epistemic access.

Another wrote $(-x)(-x) = -x + x$ perhaps emanating from the knowledge that $(-)(-) = (+)$ but failing to articulate it clearly. Such learners are at cross-roads as they cannot differentiate and integrate multiplication of negative numbers and addition involving negative numbers.

Other NOT errors were due to difficulty in using the functional notation, for example two learners (Learners A & B, Table 2) wrote (x+h) instead of f(x+h), however their working maintained the essence of the function notation. This seemingly benign error is critical because if learners do not observe notation rigor, they will fail to appreciate the essence of mathematical argument.

OT errors occurred in 60% of the cases. A learner wrote $-2(x+h)^2$ for $f(x+h)$ (Learner D, Table 2). This error shows the learner thinks differentiation from first principles is just a formality as he knows the derivative is $-2x$. The learner's misconception lies in failure to appreciate the mathematical argument, in other words the learner is failing to appreciate the mathematics problem. Such learners being far from understanding the mathematical problem itself, would not appreciate the conceptual grounding necessary for its solution processes.

Another set of students (Learner Group G, Table 2) wrote the following on differentiating $-x^2$ from the first principle: $\dfrac{\lim\limits_{h\to 0}[f(x+h)+f(x)]1}{h}$. The first error is quoting the difference quotient as a sum. This is an error due to lack of understanding that the derivate is but a gradient; a *rise* over a *run*; the vertical increase in f over the small x horizontal increase; h. It is caused by a lack of appreciation of the geometrical basis of the derivative and instrumental understanding. Some wrote $f(x)=\dfrac{1}{x^2}$ rather than $-x^2$. Later they recovered to work with the correct expression $-f(x) = x^2$ but which is wrongly simplified, because of algebraic errors.

Also, most learners often think that most mathematical processes are linear. This is exemplified by learners who differentiate products in brackets or differentiate numerator and denominator separately instead of multiplying out the brackets or carrying out the division before differentiating. For instance, the working of Learner Group C, Table 2 suggested that *(f/g)' = f'/g'*

Some students on answering a question that only required them to differentiate a function; differentiated correctly but went on to equate the derivative to zero. They mixed differentiation (a process) and how to find turning points (an application of the derivative) without consciously being aware of what they were doing. It could be that learners have seen the teacher equating derivatives to zero while establishing turning points of functions while working of calculus problems for the class. The learners are mystified by this; hence they simply imitate equating the derivative to zero without appreciating the specific contexts where the derivative vanishes. That way they have a misconception as they cannot differentiate between differentiating *and* how to use differentiation to optimise or minimise.

Learners sometimes had good calculus concepts but their weak mastery of algebra made operationalising their knowledge difficult particularly on the topic of turning points. For example one learner, in finding the turning points of

$f(x) = x^3 + x^2 -5x + 3$, $f'(x) = 3x^2 + 2x-5 = 0$; to get x +1, or x+2

Then wrote turning points (TPs) are (1,-6) and (2,-18). The learner understood that for TPs the derivative is zero. Having found the derivate correctly, however, the procedural knowledge of factorizing was inadequate leading to wrong answers.

Some learners used the method of calculating limits on the turning point problem. One learner used the co-ordinates for the intercepts to calculate the turning points. Thus we note that while at other times learners have sound conceptual knowledge, their inadequate procedural knowledge affected their performance. This means that in the teaching and learning of calculus conceptual and procedural knowledge do complement each other and that it is crucial to teach them together.

Some learners displayed insufficient knowledge of calculus terminology such as confusing turning points with the axial intercepts. Thus some difficulties are due to non-conflicting but parallel calculus conceptual knowledge.

In the interview some learners at first stuck to their errors but after some probing it became obvious that learners possessed both misconceptions and correct concepts on some ideas which to them did not conflict. Further probing helped learners to acknowledge their submerged misconceptions. As a result, some learners were able to link their IU and RU and re-consider their misconceptions in new light.

Conclusion

This study has highlighted and characterised some misunderstandings displayed by a group of grade 12 learners in calculus. Their performance in calculus was undermined by weak pre-calculus skills on factorisation, directed numbers, solving equations and simplifying indices among others. This algebraic incapacity presented clear epistemological obstacles that undoubtedly had a negative impact in learning calculus. This position unreservedly implies the need to equip pre-calculus learners with solid algebraic skills. Also, learners need to particularly appreciate the geometric/graphical basis of the derivative as well as perform calculator-based numerical investigations that enable them internalise the notion of the derivative. The gradual and progressive turning of external actions into processes; reifying the processes into mathematical objects, that are internalised into schemas, constitute the APOS theory (Dubinsky, Assiala & Cottrill, 1997) – that learning occurs from specific concrete, contextual situations to abstract, context free and generic understanding. This theory helps to bridge constructivism and pedagogy. Similarly graphical and calculator-based investigations could be regarded as “horizontal mathematisation” (Freudenthal’s, 1991) in a geometric and numerical real context that prepares learners for calculus “vertical mathematisation” in an algebraic and abstract context.

The noted errors and misconceptions show that there is structure in the misconceptions learners have and that these misconceptions emanate from prior acknowledge as learners attempt to construct mathematical meanings (See Table 1). Learners’ errors are therefore a result of naïve concept images that do not measure up to the concept definitions (Tall & Vinner, 1981), characterised by expert concepts. The noted errors are of two kinds, fragmented and elaborate (Chi, 2005). An example of an elaborate error is that differentiation of composite functions is a linear operation, while an example of a fragmented error is that $x^{\frac{1}{2}}x^{-\frac{1}{2}} = x^{-\frac{1}{4}}$. Thus the errors that learners exhibited in their work are varied.

This report recommends that further research be done to

(a) determine how learner errors and misconceptions in calculus evolve and,

(b) assess how far competency and performance in calculus can be enhanced if educators target the errors and misconceptions that have been identified in their learners and students.

Such research increases knowledge to enhance powerful calculus teaching and learning.

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