Folding Back and the Growth of Mathematical Understanding in Workplace Training

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Abstract

This paper presents some initial findings from a multi-year project that is exploring the growth of mathematical understanding in a variety of construction trades training programs.[1] In this paper, we focus on John, an entry-level plumbing trainee. We explore his understandings for multiplication, fractions and units of imperial measure as he attempts to solve a pipefitting problem. We consider the apparently limited nature of his images for these concepts and the role of 'folding back' in enabling his growth of understanding. We contend that it cannot be assumed that the images held by adult apprentices for basic mathematical concepts are flexible, deep, or useful in specific workplace contexts. We suggest that folding back to modify or make new images as needed in particular contexts is an essential element in facilitating the growth of mathematical understanding in workplace training, but also offer a note of caution about ensuring that this is genuinely effective in its purpose.

Mathematical Understanding and Workplace Training

Although recent years have seen an increase in the attention paid by researchers to mathematics in the workplace there is still only a limited body of work that considers cognition and understanding in a vocational setting. Research that does exist in the area of adult mathematical thinking within a vocational setting is primarily concerned with the use of informal mathematics (e.g., Noss, Hoyles, & Pozzi, 2000), or the place of mathematics within work (see Bessot, 2000a, 2000b; Eberhard, 2000; Harris, 1991; Millroy, 1992; Scribner, 1984a, 1984b, 1984c; Smith, 1999a, 1999b; Wedege 2000a, 2000b), rather than with the process of coming to understand mathematical concepts for the individual learner in the workplace.

Straesser (2000a) in his discussion of workplace vocational mathematics instruction noted two distinct kinds of pedagogies, the first being "modelling" and the second being "legitimate peripheral participation" (see also Lave & Wenger, 1991). He notes that the first is usually employed in a classroom setting, where:

> the situation is to come from the workplace, the mathematical model rests upon mathematical structures and algorithms known before or taught on the spot and the solution of the model hopefully can be interpreted in a way to cope with the given professional situation (p. 70).

In contrast to this, legitimate peripheral participation relates to training on the job, where "learning takes place at the workplace whenever it is needed by the workplace practice and its problems" (p.70). In this study we are interested in the first of these two learning situations, that of the workplace training classroom, where apprentices engage with contexts and problems from the workplace, but which require mathematical models in their solution. As Straesser (2000a) states "in most cases, modelling vocational problems by applying mathematics is a major difficulty for the future worker – especially the extraction of the mathematical model from a professional situation at hand" (p.70). He does not, however, offer any detailed analyses of the kinds of difficulties faced by apprentices in extracting the mathematics and working with it, nor consider the ways in which they go about developing and using appropriate mathematical models.

Pozzi, Noss, and Hoyles (1998) in discussing the use of artefacts and tools in the workplace note that the need for workers to understand "the models which underlie their artefacts, to sort out what is happening or what has gone amiss … typically occurs when there is a breakdown, and in such a situation, people need to represent to themselves how the underlying structures work" (p.118). They also suggest that modelling is "a dynamic process of building connections between a situation and its mathematisation, rather than a process of decontextualisation" (p.119). We consider the growth of mathematical understanding in workplace training as a dynamical process, and in this paper focus in detail on the way that one apprentice, when in difficulty, attempts to understand the mathematical concepts required by the task and those embedded in the use of a measuring tape.

The nature of mathematical understanding

The research reported in this paper is framed by the Pirie-Kieren theory for the dynamical growth of mathematical understanding (Pirie & Kieren, 1989, 1992, 1994). This theory views understanding not as a static product, nor as something to be acquired and then applied, but instead it is characterised as occurring in action and thus constantly evolving. Therefore, rather than being seen as a linear process, the growth of mathematical understanding is characterised as:

> not simply a matter of acting in abstract ways with more and more abstract mathematical objects. Such growth in fact entails a dynamic and a connection between more and less formal, abstract and sophisticated activities. Because growth in understanding in action occurs in contexts, a study of the growth of understanding must necessarily take into account the interactions that a person has with and in such contexts, including interactions with materials, other students and teachers. (Kieren, Pirie, & Gordon-Calvert, 1999, p. 229)

We agree that such a view of the nature of mathematical understanding is a vital one for the workplace training setting, where understandings are continually situated in specific problem-solving contexts, being dependant both on specific materials and on interactions with other workers and trainees.

This location of understanding in the "realm of interaction rather than subjective interpretation" and a recognition that "understandings are enacted in our moment-to-moment, setting-to-setting movement" (Davis, 1996, p.200) allows and requires the discussion of understanding not as a state to be achieved but as a continuously unfolding phenomenon. Hence, it becomes appropriate not to talk about 'understanding' as such, but about the process of coming to understand, about the ways that mathematical understanding shifts, develops and grows as a learner moves within the world.

The Pirie-Kieren theory provides a way of considering the socio-cultural environment of the learner, through seeing the individual not merely as existing within a particular context or discourse (as suggested by Lave, 1996; Walkerdine, 1988) but as co-constituting, co-existing and co-emerging with the context. As Davis (1996) writes ‘the world's relationship to the organism is not merely uni-directional and constraining; the organism also initiates or contributes to the enactment of its environment” (p.10), and to re-emphasise the notion of embodied action, “our sensorimotor capacities are embedded in and continuously shaped by broad biological, social, and historical contexts” (p.11). Thus, this notion of understanding, rooted in an enactivist position, acknowledges the concerns of those working from a socio-cultural perspective, and does not dispute the significance of context in considering learning. Indeed Davis (1995) states:

> enactivism does not dismiss the varied critiques of mathematics education. Rather, the framework offers a means of incorporating cultural commentary with discussions of individual cognition. It does so by arguing there is a certain self-similarity between processes of individual cognition and of collective action. (p. 8)

The theory thus offers a way to account for the growth of personal dynamical mathematical understanding in a very detailed way, whilst still recognising the vital role of the context in which this growth is occuring.

The Pirie-Kieren theory posits eight layers of understanding together with the cognitive activity of ‘folding back’ as crucial to the growth of understanding. A diagrammatic representation of the theory is provided in figure 1. The theory and the model can be used to “capture growth in mathematical understanding as *pathways of understanding* that are unique to persons and topics and co-emerge with the particular situations in which they find themselves” and also “to observe activity over many time-scales” (Kieren, Pirie, & Gordon-Calvert, 1999, p.219). It is thus a tool that offers the user a means to observe and describe the growth of mathematical understanding for a chosen learner, for a chosen mathematical concept, over a chosen period of time. These choices are made by the observer, and are driven by their particular focus of interest.

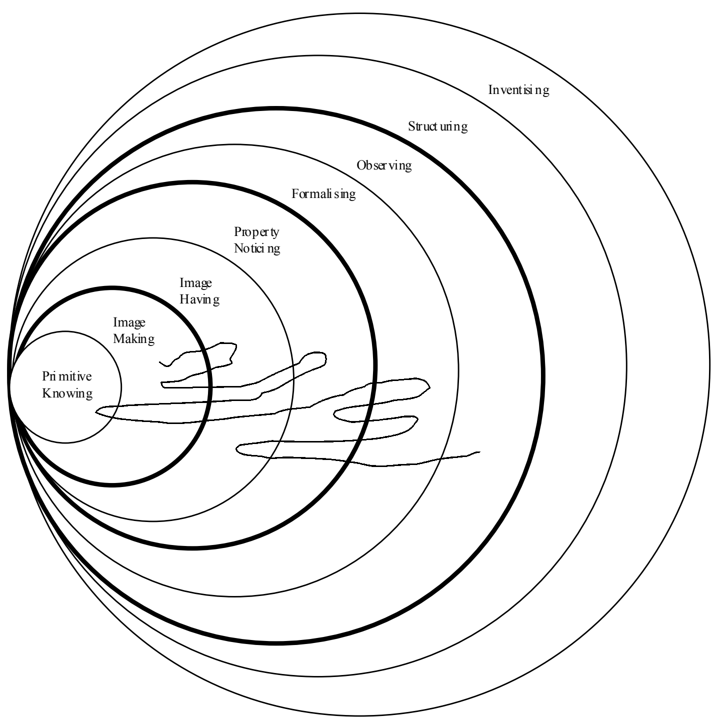

Figure 1. The Pirie-Kieren model for the Dynamical Growth of Mathematical Understanding.

Two of the inner layers are defined as *Image Making* and *Image Having*, and it is these layers that are relevant to our discussion in this paper. At Image Making learners are engaging in specific activities aimed at helping them to develop particular ideas and images for a concept. By “images” the theory means any ideas the learner may have about the topic, any “mental” representations, not just visual or pictorial ones. Image Making often involves the drawing of diagrams, working through specific examples or playing with numbers. However, it does not have to have an observable physical manifestation, it is the thinking and acting around the concept that is the actual process of making an image.

By the Image Having stage the learner is no longer tied to actual activities, they are now able to carry with them a general *mental* plan for these specific activities and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples. At this layer, the learner *has* an understanding, although this may still be very specific, mathematically limiting and context dependent, which they are able to employ when working on mathematical tasks.

The notion of folding back

Although the layers of the Pirie-Kieren theory develop from the concrete to the more abstract, or from the specific to the general, it is important to recognise that the growth of understanding does not occur through a simple linear pathway through the layers. Instead, mathematical understanding is seen to emerge through the continual movement back and forth through the layers of knowing, as individuals reflect on and reconstruct their current understandings (See figure 1 for a representation of a possible hypothetical pathway of growth).

A key feature of the theory is the idea that a person functioning at an outer layer of understanding, when faced with a problem that is not immediately solvable, needs to return to an inner layer of understanding to examine and modify their existing ideas and thinking about the concept. This process is known as ‘folding back’ implying that when a learner revisits earlier images and understandings for a concept he or she carries with them the demands of the new situation and uses these to inform their new thinking at the inner layer, leading to what may be termed a ‘thicker’ understanding for the concept.

As suggested, folding back occurs with a purpose, namely to extend one’s existing understandings which have proved to be inadequate for handling a newly encountered problem. It is in response to an obstacle that the learner re-visits earlier understandings, aiming to modify, collect, or build anew conceptions which will allow the difficulty to be overcome through an extended understanding of the topic.[2] Hence the inner layer activity is informed by what the learner already knows and by what they need to be able to do.

It is the fact that the outer layer understandings are available to support and inform the inner layer actions which gives rise to the metaphor of *folding* and *thickening*. Although a learner may well fold back and be acting in a less formal, more specific way, these inner layer actions are not identical to those performed previously. Folding back can be visualised as the folding of a sheet of paper in which a thicker piece is created through the action of folding one part of the sheet onto the other. The learner has a different set of structures, a changed and changing understanding of the concept, and this extended understanding acts to inform subsequent inner layer actions.

Folding back then is a metaphor for one of the processes through which understanding is observed to grow and through which the learner builds and acts in an ever-changing mathematical world. Folding back accounts for and legitimates a return to localised and unformulated actions and understandings in response to and because of this changing world. For example, folding back to the layer of Image Making might again involve such physical actions as drawing diagrams, working with manipulatives or playing around with numbers. However, it should be noted that:

> such image-extending inner-layer activity to which a person returns is reconstructive (or recollecting) in nature. It is not simply the instant recall of a known piece of information or action

> sequence. It involves remembering or recombining or extending actions, images, formalizations or theorems. (Kieren, Pirie, & Gordon-Calvert, 1999, p. 218)

Whilst it might be initially be seen as a step backwards in terms of the observable actions of the learner, folding back is likely to lead to a thicker and deeper understanding of the mathematical concept, and is an essential stage in the dynamical growth of mathematical understanding.

Images and Folding Back in Workplace Training

Apprentices in workplace training are often re-learning mathematics that they have already met and have existing images and understandings for, whether these are ones developed in school or elsewhere and whether formal or informal in nature. As they engage in mathematical activity during training apprentices may need to re-visit these understandings and images, make sense of them again in specific trades situated applications, and if necessary construct new understandings which will allow the mathematics to be useful in the new context.

In the data drawn on in this study, the mathematics is presented within a problem context, where the production of a physical solution is required – that of a piece of pipe cut to a correct length. To be able to apply appropriate mathematics and perform suitable calculations is something of a secondary aim, what matters in the workplace is the resulting product of the mathematics. This is in contrast to the purpose of the same problem were it to be posed in the school classroom. Here, it would be the mathematics that would likely be the focus, the problem merely being a context in which to set this. The conclusion to the problem would likely be a calculated answer, rather than a piece of pipe suitable for the task in hand. Although the mathematics required in both situations is quite possibly the same or similar, it is the application (and therefore the understanding) of this mathematics that differs in the two contexts. Also, of course there are very different consequences for the correctness of the answer. In the school classroom, an incorrect answer will likely result in nothing more than a mark on a piece of paper, whereas in the workplace there are real costs associated with such errors. Thus different images may be required for the same concept when used in the workplace rather than the school classroom (see Forman & Steen, 2000), or a previously held image may need to be modified or broadened.

For example, in the elementary school fractions are often taught with reference to parts of a whole circle (usually described as being a pizza or pie) and the image here would be quite specific and one based on an area representation. However, when using fractions in the context of measurement, it is more appropriate to see a fraction as a point on the number line – although of course to be able to read a fractional unit of measurement still requires an understanding of the part-whole relationship. This number-line image is particularly important for working with measurements in imperial units, where lengths are stated in fractional units, unlike in metric units where decimals are more commonly utilised. (For example, one would rarely talk about three and seven tenth centimetres, though it is of course mathematically valid and correct). As Martin, Pirie, and Kieren (1994) note:

> fraction learning involves constructing an ever more elaborate, complex, broad and sophisticated fraction world and developing the capacity to function in more complex and sophisticated ways within it. Such an achievement will prove impossible if the foundations laid by the images the learners hold are not adequate to the task (p. 248).

However, it is important to recognise that merely engaging in an appropriate act of folding back (e.g. to make a new image for fractional units) does not automatically guarantee that the learner will be able to immediately overcome the problem which prompted the invocative shift. For a learner who is able to use their extended understanding to overcome the original obstacle we term such folding back 'effective'. It is important to note that this does not imply that the learner now has a complete solution and acknowledge that further folding back may be required before a sufficiently extended understanding exists. The key feature of effective folding back is that the learner is able to return to the outer layer and apply a newly extended understanding to the original problem in a useful way. Continued working may yield another new and different obstacle for the learner, necessitating further back and forth movement, but this is distinctly different from being unable to make use of the new constructs at the outer layer. In the former case the understanding of the learner is still growing through a continual back and forth movement, whilst in the latter it has been temporarily halted and is termed 'ineffective'.

Methods and Data Sources

The larger study, currently underway, is made up of a series of case studies of apprentices training towards qualification in various construction trades in British Columbia, Canada. This paper presents some initial findings and discussion that draws on one of these case studies, and although our conclusions are specific to this case, we would suggest that there are implications that may be relevant to other areas of workplace training and the use of essential skills. The trainees and their instructor were observed and video-recorded over a number of sessions. The episode on which this paper focuses involved a small group of apprentices in the shop working to calculate the length of a pipe component required for a threaded pipe and fitting assembly to be built to given specifications. This followed a formal lesson on this procedure in the classroom. The second author acted as a participant observer in this session and worked closely with individual trainees as they engaged with the task. The video recording of this episode was analysed using the Pirie-Kieren theory with a particular focus on identifying the mathematical images held, accessed, made, modified and worked with by John as he engaged with the task. It should be noted that the transcript offered below represents a very short extract from a number of hours of taping, and some of the comments and conclusions we offer draw on data beyond that presented here.

John and the pipefitting task

We now move to consider John's understanding of mathematical concepts such as fractions, measurement and number and use elements of the Pirie-Kieren theory to describe the way that he is seen to engage in acts of folding back as he works on making appropriate images for these concepts.

A drawing of the pipe assembly to be constructed is shown in figure 2. The apprentices were assigned the task of constructing this assembly with a *centre-to-centre measure* (C-C) of ten inches. There are two ways to approach the calculation of length of the straight pipe (P) required to join the two fittings to meet the given specification:

Method one: Pipe length (P) = the centre-to-centre measure (C-C) minus the *take-off,* where the *take-off* = two times the *fitting allowance* (A) minus two times the *thread makeup* (E).

Method two: Pipe length (P) = the centre to centre measure (C-C) minus two times the fitting allowance (A) plus two times the thread makeup (E).

The values for the fitting allowance (A) and thread makeup (E) were provided elsewhere.

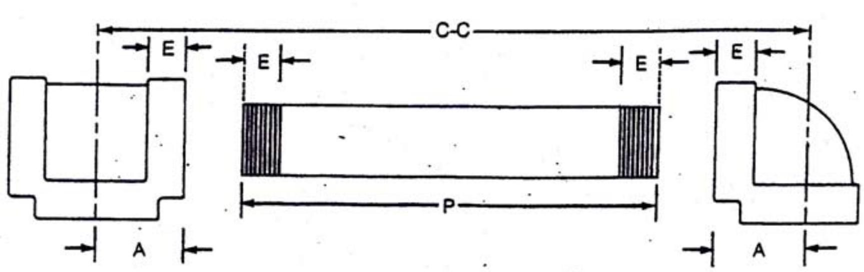

Figure 2. Pipe assembly to be constructed by John using standard pipe fittings and a cut piece of pipe.

Episode 1: Knowing How Without Knowing Why – Ineffective Folding Back

The following episode begins at a workbench in the shop as John works to make sense of the calculations needed for this task.
(John and Researcher)

J: Ok. So what I do now is, I know that it's ten, what I've got to have total.
R: Yeah.
J: I got to multiply the take-off twice, because on each end, right? *(Here John uses the term 'take-off" incorrectly to refer to fitting allowance (A))*
R: Yeah.
J: So what's got me.
R: So you multiply one and three quarters twice?
J: Yeah. One and three quarters gives me three and one sixteenths. Right?

In this first extract John has recognised the need to obtain the total *take-off* amount to accommodate fittings to be attached at each end of a pipe to make a ten-inch centre-to-centre pipe assembly. He has obtained the correct measurement for the *fitting allowance* (1 3/4") from an industry standard reference table for his calculation and written down:
1 3/4 x 2 =
on his sheet of paper. He has an understanding of what is involved in solving the problem, that he needs to remove a length of pipe from each end, and knows that he now needs to translate this understanding into a numerical calculation that can be carried out on a calculator. Once John has translated his understanding of the fitting allowance for both ends into an appropriate mathematical calculation, he uses his calculator to find this product. He enters:
$1\frac{3}{4}\times 1\frac{3}{4}=$ on his calculator a number of times while he works on the problem, getting an answer of $3\frac{1}{16}$ each time. While we cannot say with certainty why he chose to perform the calculator operations that he did, we would suggest that one factor is the mathematically ambiguous way (from our perspective) that he frames the required operation for himself, and the image, or lack of image, which underlies this.

John says, "I got to multiply the take-off twice, because on each end, right?" Here he is shifting from his appropriate (non-mathematical) pictorial and physical image for the problem – of an amount to be taken off each end – to one that is exclusively symbolic or numeric in form. However, in using language to re-formulate his understanding symbolically, he states that you have to "multiply the take-off twice" which easily lends itself to a symbolic representation of "$1\frac{3}{4}\times 1\frac{3}{4}$". It would seem that even although at one point he wrote $1\frac{3}{4}\times 2$, he may have been reading this as "one and three quarters multiplied by itself", seeing the symbols as a representation of the visual problem rather than as a calculation to be performed.

We contend that John's difficulty lies in the images that he has for the mathematical concepts being used here, especially those for multiplication. In trying to solve the problem, we see John having a viable visual and physical image of what is required for the task, and then needing to find or construct an appropriate mathematical model. To do this, he needs to have an image for multiplication, specifically one that recognises the idea that "when you multiply something by a natural number, it is the same as adding it to itself that many times." Having such an image would have allowed him to choose the correct calculation to then apply to the question at hand. We do not see John having an image for multiplication as "repeated addition" he does not access this here, nor does he generate it from his pictorial image for the problem, nor is it indeed implicit in his choice of operation.

John does not see the problem in terms of "putting" the two take-off lengths together to produce a single piece to be cut from the pipe, something that would lend itself to thinking in terms of an addition sum rather than a multiplication. We suggest there is a need for John to fold back to Image Making and through performing some actions at this layer, to thicken his image for multiplication in such a way as to allow him to apply this to the new context. This image making might include working with similar examples using whole numbers exclusively, prior to considering fractions, or working with grouping sets of objects. What is interesting here is that John is not convinced that his answer is correct, and he expresses this concern to the researcher, who then works with John on the problem.

R: Let's pull out a ruler. Here. Show me an inch and three quarters. *(John pulls out his tape measure)*
J: Ok.
R: Just show me with your finger.
J: That's an inch. That's my inch and a half. That's my inch and three quarter right there. Am I not correct? (*John indicates the correct points on the tape)*
R: Just put your finger there so I can see it. Ok. So there's your inch and three quarters right there. Add an inch and three quarters to that. And I'd go one step at a time. Like add an inch, and then add another three quarters.
J: Ok, so, I go, I've got an inch and three quarters right here. Which is right here. *(Pointing to* $1\frac{3}{4}$ *point on tape measure)* So to add another an inch, and another three-quarters to it?
R: Yeah.
J: *(pause)* Ok, hold on. Right here *(pointing at tape measure with pencil)*
R: Yeah.
J: *(long pause)* That would be three right here, right? No. That's one?
R: Right. That's one and three quarters, clearly. That's an inch and three quarters. Now if you add another inch and three quarters to that?
J: Ok. I'm stumped.
R: Ok.
J: I'm stumped. It's simple math here, needed. That's all. I'm not doing it.
R: Give me your right hand. Replace the finger on the tape measure. If that's an inch and three quarters *(indicating this point on the tape measure)* then another inch would be to?
J: Add?
R: Another inch would be to where?
J: Well wouldn't it be to here?
R: To there.
J: Right.
R: And now add three quarters. *(Indicating intervals on the tape measure)* One quarter, two quarters, three quarters. How much?
J: It would be three and a half.
R: Yeah. *(Then a long pause, with no response from John)*

Here we see the researcher taking John out of the context of the problem, and working with him to explore multiplication as repeated addition. The researcher has engaged John in a folding back activity, returning him to the Image Making layer. The hope is that through using a tape measure, a familiar and readily accessible workplace tool, John will make an image for multiplication as repeated addition through engaging with an activity of adding two equal quantities together on the tape measure. This is of course a useful mathematical image for the physical actions involved in cutting the required pipe length, as it encapsulates the idea of putting the two pieces of pipe together, finding the total and then making one cut, on one end, to remove this amount of waste. Although John engages with the activity he struggles with it, and clearly is not sure how to count-on using the scale on the rule, though at the end of the extract he seems to have performed the required addition act.

More significantly though, we suggest that John does not know why he is being asked to do this activity. As noted earlier, he gives no indication of seeing the required calculation as one involving addition. John is certain that he needs to multiply, but is simply not sure that he has chosen the correct procedure. Whilst he is happy to work on this addition problem with the researcher, he does not indicate that he relates this in any way to the pipefitting task.

For the researcher, and perhaps the reader, with powerful and versatile images for the concept of multiplication, the link between the activity and the mathematical calculation required is an obvious one, yet there is no reason to expect that John will automatically make this connection. John and the researcher are working with two different images for calculating the total take-off length, and as such, even when the correct answer to the addition is achieved, John is not sure what he should now do with this. We do observe John folding back, and recognising that he has an obstacle to overcome. However, he does not link the image making activity he is prompted to engage with to his original difficulty, and he is not able to use his thicker understanding to solve his calculation dilemma. Indeed, we would suggest he is not aware that he may now have a deeper understanding and richer image for multiplication. Although John has folded back, the action is ineffective for him, as he still does not connect repeated addition with the act of performing a multiplication calculation. The researcher continues to probe:

R: *(Pointing to* $3\frac{1}{16}$ *written earlier on sheet)* I'm a little miffed. I don't know how you got that. Show me what numbers you put into your calculator to get that.

J: Ok.

R: I don't know how to use one of these calculators.

J: Ok. Well what I do here, I'm doing a fraction on a calculator. I go, first I go. What's the number I'm doing here.

R: One and three quarters.

J: One and three quarters. I go one, then I hit the fraction button. It gets the little ratio there, right *(referring to the character "r" on the calculator screen)* Three, hit it again. One and three over four. Times, one over three over four. *(As he points to 1r3r4 on the calculator screen)*

R: Ahh!

J: That's a problem? A mistake? Should times it by, times it by two instead of? *(He now enters* $1\frac{3}{4}\times 2$ *into his calculator)* There we go. I did it wrong.

R: Bingo!

J: Got it. Ok. *(He writes* $1\frac{3}{4}\times 2$ *on his sheet)*

R: So you multiplied?

J: Yes.

R: Because you took one and three quarters, and you needed two of them. So instead of multiplying it by two, you multiplied it by one and three quarters. And because one and three quarters is really close to two, you were in the ballpark. But, when you originally wrote down three and one sixteenth you paused, you stopped for a second. And now you've erased it right here. But you were thinking of something and I'm itching to know what was going through your mind there when you were correctly uncertain. You knew.

J: The reason is, the reason is, that I was uncertain, do I multiply it by itself or by two. That's what I was thinking. And I'm thinking maybe, maybe not, maybe.

In this extract John explains to the researcher how he obtained three and one sixteenth, and the researcher becomes aware of the error John has made. John, believing that he has made an error somewhere in his pipe length calculation because it did not match that of another apprentice quickly abandons his original procedure and now multiplies by two, carrying out the correct calculation.

However, we suggest that John still does not connect the procedure for determining the total fitting allowance to one of repeated addition, and how this image relates to the multiplication that he has performed. He knew that either he had to multiply the fraction by itself or by two, but with no understanding of *why* one is correct. For him, it was a choice between the two procedures, and, as he now knows that as multiplying one and three quarters by itself was incorrect, he decides he must instead multiply it by two without offering any explanation as to why this is the case. The researcher probes further:

R: So, *(pointing to crossed out* $3\frac{1}{16}$ *on paper)* did one and three quarters flag that for you, or no matter what you would have got you still would have been thinking about it?

J: I still would have been thinking about it, because I would have known that, I still know in the back of my head, either you times it by, like I'm thinking to myself, times it by itself or you times it by two.

R: Ok.

J: That's what I'm thinking, all the time. So, and I'm looking yesterday's, yesterday's theory, I had no problem doing that. *(Points to written calculations from previous day)*

R: Yeah.

J: I made that in as dummy's terms as I can get. Right. So all I did was change the number here. *(He points to a written computation in notes from the previous day)* The formula still stays the same. And, that's my problem. I didn't want to look at that. I go on memory.

R: Yeah.

J: And what I want to do now is get a fresh sheet of paper and start over again before I cut this pipe. Ok. So that's what I'm doing. I don't want to go any step further, even though I know the answer. That's not going to help me when I do my test.

John is content to now know which operation he should use, and as he comments to "go on memory", although not to simply copy a procedure used previously. His mathematical images for multiplication seem to be unchanged, despite his folding back actions, and we would suggest that although he has now successfully completed the calculation his understanding of *how* and *why* the mathematical operation is the correct interpretation of his visual image is unchanged. Naturally, we question the reliance of an apprentice on procedural memory and return to this in our conclusions. We would suggest that perhaps John would have benefited from a further opportunity to fold back and to initially work with whole numbers, to re-make (or make) an image for multiplication that he could see as being appropriate to the task, and that would help him work with fractional amounts.

Episode 2: Knowing How and Knowing Why – Effective Folding Back

A few minutes later in the same session, with the correct value for the length of pipe now calculated, John went on to measure out the length of pipe that he needed using an imperial units tape measure, prior to cutting it. He is talking with the researcher, and Steve, another apprentice in the group. He has called the researcher over, as he is puzzled about the length of pipe he is measuring.
(John, Steve, Researcher)

J: What we're trying to find out is, this pipe size should be, what was it again?

S: Eight and three eighths.

J: Eight and three eighths.

R: That looks perfect. *(Referring to length of pipe being measured)*

J: So, now, (*pointing to* $8\frac{1}{8}$ *on the tape measure*) that's one eighths, right?

R Yeah

J: Two eighths. (*Pointing to* $\frac{3}{16}$*)* Three eighths. (*Pointing to* $\frac{2}{8}$) Right?

Here John has the correct length of pipe from his calculation, but thinks that he is incorrect when measuring a pipe of this very length. This is because when needing to count in eighths on the tape measure, he actually counts partly in sixteenths (also marked on the tape), thus resulting in him reading an incorrect measurement from the tape. The researcher points this out to him:

R: Yeah. Now wait a sec. There's a problem. Make sure you're not counting sixteenths.
J: OK.
S: Those are sixteenths.
R: OK. This from here to here is a half, right?
J: Right.
R: From here to here is? *(Pointing to the interval on the tape measure bounded by* 8*" and* $8\frac{1}{4}$*")*
J: A quarter.
R: A quarter, yeah. From here to here is? *(Pointing to the interval on the tape measure bounded by 8" and* $8\frac{1}{8}$*")*
J: *(Pause)* That's, the little, the little. *(Pointing to the mark on the tape measure indicating* $8\frac{1}{8}$*")* That's one eighth.
R: Yeah, like if this is a quarter, an eighth is.
J: I know that's a sixteenth. *(Pointing to a small vertical line on the tape measure indicating a sixteenth between eighths)*
R: Half of an eighth. Or, sorry, an eighth is half of a quarter. That's a quarter, then an eighth is from here *(indicating on tape measure)* to here.
S: So every second line is one sixteenth?
J: Right.
R: Every second line. And every single line here is a?
J: Eighth.
S: Thirty second. *(Spoken simultaneously with John)*
R: Every single line?
J: Every line is one is an eighth, is it not? The small line?
S: Every line is a thirty second.

Here we see John and Steve giving different answers to the questions, using their existing images for fractions. However, while both correctly use the language of fractions in this context, it is not clear at any point that John is actually thinking in terms of equal parts of a whole inch, and thus that to measure involves a comparison with the various fractional units that are superimposed upon one another on the tape. Instead, John seems to have and be using an image based on the fact that measuring tapes use different length lines to represent the different fractional divisions of each inch (i.e. very short vertical lines for thirty-seconds, becoming progressively longer towards the half inch), which does not encapsulate an understanding based on a part-whole fractional relationship.

John's confusion is apparent at the end of the extract when he incorrectly states that every single line is equivalent to one eighth, whereas Steve seems more comfortable with the fractional scale and gives the correct answer of one thirty-second. At no time does John talk about an inch being divided into a given number of equal sized parts, which is an image that could help him to more easily work with the complex measuring tape. John continues his explanation to the researcher:

J: Each line here, *(pointing to the fine makings on the tape measure)* see how, differences between the small and the little taller ones, right? I'm trying to make it as, here we go. *(He grabs a piece of paper and holds it over the tape measure so that only the end points of the marking lines are visible above the paper)* Here we go, ok. See how every line is different here, right?
R: Mmhu.
J: Cause one line that's smaller than the others. So.

R: Yeah.
J: That's one sixteenth, a big one is eighth, correct?

Here again John explains the different fractional units by referring to the size of the line used to make the sub-divisions on the tape measure. Sensing that this might be problematic, and that it is John's limited image for fractional units that might be causing the difficulty, the researcher introduces a set of ruler scales printed on acetate that can be stacked to illustrate how fractions are represented on a measuring tape (See figure 3).

Each acetate rule layer has the inches divided up into a different fractional unit, so the first rule has only inches on it, the next shows half inches, the next has quarter inches and so on. This provides a visual image for how a standard tape measure actually incorporates a number of different fractional units superimposed onto one scale.

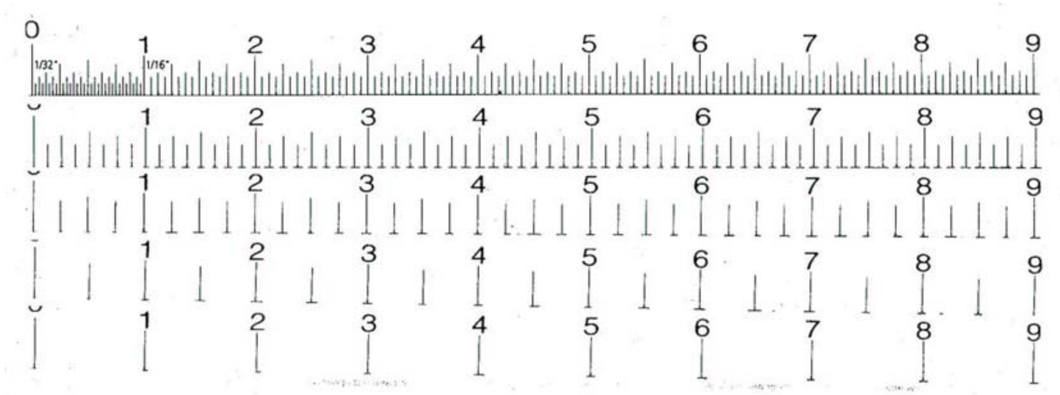

Figure 3. The set of acetate rulers.

The researcher offers the rulers to John, and they start to work with them.

R: Ok. So eight and a half is there. Halves are this ruler here. *(He selects the acetate with only half inch intervals indicated)* Then, I just want to, ok, ok. Line those up. *(The one inch interval acetate and the half inch interval acetate)* I should actually tell you, it's not quite, quite perfect. *(Referring to the acetates teaching tool)* It's out by a thirty second over nine inches, but. Ok. The quarter of an inch is there and it lays overtop like that. Yeah? Eighth of an inch, just look at the eighth of an inch *(indicating on the acetate ruler)* there. *(He positions the eighth inch interval ruler on top of the whole inch, half inch and quarter inch acetate rulers already stacked, one on top of another, on the workbench)* See how, how you're counting?
J: Right, ok.
R: That lays on top like that. But you only need eighth of an inch, right?
J: Right, I need eighth of an inch.
R: So, lets look at your ruler *(the tape measure*) up against this. *(The stack of acetates-teaching tool)* So I'm going to line up the eighth of a, lay it down flat.
J: OK.
R: Look at where your eighth?
J: So that's one, two, three. *(Counting off lines at* $8\frac{1}{8}$*,* $8\frac{3}{8}$ *and* $8\frac{5}{8}$ *on the acetate ruler, shown in figure 4)* Correct? Am I counting that correct?

Figure 4. John indicating what he thinks is $\frac{1}{8}$, $\frac{2}{8}$ and $\frac{3}{8}$ of an inch.

The set of acetate rulers are used here to offer John a valuable image making tool, allowing him to physically manipulate a set of representations of the different fractional units and to see how these relate, both to each other and to the standard measuring tape. Using the resource also exposes the problem that John is having in working with these fractional units. When he comes to count three-eighths, he points to one-eighth correctly, but then continues to point to the lines of the same length as this, i.e. three-eighths and five-eighths. He does not count two-eighths or four-eighths, as these are equivalent to one quarter and to one half respectively, and thus on the ruler are represented by lines of different lengths.

We see clearly here that John did not have an image for a fraction as a location on his measuring tape, as a point on a number line, and he does not see the relationship of the numerator and denominator of a written fraction to the part-whole of an actual inch. As suggested, John's image here for units of measurement seems to be that the lengths of the lines on the ruler scale correspond directly to particular fractional units, and is thus based on the labelling of points on the scale rather than on a continuous scale. The researcher responds:

R: No, you're doing spaces. Count spaces.
J: Ok. So that's one.
R: One, Yeah.
J: Two. *(Now counting eighth inch intervals on the acetate eighths ruler)*
R: Two, and at the end of that space is your?
J: Is this right here? *(He indicates $8\frac{3}{8}$ point on his tape measure)*
R: Right there.
J: Ok. Ok. So. *(He re- enacts the stepping process with his pencil tip from 8 inches on his tape measure)* Ok. Didn't see it, here we go. I got you. *(Now explains it back to the researcher)* Ok, so, this is our one, our one-sixteenth right here, correct? *(Pointing to $8\frac{1}{8}$ point on tape measure)* No, our one eighth, am I out?
R: Which is it?
J: This one right here, the big one? *(Pointing to $8\frac{1}{8}$ on the tape measure)*
R: Which is it, eighth or sixteenth?
J: That's eighth.
R: Ok.
J: Ok. This one here is sixteenth. *(Pointing to $8\frac{1}{16}$ point on tape measure)*
R: Yeah.
J: This one here is thirty-two? (*Indicating $8\frac{1}{32}$ on the tape measure*)
R: Thirty-second. Yeah.

J: Thirty-second. Am I on the ball with that?
R: Absolutely.
J: Ok.

In this extract the researcher engages John in a directed image making activity, he tells him to count spaces, and John carries out this physical action. In doing this, we suggest that John has now folded back to do something physical that will allow him to modify his inappropriate image for the fractional scale. He returns to Image Making, and is trying to make sense of how the ruler is to be correctly read, and *why*. After counting with his pen on the ruler and reaching eight inches and three-eighths, he re-enacts this stepping process, suggesting that he is now actively making an appropriate image, and is able to count-on in fractions whilst making the correct one-to-one correspondence with the ruler scale. The physical act of working with a manipulative allows him to understand what he is doing, and towards the end of the extract it seems that he now has an image for a fraction as a point on a continuous number line, as printed on the ruler. He is able to confidently identify the other fractional units, though we do not see him count on in these. We contend that John now has a more appropriate and useful image for the concept, though not necessarily complete, that will allow his understanding to grow, and enable him to correctly complete measurement tasks. The researcher probes a little more:

R: Because, that's how many of those little spaces would fit in a whole inch.
J: Right. So if we went one.
R: What are we counting now? What kind of fraction?
J: Eight, eight. Right?
R: Ok. Eighths.
J: So that's one-eighth from here *(indicating the 8 inch point)* to here. *(Indicating* $8\frac{1}{8}$ *inch point on tape measure)* That's two eighths *(pointing to the interval between* $8\frac{1}{8}$ *inch to* $8\frac{1}{4}$ *inch)* and that's three eighths. *(Pointing to the interval between* $8\frac{1}{4}$ *inch to* $8\frac{3}{8}$ *inch)* Gotyah. Ok. I understand now. Now that I can actually express it and point it out, ok.

Again in this final extract we see John confidently counting on in eighths, and recognising that he has an understanding of what he is doing. It is not totally clear here whether he fully grasps the idea of "how many of those little spaces would fit in a whole inch" that is articulated by the researcher. However John does agree when the researcher offers this explanation, so it does seem to make sense to him. We suggest that the future exploration of different fractional units will help him to develop his possibly partially formed image to be one that he can confidently use and apply whatever he is asked to measure, or indeed to work with fractions in another context.

In this episode John's folding back to Image Making has been effective and through his physical act of working with the measuring tape he now has a useful and appropriate image for imperial units of measure, based on a deeper understanding of the part-whole relationship of fractions. This image allowed him to successfully overcome his difficulty with measuring and enabled him to correctly measure the length of pipe, something that he was not able to do at the start of the episode.

Discussion

It is beyond the scope of this paper to comment in any depth on the complex role that mathematical understandings, images and folding back play in the trades training process, but we suggest that trades educators should expect that their trainees may not come with a useful and easily applied range of images for required mathematical concepts needed in their training.

Observing mathematical actions and activity through the lens of the Pirie-Kieren theory, and particularly through attending to the notion of folding back offers a way to both identify potential moment of breakdown, or disconnection in a pathway of growing understanding, and also to anticipate and potentially to overcome these. Recognising where learner held images are localised or limited, and unlikely to be helpful in the new specific workplace context identifies points where apprentices may need a space to be created for folding back to occur.

It would seem that offering opportunities for apprentices to fold back and to engage in appropriate image making activities for some mathematical concepts would be an appropriate way to occasion their growth of understanding, and enable the development of more widely applicable skills. However, our discussion of John also highlights the importance of ensuring that any such folding back act is effective and that the learner is able to recognise not only the limited nature of their existing understandings but also how any inner layer action is relevant to the overcoming of the initial difficulty. Whilst we recognise that in some ways, returning to "play around" with physical manipulatives might seem to be both a backward step and time consuming, we do believe that there is a need to re-engage with some basic mathematical concepts, but within the new context of the workplace. As Forman and Steen (1995) noted, there is a need in the workplace for "concrete mathematics, built on advanced applications of elementary mathematics rather than on elementary applications of advanced mathematics (p. 228). Certainly for John, he was being asked to use relatively elementary mathematical concepts but to use these in problem solving contexts that are very different from those in which the concepts will have been taught or used in school, and for which deep and flexible understandings and images are likely to be required.

It is not clear whether John had some existing images for the mathematical concepts required to successfully complete the pipefitting task (even though we can assume he will have met these concepts in school), but it is clear that even if he did, he did not see these as appropriate or useful in the creation of an appropriate mathematical model for the problem. We would suggest that in the first episode John would have benefited from an opportunity to fold back and to work with whole numbers, to re-make (or make) an image for multiplication as repeated addition that he could see as being appropriate to the task, and that would help him work with fractional amounts. We do see John folding back to image making in the second episode, and through working with the set of acetate rules and having an opportunity to play around with the different fractional units he does seem to have an appropriate image for imperial units at the end of the session. This kind of activity and accompanying learning tools are invaluable for encouraging mathematical understanding that goes beyond being able to merely read a scale or operate on numbers.

Clearly, the measuring tape is a fundamental part of working in pipe trades, and the ability to use this, and to understand the mathematics that is captured by this tool is essential for a worker. Whilst we acknowledge that such understandings are not likely to be made explicit during every task, the possession of a powerful and flexible set of mathematical images related to this offers something to fold back to, should memory fail, or the need arise to work in a new application. Such situations are examples of the kind of "breakdown" identified by Pozzi, Noss, and Hoyles (1998), and are cases where, quite suddenly, the mathematics captured within the tools of the workplace needs to be made explicit. Folding back offers a means through which the encountering of such a need does not necessarily need to result in a breakdown in the growth of understanding, but instead can act as a stimulus for continued growth and progress. Certainly for John, being able to connect multiplication with the image of placing the two cut-off pieces of pipe together, and of then understanding how this can be represented on a measuring tape could have been a valuable and growth enabling experience, in a similar way that was seen to occur when working with the acetate rules in the second episode.

We would also note that it is unlikely that a single act of folding back will ever lead to an image that can immediately be recognised as complete in some way. In our view of mathematical understanding, we suggest that images are made over time and across contexts, and as such should be constantly evolving to meet the new demands being placed upon them. As discussed, we do see John starting to make and have an appropriate image for the task he is engaged with, but for this image to become more flexible and general, and one that he can easily and comfortably use, repeated folding back is likely to be necessary. We hope that this would occur through being asked to work with different fractional units of imperial measurement in a range of different tasks and contexts.

Although we have not discussed them in this paper, the layers of the Pirie-Kieren beyond Image Having are those of Property Noticing and Formalising. At these layers, a learner is likely to have an understanding of a concept that is more general and abstract, and as such applicable to a wider range of contexts. (Working at these layers involves recognising connections across one's images, seeing general properties of these, and then being able to state these in more formal terms – perhaps as an algebraic rule, but with an understanding of why such statements are true). We do not see John yet having such an understanding for fractions, though through folding back he is working towards this more powerful way of thinking.

We contend that in the technical training classroom there is a need, through folding back, to re-visit concepts such as addition, multiplication and fractions and to go beyond learning merely how to operate on and with numbers. As Wedege (2000a) noted there are three different levels at which to consider mathematics in the workplace; the level of skills; the level of understanding; and the level of identity. Our research suggests that apprenticeship training focuses primarily, and successfully, on the first of these levels, whereas there is also a need to develop more general mathematics knowledge that is not so tightly tied to a specific workplace task.

The importance of developing mathematical understanding during workplace training is echoed by Straesser (2000b) who writes "if society is interested in highly qualified, self-reliant workers it may be worthwhile to continue training and education beyond the narrow necessities of the actual workplace" (p. 244) and more specifically that "if mathematics is taught as a bridge between the concrete, may-be vocational situation and the abstract may-be systematic structure, even classroom vocational education can show mathematics as a 'general' tool which is larger an importance that just coping with the narrow tasks of the everyday work practice or the inculcation of algorithms" (Straesser, 2000a, pp. 72-73). Our study supports this view and finds that in particular, there is a need to explore the existing understandings that trainees bring with them, to consider the appropriateness of these images for vocational related tasks, and through the provision of opportunities for effective folding back enable the construction of new images as needed, drawing upon the use of common workplace tools and resources as appropriate, whilst also ensuring that such images are flexible and adaptable.

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[1] The research reported in this paper is supported by the Social Science and Humanities Research Council of Canada, (SSHRC) through Grants #831-2002-0005 & 501-2002-005. We would also like to thank United Association of Journeymen of the Plumbing and Pipefitting Industry Trade School, Local 170, Delta, BC for their assistance with this project. We would also like to acknowledge John for his willingness to be involved in the study.

[2] For more detail on the different forms of folding back see Martin, Pirie & Kieren, 1996; Martin, 1999; Pirie & Martin, 2000.