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The Concept of Slope: Comparing Teachers' concept Images and Instructional Content

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Abstract

In the field of mathematics education, understanding teachers' content knowledge (Grossman, 1995; Hill, Sleep, Lewis, & Ball, 2007; Munby, Russell, & Martin, 2001) and studying the relationship between content knowledge and instructional decisions (Fennema & Franke, 1992; Raymond, 1997) are both crucial. Teachers need a robust understanding of key mathematical topics and connections to make informed choices about which instructional tasks will be assigned and how the content will be represented (Ball & Bass, 2000, Fennema & Franke, 1992). Ma (1999) described this *profound understanding of fundamental mathematics* as how accomplished teachers conceptualize key ideas in mathematics with a deep and flexible understanding so that they are able to represent those ideas in multiple ways and to recognize how those ideas fit into the preK-16 curriculum.

Slope is a fundamental topic in the secondary mathematics curricula. Unit rate and proportional relationships introduced in sixth grade prepare students for interpreting equations such as $y = 2x-3$ as functions with particular, linear behavior in eighth grade (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CSSO]), 2010; National Council of Teachers of Mathematics [NCTM], 2006). The focus on relationships with constant rate of change leads to distinctions between linear and non-linear functions (Yerushalmy, 1997) and

the idea of average rate of change in high school (NGA Center for Best Practices & CCSSO, 2010). Ultimately, these ideas prepare students for instantaneous rates of change and the concept of a derivative in calculus. The diversity of conceptualizations and representations of slope across the secondary mathematics curriculum presents a challenge for secondary teachers. These teachers must work flexibly and fluently with various representations in many contexts in order for their students to build a coherent, connected conceptualization of slope. Since secondary mathematics teachers need a deep understanding of slope to mediate students' conceptual development of this key topic, the study reported here investigates both how teachers think about and present slope.

The Concept of Slope

Tall and Vinner (1981) described *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152); it is the way that an individual conceives of a concept, which is built over years as a result of various experiences and impressions (Vinner, 1992). Research suggests that students' concept images of slope are fragmented. In particular, researchers have indicated that students do not make connections between the various representations of slope nor do they relate the concepts of slope and rate of change (Stump, 2001b; Teuscher & Reys, 2010). It has also been reported that students struggle to use slope in real world applications (Lobato & Siebert, 2002; Lobato & Thanheiser, 2002).

Conceptual knowledge involves a flexible understanding of governing rules that can be transferred to various problems and situations while procedural knowledge entails a knowledge of rules and processes linked to executing specific tasks (Rittle-Johnson, Siegler, & Alibali, 2001). Findings (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Harel, Behr, Lesh, & Post, 1994; Lobato & Thanheiser, 2002; Reiken, 2009; Stump, 2001b; Teusher & Reys, 2010) suggest students do not develop the rich conceptual knowledge required to relate alternate representations of slope nor to apply notions of slope in real world situations. Many students hold only procedural knowledge of the algorithms related to slope and the interpretation of slope in specific situations. Memorization of phrases like "change in y over change in x" to find the slope of a line given two points through which it passes or "rise over run" to consider the ratio of the vertical and horizontal displacement of a line are commonly used in educational settings.

Some of students' difficulties with slope may be due to the variety of ways that the concept can be conceptualized. Moore-Russo, Conner, & Rugg (2011) have suggested 11 conceptualizations of slope by refining and extending those initially offered by Stump (1999, 2001a, 2001b). Stump's (1999) original seven categories suggested that slope could be conceptualized as a *geometric ratio, algebraic ratio, physical property, functional property, parametric coefficient, trigonometric conception,* and *calculus conception.* Stump's later work (2001b) addressed an eighth category, *real world situations.* The ninth category, *determining property,* refers to the idea that slope determines the line and its relationship with other lines. It is this conceptualization that helps determine if lines are parallel or perpendicular; it also addresses the idea that in the plane a unique line is determined if you are given a point on a line and the slope of that line. The tenth category addresses slope as the *behavior indicator* of a line. Under this conceptualization, the slope of a line indicates whether it is increasing, decreasing, or constant; the absolute value of slope indicates the severity of the line's inclination. The eleventh category refers to slope as a *linear constant* in that it is the property that gives a line its "flatness" or "straightness" keeping it void of any curvature. Under this conceptualization falls the idea that slope is a constant property, independent of the region of the linear graph that is under consideration. Table 1 summarizes the 11 conceptualizations of slope.

Table 1.
Conceptualizations of Slope [author-identifying citation]

Category	Slope conceptualized as …
Geometric Ratio (G)	"rise over run" or the vertical and horizontal displacements, which is often marked on a graph
Behavior Indicator (B)	indication of a line's increasing, decreasing, or constant behavior as well as the amount of increase (or decrease)
Determining Property (D)	the property that determines either a line, when given a point, or the relationships between lines (e.g., parallel, perpendicular)
Algebraic Ratio (A)	"change in y over change in x" or a representation of ratio involving algebraic expressions such as $(y_2 - y_1)/(x_2 - x_1)$
Parametric Coefficient (PC)	m, the coefficient in either the point-slope or slope-intercept equation
Functional Property (F)	(constant) covariation often represented by the expression "rate of change"
Linear Constant (L)	the constant property of a line that creates the "straight" or "flat" absence of curvature; often mention that any two points on a line may be used to determine slope
Real World Situation (R)	an application of situations, including both physical (static) and functional (dynamic) situations, that have linear conditions
Physical Property (P)	the steepness of a line often referred to as the "slant", "tilt", "steepness" or "incline" of a line
Trigonometric Conception (T)	the angle of inclination of a line with the horizontal; the tangent of the angle of inclination; the direction component of a vector
Calculus Conception (C)	the measure related to derivative either specifically as the slope of a secant or tangent line to a curve or as relating to the instantaneous rate of change for a nonlinear function

The conceptualization categories are not discrete. Nagle & Moore-Russo (2012) provide the following example where one might think of slope as both a *functional property* and *a real world situation.* For the situation where an employee earns $10/hour, a person might simultaneously consider that for each change in input there is a constant change in output and that for every additional work hour, 10 additional dollars are earned. Hence, there is a constant rate of change. But understanding the impact of the independent and dependent variables in this applied situation requires consideration and understanding of the units involved (hours and dollars).

Slope in the Curriculum

Textbooks and teachers' knowledge must be considered as possible sources for discrepancies between national recommendations (NGA Center for Best Practices & CSSO, 2010; NCTM, 2000) and students' performance related to slope. Arnold and Son (2011) investigated the presentation of linear relationships in textbooks from the United States. Just as we (2012) found significant variation between states' standards documents, Arnold and Son (2011) found significant differences in content and problems between textbooks. They concluded that their results:

> ...indicate that though students are being asked to grapple with linear relationships at increasingly younger ages, that they are limited in the models they are asked to use for linear relationships, that discussion of concepts and connections is often limited, that the majority of problems lack a real world context, and that a majority of problems address lower levels of cognitive expectation. (p. 388)

The treatment of the concept in textbooks may limit students' development of rich, connected concept images of slope. However, adoption of textbooks that emphasize such a development of slope does not guarantee that this emphasis will carry through into the enacted curriculum (Brown, 2009; Mesa & Griffiths, 2012; Remillard, 2005).

Teachers' knowledge and pedagogical practices also impact the intended and enacted curriculum (Brown, 2009; Remillard, 2005). Walter and Gerson (2007) suggest that the emphasis of the mnemonic phrase "rise-over-run" has contributed to a procedural knowledge of slope to the extent that students are poorly equipped to connect slope and either line position or rate of change. Others have shown that teachers' tendencies toward procedural versus conceptual knowledge can impact their instructional decisions (Morton, Manouchehri, & Owens, 2011; Thompson, 1994).

Teachers' Knowledge of Slope

Because teachers' knowledge is a critical factor in how content is represented, it is important to consider the conceptualizations of slope that teachers hold and emphasize. Studies in the U.S. suggest that secondary mathematics teachers most frequently conceptualize slope in terms of a *geometric ratio* (Stump, 1999) and hold various conceptualizations of slope, but different conceptualizations are evoked depending on the task they are given (Moore-Russo, Conner, & Rugg, 2011).

We (2012) have investigated and compared the conceptualizations of slope used by college instructors and students. Findings indicated that instructors tend to have more diverse conceptualizations of slope compared to students, but found that *behavior indicator* (i.e., the use of slope to indicate the increase or decrease as well as the severity of the increase or decrease of a line) was infrequently referenced by college instructors despite frequent references by college students. The results suggested a gap between the conceptualizations commonly used by instructors and those commonly used by students (Nagle & Moore-Russo, 2012)

When considering practicing secondary teachers' content knowledge of slope, Coe (2007) found that teachers showed difficulty working with average rates of change and could not explain the use of division in the slope formula. Stump (2001a) described a program intended to develop preservice teachers' understanding of slope. Findings indicated that preservice teachers developed lessons incorporating real world situations, but their actual instruction focused more on graphs and equations. Additionally, they demonstrated hesitancy when teaching slope as a measure of steepness and rate of change (Stump, 2001a).

Present Study

In order to teach slope effectively, teachers must have a robust knowledge of the concept. Thus, the first question is: Which and how many conceptualizations of slope do teachers evidence? To investigate the extent to which teachers are prepared to teach slope in a way that promotes development of and connections between its various conceptualizations, it is important to study which conceptualizations of slope teachers emphasize. Just as a teacher's subject matter knowledge alone does not indicate her capacity to teach the concept, neither does the content of instructional materials portray the teacher's full understanding of the topic. By attempting to explore teachers' understanding (i.e., concept images) of slope as well as their emphases when preparing materials related to slope, we hope to explore as well as their emphases when preparing materials related to slope, we hope to explore the relation of the two. It is the connection between how teachers understand and portray slope that leads to our second research question: What is the relationship between pre- and inservice teachers' understanding of slope and the content they include in instructional materials on slope?

Methodology

Participants

Data for this study came from 19 individuals enrolled in a graduate course in the fall 2011 semester at a research university in the northeastern United States. All participants either held mathematics degrees and were enrolled in a teacher certification program or held mathematics education degrees and were taking additional graduate course work. Five were preservice teachers working on initial teaching certification. Six were preservice teachers who held initial teaching certification and were seeking permanent employment while working on the next level of certification. The remaining eight were inservice teachers in full-time, permanent teaching positions; five of whom were novices with less than two years of experience in the classroom and three of whom were veterans with more than five years of classroom experience.

The course in which the participants were enrolled is a three-credit hour, graduate-level course with a primary goal of introducing future and practicing middle and high school mathematics teachers to mathematics-specific and general educational digital tools that can be used in secondary mathematics education. When course instruction is related to general digital tools, instructional tasks always make use of these tools in the context of teaching or considering mathematical concepts. Prior to data collection, the participants in the course had worked with GeoGebra software and applets found in the National Library of Virtual Manipulatives (http://nlvm.usu.edu/).

Data Collection

Data was collected from two instruments that were given to the preservice and inservice teachers (henceforth simply called "teachers") enrolled in the course. According to Schmittau (2009), concept mapping is a valuable tool that can indicate the degree to which preservice teachers have developed an interconnected understanding of a topic and the likelihood that they will be able to mediate the concept to students. Thus, in the first task, the teachers were introduced to Prezi (Hansen, 2011) zooming presentation software and were instructed to use the Prezi environment to create a concept map presentation for slope that they would use for instruction with secondary students. Since the directions specified that this concept map was intended for use in instruction, this task provided insight into the teachers' instructional intent.

In the second task, the teachers were introduced to Wordle™ (Feinberg, 2012), an environment that allows users to input words and phrases thus creating a word cloud. The teachers were informed that word clouds can be used to shed light on an individual's thoughts about a topic. The teachers were instructed to use Wordle™ to create a word cloud that represented their understandings of slope. They were informed that they would be graded on their thorough completion of the task (rather than being graded on stylistic elements like color or font). Since the teachers were instructed that the word cloud should reflect their own view of slope, this task provided insight into the teachers' personal concept image of slope.

Data Analysis

During the first stage of data analysis, both research team members independently coded all responses for both sets of data using Moore-Russo, Conner, & Rugg (2011) 11 conceptualizations of slope as outlined in Table 1. Both researchers had worked previously with the conceptualizations of slope and the coding of data using these conceptualizations.

Each teacher's product (i.e., each concept map and word cloud) was coded as either providing or not providing evidence of each of the 11 conceptualizations. The 19 teachers, 11 conceptualizations of slope, and two products from the two tasks (concept map and word cloud) provided 418 units of analysis. For each teacher, conceptualization, and task combination, evidence of the conceptualization was coded as present or absent. The Krippendorff's alpha (a measure of inter-rater reliability) for all 418 units of analysis was 0.99. In the rare instances when there was a disparity, the two research team members discussed coding until consensus was reached. The research team generated descriptive statistics to compare the teachers' conceptualization use on both tasks.

During the second stage of data analysis, the research team revisited each of the teachers' concept maps and word clouds a minimum of 10 times. Employing an inductive process of constant comparisons (Glaser & Strauss, 1967) of data from the concept maps as well as from the word clouds, the research team then reconsidered the data to determine how teachers think about and present slope. In the final stage of data analysis, the research team made additional passes through the data to determine if other observations not taken into consideration by the conceptualization coding were present.

Table 2.
Number of Teachers (n = 19) Who Demonstrated Each Conceptualization

	Number of Teachers Demonstrating the Conceptualization…			
Conceptualization	On Concept Maps	On Word Clouds	On Both	On Either
Behavior Indicator (B)	16	19	16	19
Geometric Ratio (G)	16	18	16	18
Physical Property (P)	15	18	14	19
Algebraic Ratio (A)	17	15	15	17
Parametric Coefficient (PC)	15	11	9	17
Real World Situation (R)	14	7	7	14
Functional Property (F)	6	12	6	12
Determining Property (D)	4	15	4	15
Linear Constant (L)	4	12	3	13
Calculus Conception (C)	2	4	0	6
Trigonometric Conception (T)	0	4	0	4

Result

The research team looked at the concept map data, the word cloud data, and the combined data. When combining the concept map and word cloud data, teachers demonstrated between 4 and 10 conceptualizations of slope with a mean of 8.1 conceptualizations per teacher. The number of teachers using each of the 11 conceptualizations is summarized in Table 2.

Results from Concept Map Data

A sample concept map is provided in Figure 1. This particular concept map was coded as highlighting *real world situation, parametric coefficient,* and *algebraic ratio* conceptualizations of slope. All teachers demonstrated

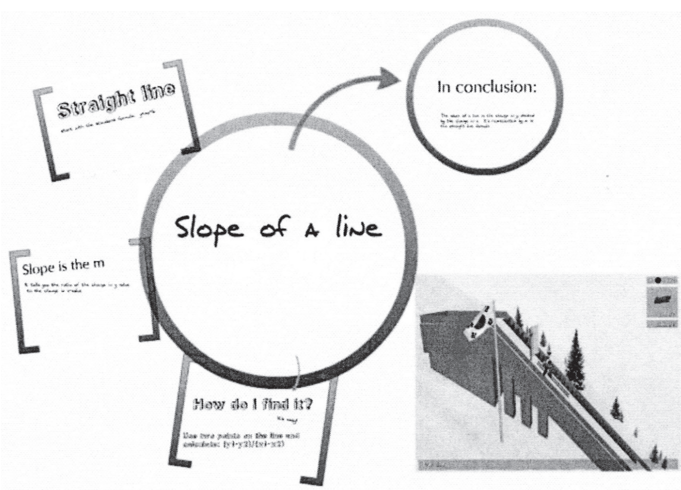

Figure 1.
Example of one teacher's concept map.

between three and nine conceptualizations of slope in their concept maps, with a mean of 65.7 conceptualizations per teacher. *Algebraic ratio* was used by the greatest number of teachers (17), followed closely by *geometric ratio* and *behavior indicator* (16 each). The *trigonometric conception* was not used by any of the teachers, while only two teachers used a *calculus conception.*

In total, 15 of the 19 teachers used *m* to denote slope in their concept maps. However, the teachers frequently used the notation in an example or equation without first presenting that this particular letter would be used to represent slope. In fact, more than half of the teachers (n = 8) who used *m* in their presentations did so before indicating that *m* stood for slope.

The concept maps also allowed teachers to use a variety of representations. The representations used included words, pictures of real world situations, graphs, videos, and equations. The pictures included a variety of situations intended to model slope, including mountains, snow skiing, roller coasters, roofs, landscapes, stairs, and road signs. However, while some of these illustrations provided clarification and elaboration on the idea of slope, others seemed to be incorporated more for visual impact than for clarity. For example, one teacher included a drainage pipe, but the photograph only showed the end of the tube where waste water was falling out, not a side view that showed the slope of the pipe. Many of the photographs included situations that were not linear. For example the view of the roller coaster displayed its curved rails both increasing and decreasing. No additional markings were given for clarifications as to which sections of the roller coaster were locally linear or if it was the teacher's intention that students should look at the average rate of change. Another example showed a staircase whose sections were either vertical or horizontal. The staircase example also failed to show a superimposed line on the photograph to clarify why the photograph represented slope, indicating there was no mathematization of the relationship between height and length.

Most of the teachers also incorporated examples and equations into their presentations. Standard form ($ax + by = c$), slope-intercept form ($y = mx + b$), and point-slope form ($y - y_1 = m(x - x_1)$) representations of linear equations appeared in the concept maps. Slope-intercept form occurred most often, in 11 teachers' presentations. Not only were standard form and point-slope form used by fewer teachers (n = 3 for each), but the teachers who used either of these representations always presented the slope-intercept form as well. No teacher incorporated all three equations into the presentation.

Most teachers also included numeric or graphic examples in their concept maps. Fifteen teachers provided at last one example highlighting *behavior indicator, geometric ratio, real world situation, algebraic ratio, linear constant,* or *determining property* conceptualizations. Examples were also used to represent linear equations in both point-slope and slope-intercept forms.

The most common examples provided the equation or graph of a linear equation with positive slope in contrast to another with negative slope to demonstrate increasing and decreasing behavior or provided two points and determined the slope between them.

Results from Word Cloud Data

The teachers demonstrated between three and ten conceptualizations of slope in their word clouds, with a mean of 7.1 conceptualizations per teacher. The *behavior indicator* conceptualization was evidenced in all 19 teachers' word clouds, followed closely by *geometric ratio* and *physical property* (18 each). "Rise" and "run" were the most common words used in the word clouds appearing in some form in 18 of the teachers' word clouds. The word "steepness" appeared in fifteen of the word clouds. *Trigonometric ratio* and *calculus conception* were used least often, with only four teachers using each. The number of teachers using each of the 11 conceptions on the word cloud task is provided in Table 2.

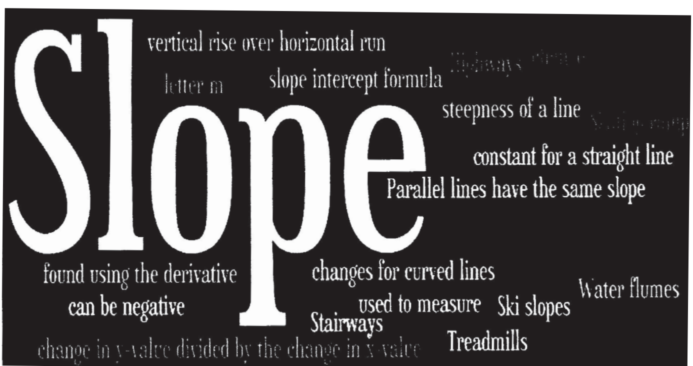

Figure 2.
Example of one teacher's word cloud.

Table 3.
Coding Assigned to Word Cloud in Figure 2.

Conceptualization Coded	Phrase in Word Cloud
Behavior Indicator (B)	can be positive, can be negative
Geometric Ratio (G)	vertical rise over horizontal run
Physical Property (P)	steepness of a line
Algebraic Ratio (A)	change in y-value divided by the change in x-value
Parametric Coefficient (PC)	letter m
Real World Situation (R)	Highways; Skating ramps; Water flumes; Ski slopes; Stairways; Treadmills
Determining Property (D)	Parallel lines have the same slope
Linear Constant (L)	constant for a straight line
Calculus Conception (C)	straight line tangent to a curve; found using the derivative
None Assigned	change; slope-intercept formula; changes for curved lines

The same teacher who submitted the concept map shown in Figure 1 submitted the word cloud shown in Figure 2. Table 3 shows how the phrases used in the word cloud displayed in Figure 2 were coded. Phrases assigned the same code are grouped together. Notice that this participant used six additional conceptualizations (*behavior indicator, geometric ratio, physical property, determining property, linear constant, calculus conception*) on the word cloud compared to the three used in the concept map (*real world situation, parametric coefficient,* and *algebraic ratio*).

Comparison of Results from Concept Map and Word Cloud Data

The next phase of analysis involved comparing results from the concept maps and word clouds. Teachers used a mean of 1.4 more conceptualizations on the word clouds (mean = 7.1) compared to the concept maps (mean = 5.7). Although the teachers tended to use more conceptualizations on the word clouds, this trend was not evenly distributed across the conceptualizations.

While some conceptualizations were much more prevalent on the word clouds, others actually appeared more often on the concept maps. The summary of use of each conceptualization on the two tasks is provided in Table 4. This table indicates the number of teachers using the conceptualizations on both tasks, the number of teachers who used the conceptualization on the concept map only or on the word cloud only, the total number of teachers who used the conceptualization on one but not both tasks (labeled as inconsistencies), and the "net difference" between use on the concept maps and use on the word clouds. This "net difference" was attained by subtracting the number of teachers who used the conceptualization on the concept map only. Net differences with large absolute values indicate one task prompted much more use of the conceptualization than the other. Negative values indicate that the word cloud prompted more use while positive values indicate that the concept map prompted more use.

Table 4.
Number of Teachers Using Each Conceptualization by Task (*n* = 19 teachers)

Evidence of Use	Conceptualizations										
	R	PC	A	G	C	B	P	T	F	L	D
On Concept Map	14	15	17	16	2	16	15	0	6	4	4
On Word Cloud	7	11	15	18	4	19	18	4	12	12	15
On Concept Map Only	7	6	2	0	2	0	1	0	0	1	0
On Word Cloud Only	0	2	0	2	4	3	4	4	6	9	11
Number of Inconsistencies in Use	7	8	2	2	6	3	5	4	6	10	11
Net Difference (Concept Map Use – Word Cloud Use)	7	4	2	-2	-2	-3	-3	-4	-6	-8	-11

From Table 4, we notice that *real world situation, parametric coefficient,* and *algebraic ratio* were evident on more teachers' concept maps than their word clouds, indicating increased instructional attention. Each of the other eight conceptualizations was more likely to occur on the teachers' word clouds. Among these, *determining property* had the largest discrepancy, with 11 teachers using the conceptualization on the word cloud but not on the concept map presentation. The *linear constant* and *functional property* conceptualizations were also used by many teachers on the word cloud only.

Table 5.
Number of Teachers With Discrepant Conceptualizations

	Number of Conceptualizations Possible (n = 11)			
Number of Teachers (n = 19)	Conceptualization Evidenced on Concept Map only (mean = 0.95)	Conceptualization Evidenced on Word Cloud only (mean = 2.26)	Number of Inconsistencies of Use (mean = 3.21)	Net Difference between Number of Conceptualizations on Concept Map and Word Cloud (mean = -0.79)
1	0	1	1	-1
3	0	2	2	-2
1	0	4	4	-4
1	0	5	5	-5
2	1	0	1	1
1	1	1	2	0
2	1	2	3	-1
3	1	3	4	-2
2	2	2	4	0
3	2	3	5	-1

It is also useful to see how conceptualization use varied by task for each teacher. These results are presented in Table 5. The table indicates that teachers used zero, one, or two conceptualizations on the concept map only. Meanwhile, teachers used between zero and five conceptualizations on the word cloud only, with a mean of 2.26. These results indicate that overall teachers showed a fairly large inconsistency between the two presentations. An average of 3.21 conceptualizations were used on one but not both tasks. The negative average net difference indicates that, in general, teachers were more likely to use a conceptualization on the word cloud versus the concept map.

Discussion

The results described above provide insight into how teachers think about and present slope. In particular, the directions for the concept map specified that it was for use with secondary students while the word cloud represented a collection of personal ideas related to slope. Both combine to provide at least a partial description of the teachers' concept images for slope, and both give suggestions as to what teachers emphasize related to slope. Comparing the conceptualizations that teachers evidenced on each of these products highlights the relationship between how teachers think about slope and how they present the concept. Connections to previous research and the practical implications of the findings are discussed below.

Teacher's Concept Images and Instructional Focus

The first research question considered which and how many conceptualizations of slope the teachers evidenced. The word cloud data (representing a window into the teachers' personal concept images of slope) suggests that secondary teachers hold a variety of conceptualizations. *Behavior indicator* was used by all 19 teachers, followed closely by *geometric ratio* and *physical property* (n = 18). The findings for *geometric ratio* support earlier findings by Stump (1999) that *geometric ratio* was a dominant conceptualization among secondary teachers (*behavior indicator* had not yet been identified and was not used in Stump's study).

The second research question considered the relationship between the teachers' understanding of slope and the content emphasized, or at least included, in their instructional materials prepared for students. The concept map data suggests that the intended instructional emphasis does not always reflect the teachers' own concept image. When creating concept maps intended for secondary students, teachers were more likely to emphasize *algebraic ratio, parametic coefficient,* and *real world situations* compared with the use of these conceptualizations in their own concept images. On the other hand, teachers were much less likely to include *determining property, linear constant,* and *functional property* in developing a presentation for students.

NCTM's (2006) Curriculum Focal Points describe conceptualizing slope as a *functional property* as a goal of eighth grade instruction, recommending that students interpret "...slope as a constant rate of change, so if the input, or x-coordinate, changes by a specific amount, $a,$ the output, or y-coordinate, changes by the amount ma" (p. 20). Recommendations related to the *determining property* and *linear constant* conceptualizations of slope are found in descriptions related to system of equations with parallel and perpendicular lines and using the similarity of "slope triangles" to understand the constant slope of a line, respectively (NCTM, 2006).

Not only does teachers' omission of these ideas in planning instructional activities fail to align with NCTM's recommendations for instruction, it appears teachers omitted the more conceptual notions of slope that were part of their own concept images. Their instructional materials often emphasized procedural knowledge of slope by emphasizing conceptualizations like *algebraic ratio* ("change in y over change in x") and *parametric coefficient* (m in $y = mx + b$) to calculate or identify slope without interpreting its meaning. Although the emphasis of *real world situations* in instruction may indicate an attempt by teachers to build students' conceptual understanding for the use of slope in applied settings, recall that many of the uses of these real world situations in the concept maps involved pictures or videos without mathematical interpretations. This emphasis may also be a result of the product itself since the Prezi concept map allowed the use of pictures and video, features that are not possible when creating a word cloud. Thus, the apparent increased emphasis of *real world situations* in instructional materials should be interpreted cautiously.

Two additional conclusions may be drawn from the study's results regarding the second research question. First, the use of *behavior indicator* by every secondary teacher supports the findings from other research that this conceptualization was prominent among college precalculus students despite its infrequent use by college instructors Nagle, Moore-Russo, Viglietti, & Martin (2013). Together, these results suggest that students' emphasis on slope as a *behavior indicator* might stem from the focus of secondary instruction. Second, the contrast between secondary teachers' and college instructors' use of this conceptualization supports a link between their dominant conceptualizations and the content they teach. Rather than citing the relationship between slope and the behavior of a linear function as emphasized in the secondary curriculum (NGA Center for Best Practices & CCSO, 2010), the college instructors were more likely to reference the relationship between slope and the increasing or decreasing nature of any differentiable function as given by the first derivative test in calculus Nagle, Moore-Russo, Viglietti, & Martin (2013).

Implications and Limitations

The results of this study suggest a fairly complicated role between a teacher's concept image and the content of their lessons on slope. Along with previous findings Nagle, Moore-Russo, Viglietti, & Martin (2013), this study suggests that the conceptualizations of slope held by teachers may be linked to the content of the courses they teach. However, the findings also indicated that the emphasis of instructional materials may not always align with a teacher's understanding of slope. In particular, while teachers may conceptualize slope in terms of a *determining property* and *linear constant*, they are not likely to present these conceptualizations to students. Furthermore, it appears that teachers emphasize an image of slope that is more procedural and less conceptual than their own image of the concept.

While this study investigates the relationship between teachers' concept images and the emphasis of their instructional materials, it must be cautioned that actual instruction may still emphasize different conceptualizations of slope. In particular, Stump (2001a) previously found that preservice teachers prepared lessons that promoted *real world situations* but emphasized graphs and equations in the actual instruction of the lesson. The examination of teachers' instructional materials in this study provides insight into teachers' intended instructional focus, but does not indicate the focus of enacted instruction. Future research should investigate the three-way relationship between teachers' concept images, the focus of instructional materials, and the emphasis of the actual, enacted lesson.

The findings are also limited to a small group of teachers in a narrow geographic region. Cultural differences may occur, and thus the results cannot be generalized to other teachers in different geographical locations.

Conclusions

Slope is a critical topic based on its prominence throughout the secondary curriculum, the multitude of ways it can be conceptualized, and the role it plays in the development of more advanced mathematical ideas. This study has described the complicated interaction between teachers' understanding of slope and the emphasis of instructional materials they prepare on slope, linking teachers' concept image to the content they teach and recognizing that only a subset of that concept image is emphasized in instruction. The result suggest that teachers' content knowledge interacts with their mathematical knowledge for teaching to determine the focus of instructional materials. The findings support the notion that strong content knowledge is necessary, but not sufficient, for effective teaching; teachers also need training in choosing the focus of instructional materials.

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