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Responsive Teaching from the Inside Out: Teaching Base Ten to Young Children[1]

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Abstract

Decision making during instruction that is responsive to children's mathematical thinking is examined reflexively by the researcher in the context of teaching second graders. Focus is on exploring how the research base on learning informs teaching decisions that are oriented to building on children's sound conceptions. The development of four children's understanding of base ten over a ten-week period is tracked.

"The work of teaching orients teachers to constantly consider their next moves" (Jacobs, Lamb, Philipp, & Schapelle, 2011, p. 98).

Introduction

A recent agenda-setting document called for more attention to how research on learning can be used by teachers. Its authors argued that there is no set of materials or technology that "can replace careful attention and timely interventions by a well-trained teacher who understands how children learn mathematics" (Daro, Mosher, & Corcoran, 2011, p. 15). Such teachers would

> .... get students to reveal where they are in terms of what they understand and what their problems might be. They have to have specific ideas about how students might progress ... and how they might be expected to go off track or have problems. And they would need to have, or develop, ideas about what to do to respond helpfully to the particular evidence of progress and problems they observe. (Daro, Mosher, & Cocoran, 2011, p. 15)

[1] Acknowledgement: Sincere thanks to Vicki Jacobs for feedback on this article as I was preparing it, Luz Maldonado for enriching conversations about children's thinking, and the four childrn with whom I worked and their teachers for inviting me into their school and agreeing to work with me.

In a nutshell, *responsive teaching* entails taking into account the evidence provided during instruction about children's thinking and its advancement. To teach in ways that are responsive to children's mathematical thinking, teachers need to elicit children's thinking, interpret this thinking, and then "respond helpfully." They need to understand the mathematics children are to learn and know what progress in learning the mathematics look like. And they need to make decisions, often quickly in response to children's thinking during instruction.

In spite of its intuitive appeal, responsive teaching is not widespread or well understood (e.g., Hiebert, Gallimore, Garnier, et al., 2003; Kennedy, 2005). To appreciate the nature of this work, I decided to explore responsive teaching from a first-person perspective by immersing myself in the work of teaching a small group of children over an extended period. My goal was to document what was involved in making teaching decisions that were responsive to children's mathematical thinking during instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fraivillig, Murphy, & Fuson, 1999; Jacobs & Ambrose, 2008; Jacobs, Lamb, Philipp, & Schapelle, 2011; Philipp, Jacobs, Lamb, Bishop, Siegfrid, & Schappelle, 2012). How would I respond "to the particular evidence of progress and problems" I observed? In what sense would my decisions be research-based and in what sense not? What would be involved in using the research on children's mathematical thinking as I was interacting with children?

In this paper, I present a first-person narrative of my work with a group of children over a ten-week period, in which I foreground the challenges I encountered and the kinds of decisions I made to address these challenges. The children with whom I worked were four lively second-graders named Sunny, Daniella, Jack, and Emilio. These children were characterized by their teachers as lacking number sense and, generally, as having difficulties in mathematics. Out of all of the second graders in their school, they had scored the lowest on a benchmark mathematics test. My work with them centered on base-ten concepts and problem solving. We did not start out with this focus; after a short time of working with the children, I decided that their lack of number sense could be more precisely characterized as a lack of usable understanding of base ten. In my decision making I strived to balance sensitivity to the children's current understanding, as I interpreted it, with movement toward the "mathematical horizon" (Ball, 1993), which required balancing multiple, sometimes contradictory, goals. I wanted to document what was involved in drawing upon research into children's mathematical thinking to guide my decision making while also honoring these four children's emerging mathematical understanding.

This report can be read on two levels. On one level, it is an account of four children in the midst of advancing their understanding of base ten, whole-number operations, and problem solving and the teacher's research-based decision making in service of that advancement. It would be useful for anyone wondering how to support children to learn base ten, and could provide advanced insight into why some children in later grades struggle with more advance operations, such as multi-digit multiplication or subtraction with regrouping. On another level, it is a case study of responsive teaching, from the perspective of a teacher's decision making. As such, it does not make claims about what responsive teaching should look like for other teachers with other children. It provides an account of this approach to instruction in a particular domain - base ten and whole-number operations - involving four second graders and considers, more generally, how teacher decision making might draw on research-based findings about children's thinking in the midst of interactions.

The report is organized chronologically and reflects the threads of understanding to which I was attending in my work with the children. Children's mathematical thinking and its details is foregrounded, to provide an opportunity for teachers to think about how they might respond to similar evidence of children's thinking in their own classrooms. Not all the details of what we did are included. Rather, I include the details that stood out to me as informative during my interactions with the children and which became pertinent to my decision making.

Method

The four children with whom I worked were chosen by teachers on the basis of a mid-year paper-and-pencil mathematics assessment given to all second graders at the school. They had the four lowest scores out of all second graders at the school and their teachers believed they needed additional help in mathematics. Before meeting with the four children, I met with their teachers to learn about the children, although most of what I learned came from my interactions with them during our sessions together.

We met once per week for nine weeks, after school for one hour. During these sessions, I took field notes on the children's thinking and our interactions. After each session, I spent an additional hour alone elaborating the notes and supplementing them with additional information about what happened during our time together. My focus in these research notes was to document my own decision making and the evidence on which it was based (Lampert, 1998).

Teaching Sessions and Findings

Session 1: We Begin Our Work Together

I did not know Sunny, Daniella, Jack, and Emilio before my work with them. To prepare for our first session together, I wrote a set of word problems that involved addition and subtraction and represented four distinct problem structures (Carpenter, et al., 1999, 2014), including a mix of single-digit and double-digit quantities. I wanted to spend the first session finding out about what the children could do, what they understood, how they expressed it — both orally and in writing — and how confident they were. Not knowing what to expect, I started with fairly small numbers, but made sure to include some double-digit numbers

Table 1: Problems given during first instructional session to find out what the children understood.

Problem Type	*Problem*
Join Result Unknown	Maya has 13 jellybeans. Her brother gives her 8 more jellybeans. How many jellybeans does Maya have now?
Separate Result Unknown	Jason has 28 pennies. He loses 13 of them. How many pennies does Jason have now?
Base Ten, Multiplication	You've got 3 big bags of soccer balls. Each bag has 10 balls in it. You've also got 2 loose balls. How many balls do you have?

The first problem that I posed to the children was a Join Result Unknown (Table 1). The children either counted up from 13 to solve it or direct modeled it with cubes or pictures. Nobody used a more advanced strategy such as derived or recalled facts. (Direct modeling strategies involve the physical representation of each item in a story problem, such as with cubes or marks on paper, and the manipulation of these representations in a way that follows the structure of the story. Number-fact strategies represent an advance over direct modeling. A derived fact strategy for this problem would be something like, "I know that 13 plus 7 is 20, plus 1 more is 21." [Carpenter et al., 2014]).

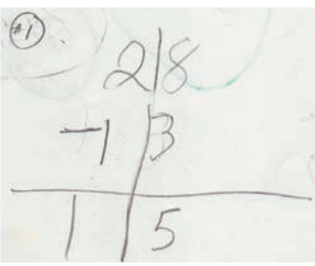


Figure 1: Daniella's written strategy for 28-31.

The second problem we worked on was a Separate Result Unknown (Table 1). Daniella solved this problem by writing 28 - 13, vertically (Figure 1).

She separated the tens and ones into two columns, and subtracted the ones first, then the tens. Her use of this procedure made me curious about what she understood about base-ten concepts, as opposed to the procedure she had used. I began to listen for evidence of base-ten understanding among all the children and noticed that, even though the problem involved double-digit numbers, none of the children had used base-ten concepts in their strategies. I was listening in particular for children's use of *ten as a countable unit* - a shorthand way of describing children's understanding that 10 ones can be represented as one 10 (Carpenter, et al, 1999, 2014; Cobb & Wheatley, 1988; Fuson, Wearne, Hiebert, et al., 1997). This concept is distinct from the place-value concept in which the value of a digit is determined by its placement in a number - for example, in the number 28, the 2 represents 2 tens. However, understanding the meaning of 2 tens depends on understanding 10 as a unit and thinking flexibly about 2 tens as 20 ones or 1 ten as 10 ones.

I decided to create a third problem, on the spot, to explore children's understanding of base-ten. After ascertaining that they all knew about and liked soccer, I posed a problem that involved groupings of 10: "You've got 3 big bags of soccer balls. Each bag has 10 balls in it. You've also got 2 loose balls. How many balls do you have?" I used hand gestures to indicate the bags were big and repeated the problem to be sure the children heard it. They set to work. Everyone but Emilio was direct modeling the problem by drawing all the balls individually (Figure 2).

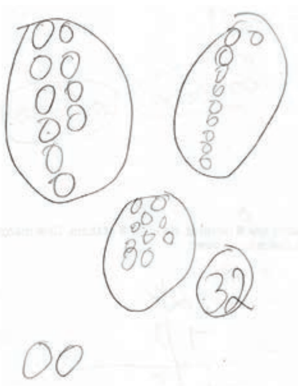


Figure 2: Drawing all the individual balls

There was no use of ten as a unit! Emilio did not appear to be doing anything, so I repeated the problem for him. "Oh," he said. "10 plus 10 is 20." I was so pleased with this insight that I emphasized to him - and for the benefit of the others - that he did not even need to draw any pictures to figure it out. It prompted Jack to remember that he, too, knew that 10 plus 10 was 20.

However neither boy was able to use his knowledge of this fact to solve the problem. Sunny was not sure whether to add or subtract the two loose balls, so I told her she had to decide for herself what made sense; she decided to subtract because, she said, the two "loose" balls could roll away. Emilio got 30 for his answer and when I asked him how he was going to count the 2 loose balls, he changed his answer to 31. And Jack got 28, because one of his bags had the wrong number of balls in it.

There was so much to talk about, but it was time to go, so I made a note to myself to return to problems like this one next time.

Session 2: I Discover a "Lack of Number Sense" is Really a Lack of Base-Ten Understanding

I had a hunch that what seemed like a lack of number sense for some of the children was actually little-to-no understanding of base ten. I decided that I wanted to find out more about what the children understood about groupings of 10 and 10 as a unit. I wrote four problems for this session that were intended to both assess and develop base-ten understanding (Table 2).

Children who understand 10 as a unit find a problem in which they have to calculate four groups of 10 easy (e.g., 4 rolls of 10 candies each). They might count by tens to solve the problem or immediately realize that 4 tens is 40. Those who do not understand 10 as a unit would find such a problem just as difficult as any other grouping problem. For example, they would solve 4 groups of 7 in the same way as they would solve 4 groups of 10.

Emilio's thinking about the Valentine's Day problem showed no base-ten understanding. First, he interpreted the context to mean he should add 10 and 4 to get 14 rolls altogether. After questioning him, unsuccessfully, about why he added, I described a context where he was the candy maker and had to put 10 candies into each of 4 boxes. The librarian handed us a roll of

Table 2: Problems given during the second session to assess and develop children's understanding of ten as a unit.

Problem Type	*Problem*
Equal Groups Multiplication (10 in a group)	For Valentine's Day, Emilio got 4 rolls of candy. Each roll had 10 candies in it. How many candies did Emilio get altogether?
Join Result Unknown (Change quantity is a multiple of 10)	Sunny has 16 pieces of chocolate. Just to be nice, her friend gives her 20 more pieces of chocolate. How many pieces of candy does Sunny have now?
Join Change Unknown (Start quantity is a multiple of 10)	Jack has 30 dollars. How many more dollars does he need to have 45 dollars to buy a new bike?
Equal Groups Measurement Division (10 in a group)	Daniella has 55 tropical fish. She wants to put 10 fish in each bowl. How many bowls does she need for all of the fish?

sweet tarts to help Emilio visualize the *10-candies-per-1-roll* relationship. I then left him to solve the problem; he solved it by drawing each box and putting 10 single cubes in the interior of each. He counted his answer by twos and got 40. Success, of a limited sort then: he used his knowledge of the context to construct a solution. But he did not use any of the knowledge of tens that was in evidence last week.

The other children performed similarly. I felt reinforced in my hunch that much of their current difficulties in mathematics could be traced to under-developed knowledge of ten as a unit and decided to focus on developing this understanding over the next few weeks. At the same time, I would be working on helping the children increase their sense of agency in problem solving and their capacity to "make sense of problems and persevere in solving them" (CCSSM, 2010, p. 6).

Session 3: I Accidentally Make the Problems Too Difficult

In my excitement about discovering a possible source of children's difficulties in second-grade mathematics, I wrote a set of problems (Table 3) to address these difficulties that, in hindsight, were too difficult. My aim was to further assess and begin to develop children's understanding of ten as a unit, so I wrote a set of problems that all involved multiples of ten.

Table 3: Problems given during the third session to further assess and begin to develop children's understanding of ten as a unit.

Problem Type	*Problem*
Separate Result Unknown (Start quantity is a multiple of 10)	Daniella has 40 chocolate chips. She ate 15 of them. How many chocolate chips does Daniella have left?
Join Change Unknown (Result quantity is a multiple of 10)	Sunny has 22 pennies. How many more pennies does she need to have 50 pennies to buy a book?
Separate Result Unknown (Start quantity is a multiple of 10)	Jack has 60 cards in his collection. He gave away 26 cards. How many cards does he have left?

Both Sunny and Daniella had some trouble with the pennies problem. Each used unifix cubes in sticks of 10 to build 22 (Figure 3), which suggested to

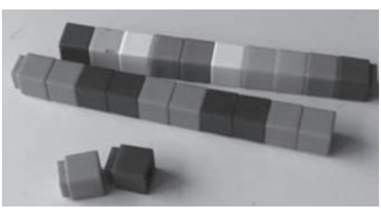


Figure 3: Daniella's model of 22

me that they have some understanding of using 10 as a unit to model 22. But after this auspicious start, they were stumped about how to build on from 22 to get to 50. When I changed the 50 pennies in the problem to 30 pennies, Daniella quickly solved it by counting up from 22 to 30 by ones.

Jack solved the pennies problem handily, although his strategy made no sense of tens. He counted up by ones from 22 using tallies to keep track.

I noticed several of the children using vertical notation for double-digit problems but not making use of tens in their solutions. For instance, Jack wrote a vertical number sentence for the chocolate chips problem (40-15), but actually solved it by drawing the 40 individual "chips" and crossing out 15 of them (Figure 4).

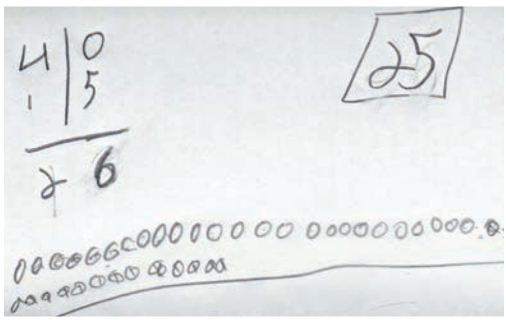


Figure 4: Jack's vertical number sentence for 40-15.

Daniella wrote something similar for the cards problem (60-26) and got 4 for her answer - a classic "bug" which suggests she does not understand, or is at least not making use of, base-ten concepts.

I decided for next time that I needed to choose numbers that were more conducive to using 10 as a unit. For example, having a start number of 20 for the pennies problem, instead of 22, would make it easier for the children to build up from the start number using unifix cubes in stacks of 10. I also decided to continue to provide the children with materials that are structured in tens (e.g., unifix cubes in sticks of tens or base-ten blocks) and to urge them to use these materials to solve problems. We would reflect on their strategies and record them using numbers to help the children connect the base-ten structure of the materials with their number symbols. I expected that developing an understanding of base-ten concept in a way that these concepts are usable in their strategies would take some time (Hiebert & Wearne, 1996).

Session 4: I Find Evidence of Emergent Understanding of Base Ten

For the next session, I wrote a set of problems without a story context to find out if children could connect numbers and context (Table 4). It seemed from previous sessions that the children had a basic understanding of addition and subtraction. Because they have been using strategies based on ones but not tens I used multiples of ten in these problems. Before the children started working on these problems, I made sure each child had 70 unifix cubes in groups of ten in front of him or her, in stacks of ten.

Table 4: Number sentences for Session 4 Problems

30 – 12 = ___
20 + ___ = 45
40 + 21 = ___

I began by holding up 3 sticks of 10 unifix cubes (Figure 5), and asking the children how many I had. All but Emilio, who was sharpening his pencil, said 30. Then I held up 52, in 5 tens and 2 ones. It was a little harder for them to see, but basically they understood the tens and ones combination. (Sunny saw 42, Daniella saw 51, and Jack saw 52.) Yet their understanding of the base-ten structure of double-digit numbers seemed fragile, because they used very little of that understanding to solve the multidigit addition and subtraction I posed to them that day.

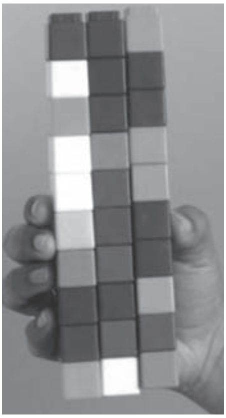


Figure 5: Three groups of 10 cubes each.

I asked the children to begin by providing a story for the first number sentence, 30 -12 = ___. The story they came up with went like this: "Jack and Daniella went to the candy store and bought 30 pieces of Valentine's gum. Jack ate 1 piece and Daniella ate 11 pieces. How many pieces did they have left?"

The children used a variety of incorrect and correct strategies to solve the problem. Emilio solved it by counting back by ones, with no miscount this time. Jack solved it, as usual, by direct modeling by ones — he made 30 tally marks, and crossed out 12 of them. Both boys got 18.

Daniella solved it using the same common buggy algorithm that she had used the week before; her original answer, erased, was 22 (Figure 6).

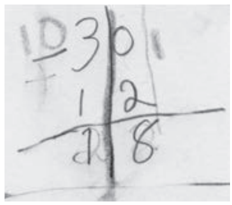


Figure 6: Daniella's buggy algorithm, with original answer (22) erased and replaced with the correct answer (18)

When she heard that Emilio and Jack had gotten 18, she erased her "22" and wrote "18."

Despite the story frame that she helped create, Sunny first added 30 and 12. But when I reminded her of the story she easily figured she would subtract.

Reflecting on each other's strategies. I decided to have the children listen to each other's strategies as a way to move their thinking forward. Although they can count by tens and can identify group of tens, they do not readily use this knowledge to solve problems; it is not very flexible knowledge for them. So my goal was to use the group discussion to help them begin to make connections and develop this base-ten understanding.

I had a big piece of newsprint that we could all easily see. I asked Jack to share his strategy first, because it was basic direct modeling. I represented his strategy using tallies. Sunny had the idea of grouping the tallies into tens to make them easier to count. I grouped the tallies and everyone said it was 30 (Figure 7). This was consistent with the very first quick activity we had done with the unifix cubes and it seemed to be a good way for the children to develop an understanding of the ten-ones-is-one-ten relationship.

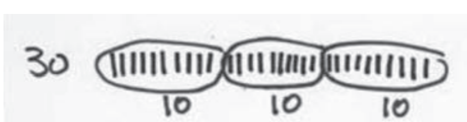


Figure 7: My representation of Jack's direct modeling strategy

But when it came to subtracting 12 from this group of 3 tens by subtracting 1 group of ten and then 2 ones, the difficulty of applying knowledge of tens became apparent. None of the children knew spontaneously what 30 take away 10 was. Daniella said 29 but it seemed to be a guess. I told them to use their unifix cubes to figure it out. They did, easily.

The ease with which the children solved this problem using manipulatives suggests a clear cognitive distinction between modeling with tens, as they had done, and working with ten as a unit mentally, which they did not do. This distinction is consistent with the research literature (Fuson, et al., 1997; Sophian, 2007; Steffe & Olive, 2010). I decided to continue working with the children to make connections like the ones they made this day between ones grouped into tens and ten as a unit. I believed that repeated opportunities to create and reflect on ten as a unit in their strategies would pay off.

Session 5: I Use Money as a Context and Some of the Children Have Difficulty

We continued to work on building a flexible understanding of ten as a unit in this session. I started by reading aloud a book called *Only One* (Harshman & Garrison, 1993), which emphasizes the mathematical big idea of thinking of several things as one thing. We discussed the big idea that one thing can be the same amount as many things, such as one dozen is 12 eggs and one dime is 10 cents. They seemed to understanding this idea in the abstract, and it gave us a point of reference when talking about tens and ones: "Oh, you mean a dime is the *same as* ten cents!" We then moved on to some addition and subtraction story problems to continue to work on developing ten as a unit (Table 5). As it turned out, the money context posed some special problems of its own.

Table 5: Problems for Session 5

Problem Type	*Problem*
Join Result Unknown (Multiples of 10)	Emilio had 5 dimes to spend. He bought a _____ that cost 20 cents. How much money did he have left?
Join Change Unknown (Multiples of 10)	Sunny had 2 dimes. She wants to buy a ____ that costs 40 cents. How much more money does she need?
Separate Result Unknown (Start quantity is a multiple of 10)	Dr. E has 70 cents. She spent 52 cents on a chocolate bar. How much money does she have left?

Emilio and Jack both solved problems in ways that showed me they are building an understanding of ten as a unit. I was especially happy to see Jack represent five dimes with 20 cents subtracted out by drawing circles for dimes (Figure 8). In my sessions with him before today, he has been representing tens with ten tally marks or something similar, so to use one circle to represent 10 things was a real advance! I was hoping that a dime would have for the children a "one-ness" and also a "ten-ness."

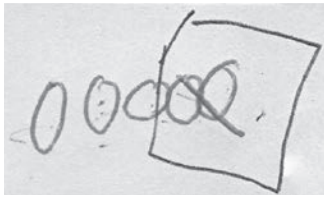


Figure 8: Jacks representation of 5 dimes take away 20 cents

However, both Daniella and Sunny struggled with these problems. In fact, Sunny did not solve a single one. She represented the 5 dimes in the first problem with 5 cubes. We talked about how much 20 cents was; she knew it was 2 dimes. But when I left her alone to work on the problem, she took all of the tens she had (unifix cubes) and broke them into ones to represent the stars in the sky that Emilio, in the story problem, had decided to buy. Daniella seemed confused about dimes and cents as different units, and how they related. She wrote "Emilio has 0 money now," because 5 of something take away 20 of something leaves you with, at most, 0 (Figure 9). Interestingly, she represented the dimes as units with the numeral "10" on it, suggesting she may know that one dime is 10 cents; but she did not seem able to use that knowledge to solve the problem.

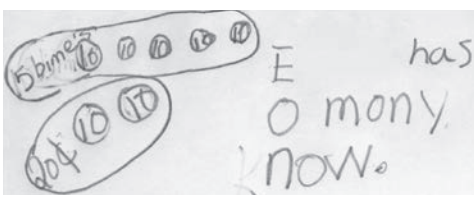


Figure 9: Daniella's written strategy for 5 dimes take away 20 cents

I think these problems were just right for Jack and Emilio, but too hard for Sunny and Daniella. For children whose knowledge of dimes and other coin denominations is not easily accessed as they reason, these problems must seem like multi-step problems with one of the steps left out ─ something like this: Henry has 4 packages. He eats 6 cookies. How much food does he have left? It does not make sense without the crucial information of how many cookies per package.

I wondered what we should do next. I knew that simply telling or showing these children that one ten is the same as ten ones would not be enough to help them learn to use this knowledge in problem solving. It is a difficult concept for young children, although the fact that it is a sophisticated mathematical idea is not obvious. I decided to continue to engage the children in problem solving with explicit discussion of how they were using tens and to write problems that involved units of 10 and units of one, to help the children learn to coordinate the two different units. I also planned to engage them in more comparisons of each other's strategies and the differences and similarities in how tens are used.

Table 6: Problems for Session 6

Problem Types	*Problems*
Measurement Division (Groups of 10)	Daniella has 42 beads. She wants to make necklaces with 10 beads on each necklace. How many necklaces can she make?
Join Result Unknown (Start quantity is a multiple of 10)	Jack has 30 cents. Emilio gives him 52 cents. How much money does Jack have now? What could he buy with this much money?
Multiplication, multistep (Groups of 10)	Sunny has 11 packages of cookies. Each package has 10 cookies in it. She also has 5 extra cookies. How many cookies does she have in all?

Session 6: Progress!

I wrote more problems involving multiples of tens, using bigger numbers in hopes that the children would find the use of individual tallies tedious. It worked, for some.

The third problem, which involved 11 packages of 10 cookies each and 5 extra cookies, proved interesting and productive. It was an appropriate challenge for most of the children. Emilio and Jack both started out by adding up the numbers to get 26 for their answer. Jack then decided on his own that 26 was not the correct answer, and easily direct modeled the problem by drawing groups of 10 ones (Figure 10).

As he was drawing all of this out, I seized an opportunity to extend his thinking. I asked him if, instead of drawing each cookie in every package, he could represent the cookies in the package by writing the numeral "10" in each one. He said that no, he could not; it would be too hard. So I let

Figure 10: Jack's drawn strategy for 11 groups of 10 cookies plus 5 extra cookies

him continue to model by ones. But I noticed when he counted the total he counted by tens instead of by ones, so again I asked him if he could represent the cookies by writing "10" in each package instead of drawing each individual cookie. I pointed out to him that the had just counted each group by tens. It seemed to make sense to him this time so I gave him a new but related problem to solve, encouraging him to use this new strategy. The problem was 14 packages of cookies, 10 in each package, and 10 extra cookies. He began, as usual, by representing each individual cookie (Figure 11).

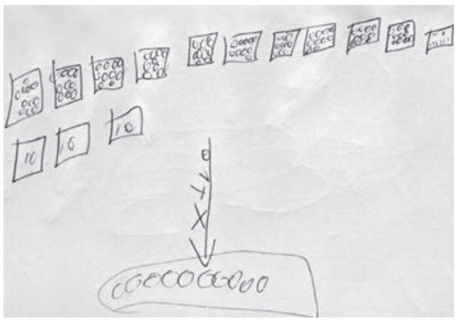


Figure 11: Jack's drawn strategy for 14 groups of 10

He said he forgot to use the strategy we had talked about (and only remembered when his hand started getting tired). Because he had so easily solved this problem, however, I felt sure he could use the more abstract counting approach. So again I posed a new but related problem: How many cookies would be in 12 packages? When I came back, I saw he had successfully represented each package of cookies with a single mark (the numeral 10), rather than drawing each individual cookie (Figure 12).

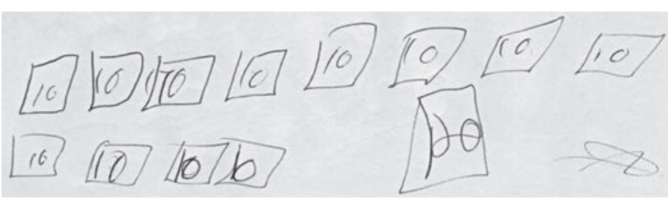


Figure 12: Jack's drawn strategy for 12 groups of 10

He agreed that this strategy was faster, as well as easier on the hand.

Emilio misinterprets the problem and I invite him to listen to Daniella's strategy to change his mind. Emilio had trouble getting started on this problem. It is not clear to me why. His initial answer was 26, which he got by adding 11, 10, and 5. I asked him why he decided to add them altogether and whether they were all cookies or packages, but he gave no clear answer.

Daniella, like Jack, direct modeled the entire situation by representing each cookie, but she confounded packages of cookies with single cookies (Figure 13) – just as she had done with dimes and pennies the week before. At first she counted her answer on by ones from the first package of 10, then she went back and counted the total of tens. (She got 160, which she later erased.)

Figure 13: Daniella's strategy for 11 groups of 10 cookies plus 5 extra (with the 5 extra cookies drawn as packages and then erased)

Because she had accurately represented the packages of cookies and Emilio had not, I decided to ignore her confusion about the 5 extra cookies for the time being and called Emilio over to compare what he was doing with what Daniella was doing. The first difference he saw was in how each of them had represented the package. His was more "realistic" (Figure 14). Daniella saw that he had six cookies in his packages and she had put 10 in each of hers. With some prompting from me to speak directly to Emilio and not me, she was further able to tell Emilio why she had drawn her packages this way. Emilio decided to start over, and at my suggestion, gathered a

Figure 14: Emilio's invalid strategy for 11 groups of 10 cookies plus 5 extra

bunch of sticks of unifix cubes in tens. He ended up with 34 sticks of 10 arrayed in front of him but did not solve the problem before it was time to go.

Sunny thinks really hard. Like Jack and Daniella, Sunny started out drawing the packages of cookies with each individual cookie represented. I encouraged her to use the cubes in sticks of ten instead, anticipating that the ten-to-one structure might support a more sophisticated strategy. She decided she wanted a bunch of sticks of four. I was not sure where she was going with it, but decided to let her create them, and even helped her. When we were done, she gathered sticks of 10 and used the sticks of four to stand for the packages! (The number of cubes in each stick of four was probably irrelevant; perhaps the long rectangular shape reminded her of a package.) Then she put 10 cookies ─ a stick of 10 ─ in each package (Figure 15) and counted the total beautifully by tens. During all of this she mentioned that she was thinking so hard she could not even think of anything else. I thought it was a keen observation because at the end I had to remind her of the 5 extra cookies. She included them but counted them as tens, as Daniella had at first.

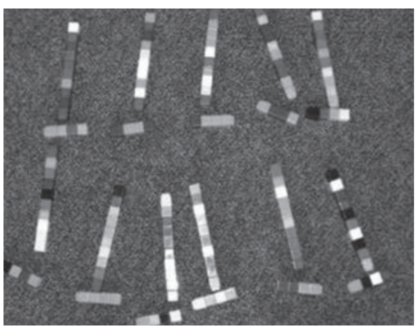


Figure 15: Sunny's strategy for 11 groups of 10 cookies

Looking forward. With encouragement, then, Jack, Daniella, and Sunny were able to move from counting by ones to counting by tens. Jack was able to represent groups of 10 by something other than a collection of 10 things ─ a real advance, if he sustains it. I was not sure what Emilio could do or how much he understood of problems like this one. He had solved them in the past. His focus on that day seemed divided so I think these problems did not get his full attention. In fact, he started out wanting to do his spelling homework.

To take advantage of the progress the children made, I decided that in our next session, we would solve more equal groups problems involving groups of 10, like the first and third problems and I would continue to push the children to represent sets of 10 with the numeral 10. Having counted 10 as a unit, I thought that the children would be more likely to use it to solve problems if the quantities were smaller.

Session 7: Frustration!

For this session, I used a story context, the Candy Factory, that involved units that were powers of ten. At the Candy Factory (McClain, Cobb, & Bowers, 1998) candies are made, which are packed into rolls of 10 candies each; rolls are packed into cartons of 10 rolls or 100 candies each (Figure 16).

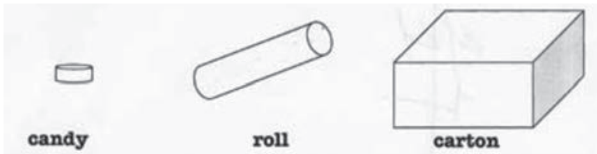


Figure 16: At the Candy Factory, the candy packing machines puts 10 candies in each roll and 10 rolls in each carton.

Table 7: Problems for Session 7, *in which a roll contains 10 candies*

Problem Type	*Problem*
Multiplication, multistep (groups of 10)	Mr. Diaz has 6 rolls of candy and 10 loose candies. How many candies does he have?
Measurement Division (groups of 10)	Principal Vegas has 110 candies. How many rolls of candy can she make?
Measurement Division, multi-step (groups of 10)	Ms. Little and Ms. Nelson are buying candy for their classes. They want each child to have only 1 candy each. If there are 38 children altogether in their classes, how many rolls of candy should they buy altogether?

I began by posing some quick problems just to check for understanding of the context. I asked how many candies were in 2 rolls, then in 5 rolls. Jack and Sunny both counted by tens to figure these problems out. Emilio too seemed to understand, although looking back, and knowing what he did later in the session, I am not sure now. (Daniella was absent.)

I limit their use of tallies. I began by reminding them how sometimes they solved problems by making single tallies (Figure 17) and told them that today, I did not want them to use tallies like these. They could use the unifix cubes in sticks of ten or use numbers written on their paper or solve it mentally. My purpose was to push them to work with ten as a unit. Al-

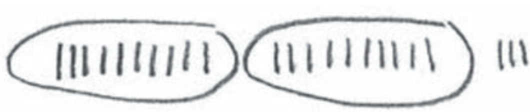
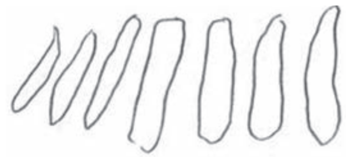


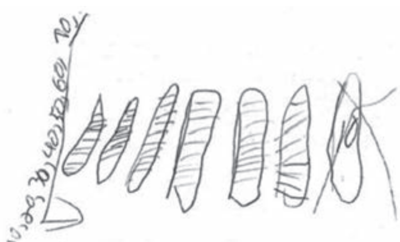
Figure 17: My illustration of using tallies to solve a problem

lowing unifix cubes meant that if they needed to count by ones they could; but the structure of groupings of ten would at least be something they had to choose to ignore.

Jack sustains the progress he made last week. As it turned out, Jack began the problem by drawing the rolls without candies (Figure 18a). He then decided to represent the individual candies in each roll (Figure 18b).



(a) *Jack draws rolls without candies at first*



(b) *Jack presents individual candies in each roll but the last*

Figure 18: Jack's drawn strategy for 6 groups of 10 candies plus 10 individual candies

Just as he was up to the last roll, I asked him if he needed to show those candies in order to count them. This conversation was just like the one we had had the week before. He quickly said "no" and wrote "10" in the last roll. When I asked him later to write a number sentence or write numbers to show how he solved it, he wrote: 10, 20, 30, 40, 50, 60, 70. Progress!

It's not clear what Sunny understands about ten as a unit. Sunny was slow getting started. She seemed to be confusing the idea of 6 rolls with rolls of 6. She easily modeled the 10 loose candies with 10 single cubes. But for the rolls she had a stick of 6 unifix cubes and described it as "a roll of 6." I clarified: "6 rolls of 10, not a roll of 6," and she was off, modeling the rolls with 6 sticks of 10. There was some confusion about how to count the total of 6 sticks of 10 and 10 loose ones; she got 16 at first, but with a discussion in which I asked her to connect it back to rolls and candies, she counted appropriately. I emphasized in my revoicing of what she had done that she could count the rolls ─ 1, 2, 3, 4, 5, 6 ─ or count the candies ─ 10, 20, 30, 40, 50, 60. (Plus the loose ones, which no one has any trouble counting.) Success!

I am frustrated with Emilio. Emilio solved the first problem by adding 6 rolls and 10 candies and got 16. I asked him to solve it a second way, and he drew a stick of 6 and a stick of 10, and counted all to get 16. I asked him to talk with Jack about his strategy, and listen to how Jack solved his, but he declined to do those things. I asked him if the problem was too hard for him, but he did not answer, working instead on figuring out what time it was and when he could go home. I asked Sunny to explain her terrific direct modeling strategy, hoping he would see the difference between 6 rolls (sticks of ten) and 10 loose candies (individual cubes). It seemed like he looked everywhere but at Sunny or her strategy. At each step of Sunny's explanation, I stopped her to ask Emilio a question, trying to get him to make a connection between the cubes arranged in sticks of tens and rolls of candy, trying to get him to make sense of the problem. I thought at first he was deliberately not engaging.

However, I decided I did not want to assume that he was avoiding work. Perhaps it was his way of expressing boredom or confusion; maybe he was preoccupied with a personal problem more important to him than counting candies in rolls. So finally, as it was nearing time for our session to be over, I asked him if he wanted me to make him an easier problem. He said he did. So I turned his paper over and wrote "2 rolls, 10 candies, how many candies?" "12," he quickly replied. So I asked him to use cubes to show the rolls and the loose candies. "How many candies in one roll?" I asked him. He put his head down and said he was ready to go home. Feeling resolute, however, I told him he could not leave until he solved this problem. I was remembering the fact that he had solved problems like this in the past. "How many candies in 2 rolls then?" I asked. "20?" he squeaked out, with his head buried under his arms. "So," I continued, "if you put 10 more candies with them─why don't you represent those 10 candies with these cubes, any way you want." He picked up a few sticks of 10 and began to put them with the 2 "rolls." "Now," I pressed him, "show me the 10 loose candies." It took a while but, finally, he pulled one stick of 10 out of what he had grabbed, and put it with the 2 rolls. "How many?" I asked. "30," he said, without even counting.

In hindsight, I do not know if I was engaged in a power struggle with Emilio or helping him make a cognitive leap. I wondered what would be the *residue* (Hiebert, et al., 1996) of this interaction for Emilio? What did he take away from it? A new understanding of ten as a unit? A feeling of confidence that he can solve problems? A feeling of being forced to do something he did not want to do? I think the answer to that question − which I simply do not know − is much more important than the fact that he answered "30" in the end.

Session 8: I work on Extending the Children's Thinking

We continued our work with the Candy Factory again and the use of number sentences to represent the situation.

Table 8: Problems for Session 8

Problem Type	*Problem*
Multiplication (groups of 10, with extras)	Dr. E. has 4 rolls of candy and 11 loose candies. How many candies does she have altogether?
Partitive Division	Dr. E wants Sunny, Daniella, Jack, and Emilio to share her [51] candies equally. How many candies can each child have?

Emilio does something different. As the children worked on the new Candy Factory problems, they fell into their usual patterns, with the exception of Emilio. I sat with him first to get him started. He read the problem to himself then I asked him rephrase it in his own words out loud. No problem; he remembered the quantities in the Candy Factory. I asked him how many candies in a roll; he said 10. So he understood the context and the problem parameters. "So," I asked him, "how many candies does Dr. E. have?" His first response was 40 because, he spontaneously gave the reason, there's candy in the four rolls. When I asked him about the 11 loose ones, he got 52, at first, because he added 10 on to the 40 (nice work!), and counted up somehow to get 52. When I asked him how he added the 10 on, he did not say, and ended up solving the problem by counting up from 40 by ones. What a terrific solution! In contrast to his thinking last week, he did not seem to have any problem distinguishing groups of 10 from singletons; and once he understood the context, he had no problem applying his knowledge of multiples of 10.

I was puzzled by how easily this strategy came to Emilio, considering the struggle the last time we met. I wondered how much of his success, or lack of it, was based on whether he is preoccupied with something more pressing or more interesting than the problem at hand; after all, when our attention is divided, our capacity to reason is compromised (Glenn, 2008).

Jack uses numerals to represent his thinking. Jack direct modeled by representing the groups of 10, showing each candy. But just as he has been doing, he counted the solution by 10s. He has shown that he does not really need to represent each individual candy in the solution and so I asked him to write a number sentence that showed how he solved the problem. He wrote: 10, 20, 30, 40, 10, 1—showing the quantities separately, and not how he combined them by counting tens.

Dramatization helps Sunny. Sunny, as before, seemed to have trouble getting started. She confused rolls and candies, and at one point said there were 10 rolls, instead of 4. She also did not combine the rolls and the loose candies at first. Although her strategy was not clear to me, I think she separated out the 11 loose candies from the 4 rolls. I thought that animating the situation for her, and in particular, putting her in the problem with me, might help her visualize the context. So I dramatized the problem with her as a character asking Dr. E. about the candies she had, just as Jacobs and Ambrose (2008) reported teachers do. It worked. She decided that she needed 4 rolls of 10 and the extra 11, and counted them all by ones to get 51.

Making connections. At this point I decided to gather the children together for a group discussion. They had three different strategies but all of them had in common the use of tens in some way. A number sentence might help tie together the ideas that were in each of these strategies, and extend the children's understanding (Jacobs & Ambrose, 2008). I asked Jack to report his strategy. As he was talking about the sticks of ten, I wrote "10, 10, 10, 10" to represent what he had drawn and to make a connection. Then on big paper, we followed through with: "10 + 10 + 10 + 10 + 11 = 51" to represent the entire situation. I was anticipating that writing the number sentence in reference to the concrete strategy would help children build a connection between the quantities in the problem and the mental use of 10 as a unit.

The Last Session: I Informally Assess What the Children Have Learned

For our last session together, I wrote a mix of problems that would provide insight into what the children had learned about base-ten concepts and their use in problem solving.

Jack. Jack seemed to have made a great deal of progress. He solved the first problem (8 rolls of candy, 10 in each roll) by drawing a rectangle-like representation of each roll. At my suggestion, he wrote "10" above each one. He finished off by drawing the extra 12 candies individually. As he counted them however, he pointed out the extra 10, for a total of 90 and "3, oops, 2 more" (Figure 22).

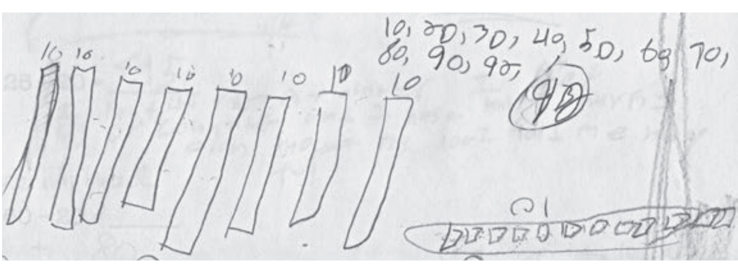


Figure 22: Jack's drawn strategy for 8 groups of 10 candies each, plus 12 extra candies

I asked him to write a number sentence and he wrote "10, 20, 30, 40, 50, 60, 70, 80, 90, 92" (as before). This strategy is significant because he no longer depends on representing the individual units (each candy) to construct 10. He can use 10 as a unit in his strategies.

I asked Jack if he could solve the second problem (30 pencils, 29 more pencils) in his head; he though for a moment, said no, and proceeded to draw this (Figure 23):

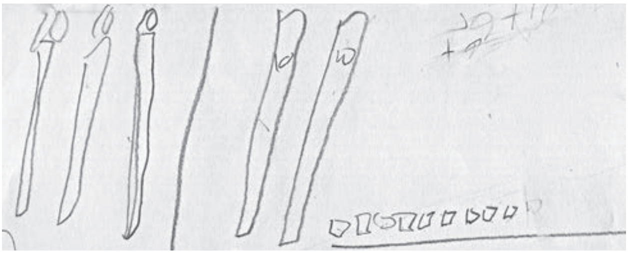


Figure 23: Jack's drawn strategy for 30 pencils plus 29 more

This time I asked him to write a number sentence using plus and equals to show how he solved it. He wrote "10+10+10+10+10+9=59." As he was writing the tens, I asked him how many tens in 50. "Five," he said. So he understands the place-value relationship between 50 and five 10s.

Jack's solution for the third problem (45 beads, 10 beads per necklace) suggests that his new knowledge of ten as a unit may be somewhat fragile. When I checked in with him, he had written some tens and ones on his paper to represent the total quantity (Figure 24).

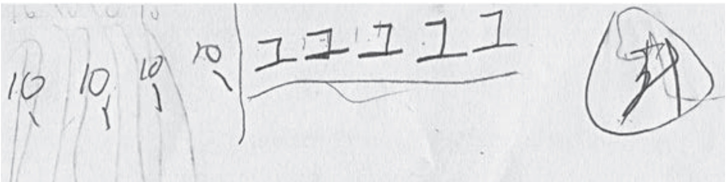


Figure 24: Jack's written strategy for 45 beads divided into groups of 10

The problem seemed to be solved. He seemed to think the answer was 4. But as I questioned him about what he had done and why he had done it, his answer changed, first to 5 (pointing to the remainder), then to 40 (the number of beads in 4 necklaces). I continued to ask him questions to help clarify his thinking and to emphasize the context of putting beads on necklaces, and the relationship between beads and necklaces. He finally returned to his original answer of 4 total necklaces. He was in the process of learning to mentally coordinate related units, such as ones and tens, beads and necklaces.

Turning to the open number sentences, I again asked Jack if he could solve the problems in his head. "Yes," he said, for 30 + 40 = ___. He counted on by tens from 30 to get 70. I skipped 25 + 20 in order to see what he would do with another problem that involved multiples of 10 only. "How about 60 - 20?" I asked. "80," he replied. I drew his attention to the minus sign. "So if it's plus," I said, wanting to reinforce his mental strategy, "the answer is 80. What if it's minus?" Jack easily counted back by tens to get 40 and likened the problem to 6 take away 2. If I were to continue to work with Jack, I would give him more problems like all of these, and support him to use more abstract counting strategies consistently. I would expect him to move toward a place-value understanding of multiple groups of ten, as in just knowing that 5 groups of 10 is 50, for example.

Sunny. Sunny's strategies were more concrete than Jack's, but I noticed that the language she was using to describe these strategies suggested an emerging understanding of base-ten concepts.

For instance, to solve the problem involving 8 rolls of candies, with 10 candies per roll, she direct modeled using unifix cubes in sticks of 10. But when she described her solution she said, "It's 80, because 8 tens is 80, when you count by 10, eight times, it's the number 80." In her explanation Sunny was making a connection between "counting by tens" a certain number of times and multiples of ten.

The emergent nature of her understanding of the base-ten structure of numbers was also apparent in her strategy for adding 30 pencils and 29 pencils. Again she direct modeled the quantities, using unifix cubes in sticks of 10. But beyond this, Sunny made little use of base-ten concepts: to count the total, she counted up by ones from 30. I think she is just arriving at understanding 30 is 3 tens and that applying this knowledge in constructing a solution such as counting on by tens is somewhat beyond her right now. If I were to keep on working with Sunny, I would continue to give her addition and subtraction story problems with double-digit quantities as well as equal groups problems involving four or more groups of 10 to help her develop strategies that made more efficient use of base-ten concepts and processes.

Daniella. Daniella was also making progress in her use of ten as a unit. In particular, in her strategies for problems that involved three or more groups of ten, she moved from adding the first two tens (10 + 10 = 20) and counting the other groups of ten by ones to counting all groups of ten by tens. She was also able keep the difference between tens and ones in mind, in contrast to earlier strategies in which she conflated the two units. For example, to determine the value of 3 dimes, 1 nickel, and 2 pennies, Daniella counted by tens to 30, then counted the other coin denominations by ones. To figure 8 rolls of candy, she counted by tens up to 80, and then counted the 12 extra candies on by ones. However, she found it harder to use this knowledge of ten as a unit to solve the division problem involving groups of ten and to count on by tens from a non-decade number, such 25. If I were to continue working with Daniella, I would give her equal groups problems to solve involving four or more groups of tens, including both multiplication and measurement division, to help her consolidate her new use of ten as a unit in her counting strategies. I would also give her double-digit addition story problems involving the addition of a non-decade number and a small multiple of 10, such as 10 or 20, to provide the opportunity for her to begin to count by tens from a non-decade number.

Emilio. Like Daniella, Emilio solved problems that involved multiple groups of ten by counting by tens and had some difficulty using ten as a unit in other problems, such as the measurement division problems (45 beads, 10 beads per necklace) even when the total number of beads was changed to 30. For the coin problem, he counted the dimes by tens to 30, and then counted the nickel and two pennies by ones. For the rolls of candy problems (8 rolls of candy, 10 candies in each roll, plus 12 extra candies), he successfully direct modeled the 8 rolls using cubes in sticks of ten and counted the total using a combination of skip counting by tens and counting by ones. I would encourage Emilio to continue to solve problems like these and I would look for evidence that he was beginning to transition from direct modeling by tens to skip counting by tens.

Looking back, I was struck by the ebb and flow of the children's advancing understanding. Constructing ten as a countable unit ─ the foundational concept of base-ten and place value understanding ─ was a protracted process for these children. Their understanding did not advance at the same rate, and when advances were made, they were not uniformly sustained. Further, I stopped my work with them with many open questions. Our time together was up, and there was no neat resolution to the most basic learning goal I had for these children, which was the development of more sophisticated understanding of base ten and the ability to use this knowledge flexibly in problem solving. Children do not necessarily learn what we may have planned in the time frames we set. Nonetheless, each child made what I recognized as progress.

Discussion

David Cohen (2011) described one of the predicaments of teaching as a paradox: no matter how expert teachers may be, they "frequently have no conclusive expert solutions, even to many basic problems" (p. 5). Because of my work as a professor of mathematics education, my knowledge base of research on children's thinking is fairly extensive. Yet this knowledge base did not contain ready-made answers to the problems of teaching mathematics to a group of four wiggly young children. In my work with Daniella, Emilio, Jack, and Sunny, I encountered many of what Cohen might call basic problems. I worked hard to decide what to do and often felt uncertain of the outcome.

My basic teaching problems centered on decisions about what problem to pose next, when to push a child to use a more sophisticated strategy and when to hold back, how to respond to children's incorrect strategies in a way that supported their thinking rather than took over their thinking (Jacobs & Philipp, 2010), how to help children move from more concrete to more abstract strategies, how to deal with a diversity of understanding among the children, how to foster children's curiosity about their mathematical thinking, and how to manage the group so that they listened to and learned from each other. Research findings on the development of children's understanding of base ten were an important resource for me, as were teaching principles centered on the development of children's mathematical agency and ownership (Barton & Tan, 2010; Turner, Dominguez, Maldonado, & Empson, 2013). Although these children had all been characterized as *struggling with mathematics,* I wanted to find their strengths ─ what they understood, rather than what they lacked. Research-based knowledge gave me a lens through which to see these things.

How did I decide what to do next in ways that were responsive to children's mathematical thinking? My moment-to-moment decision making was guided by the goal of supporting these children to work from what made sense them. To find out what made sense to them, I elicited their thinking about strategically chosen problems involving both equal groupings of ten and adding and subtracting decade numbers, without dictating that they think about these problems in a certain way. I probed their thinking, and I asked questions to support and extend their thinking (Fraivillig, et al., 2010; Jacobs & Ambrose, 2008). Thus, the information on which my decisions were based emerged almost exclusively during instruction as I talked with the children about their thinking. This talk and its teaching moves were intended to provide opportunities for children's thinking to advance. I taught these children by listening to them and, based on what I heard, providing opportunities for them to extend their thinking.

I wrote problems after each session, taking into account what I had learned about one or more of the children. The problems were informed by framework provided in Carpenter and colleagues (1999; 2014) and Carpenter, Franke, and Levi (2003). However, I decided the problem contexts (e.g., soccer balls, packages of cookies, beads on necklaces), specific numbers, the sequences in which to present the problems, and what parts of children's strategies to focus on in my conversations with the children. These choices, in turn, were based on my emerging understanding of what these children understood and where their understanding might lead them next (Simon, 1995). For example, in Session 2, I wrote a problem involving multiple groups of ten, because I was curious about how the children would think about these groupings. In Session 3, when I saw that Daniella was stumped about how to use unifix cubes in stacks of ten to solve a problem that involving building on a non-decade number (22) up to a decade number (50), I realized I needed to adjust the numbers in the problem to make it easier for her to use a more sophisticated strategy ─ modeling with and counting by tens ─ to solve the problem.

With each teaching move, I aimed to create an opportunity for one or more of the children to make critical connections (Hiebert & Grouws, 2007). Because I saw it as my role to create such opportunities but not to force a particular answer, it meant that sometimes an opportunity was not taken up by a child in a way that was obvious to me. I responded, for the most part, by not insisting that the children provide an answer that accorded with my idea of the connection I wanted them to make. Instead, I listened for the connections that were readily made by the children, so that I could capitalize on them, and I allowed myself to be comfortable with uncertainty about what exactly each child might be taking away from the interaction.

To an observer some of my decisions may have seemed counter intuitive or misguided. At one point or another, for example, I decided to ignore a wrong answer or to not show a more efficient strategy. In Session 6, when Daniella solved a problem involving 11 groups of 10 plus 5 extras by counting the 5 extra singles as 5 extra tens, I decided to momentarily ignore her incorrect answer (160) to focus on how she had modeled and counted the 11 groups (in 11 groups of 10 ones each) so that Emilio might have the chance to reflect on his own, incorrect strategy. Decisions such as these were contextualized in my larger goals of supporting each child to work from what he or she understood, to make the connections that were critical to their mathematical growth at that moment, and in the long run, to develop mathematical power. Keeping my eye on these larger, longer-team goals sometimes required making trade offs in my moment-to-moment work with the children between emphasizing procedural accuracy and building on children's sound conceptions.

What was the role of research-based knowledge in this decision making? My knowledge of research on children's thinking in the domain of number and operations oriented me to attend to the children's strategies, what I thought these strategies indicated about their understanding, and how I could use that understanding in deciding my next steps (Jacobs, Lamb, & Philipp, 2010; Philipp, et al., 2012).[2] Certain details, such as how a child used ten as a unit or counted on from 22, drew my attention because they fit into my generalized understanding of what it meant for children to understand base ten. For example, when Jack began to count his groups of individually drawn cookies by tens instead of ones in Session 6, I recognized that he was on the cusp of a significant shift from direct modeling by ones to skip counting by tens. It made sense to urge him to use a more abstract strategy at that point. Without knowledge of the development of children's mathematical thinking, details such as this one would have escaped my at-

[2] Vicki Jacobs, Randy Philipp, and colleagues call this phenomenon *teacher noticing*. In a sense, I am studying my own noticing. For more information about this growing body of work, see Sherin, Jacobs, & Philip (2011).

tention or seemed irrelevant. I would not have seen a possible advance in the making or I might have rushed children who were not ready to advance.

At best, however, research-based knowledge of children's thinking served as a guideline for me. It did not prescribe what to do in each lesson and certainly did not provide an answer for what to do next when I was in the midst of interacting with the children. I had to do the work to interpret each child's thinking, by attending to what the child did or said ─ how the child used cubes, what they wrote on paper, or how they counted, for example ─ and connecting these actions with the components of this research-based knowledge, as I understood it; my interpretation of each child's thinking then provided the basis for deciding how to respond (Jacobs, et al., 2010). Researchers who study teachers' thinking have pointed out the seamlessness with which these interconnected parts of decision making occur (Erickson, 2007; Jacobs, et al., 2010; Philip, et al., 2012).

Reading the research on learning, one might be tempted to think that the development of children's thinking follows a clear sequence that can be easily applied by teachers in practice. It's an appealing idea, because we think of learning as cumulative, which seems obvious when we view learning retrospectively. Children's thinking about base ten, for example, tends to progress from modeling quantities by ones, to modeling quantities by tens, to skip counting and adding tens, to immediate place value knowledge (Carpenter, et al., 1998; Hiebert & Wearne, 1996). Teachers who look for these general patterns in their classrooms, however, quickly realize the complexity and apparent non-linearity of children's learning. Children's thinking is not always clear, children advance at different rates, breakthroughs that were made one week seem lost the next, and what worked for one child may not work for the next.

At any given decision point, there are so many possible choices about what to do next, that the idea that there might be a single best next step or path is profoundly misleading. Research on children's learning may provide frameworks and mathematics educators may design resources, but the real work of using research on learning in instruction requires an interpretive agent ─ the teacher ─ to do the work of connecting what one child is doing with the more generalized knowledge of how children learn in order to decide how to respond. In this account of my work with four children, my goal was to document what was involved in doing this work in the domain of base-ten number and operations.

Conclusion

Responsive teaching involves new skills and requires that teachers be constantly attentive to children's mathematical understanding as they teach.

Teachers are to look for evidence of and attend to this understanding for several purposes ─ to monitor correctness, to diagnose errors and misconceptions, and to build on children's sound conceptions (Daro, et al., 2011). I focused on the decision making that occurs during instruction and is aimed toward building on children's sound conceptions, as children engaged in solving problems and expressing their reasoning. This decision making was responsive to what emerged in children's activity and informed by research on children's learning in a specific content domain.

If instruction provides opportunities for students to engage in conceptually challenging mathematics drawing on what they know, then the possible directions when students reveal their thinking are many. Research on learning can offer resources but not prescriptions. Teachers need to be able to work out for themselves, on the spot, how to respond to the evidence at hand, and they need to be able to do this continuously as they engage children in solving problems and reflecting on their solutions. A critical goal for teacher education, then, is to develop teachers' capacity for making and enacting informed, responsive decisions as a continuous feature of instruction. When it comes to teaching children, nothing replaces a teacher as the ultimate decision maker, attending to each child.

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