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SECOND-GRADERS' MATHEMATICAL PRACTICES FOR SOLVING FRACTION TASKS

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Abstract

Recently, over 40 states in the United States adopted the Common Core State Standards for Mathematics (CCSSM) which include standards for content and eight standards for mathematical practices. The purpose of this study was to better understand the nature of young children's mathematical practices through an exploratory examination of the practices of a group of second-grade students engaged in several mathematical tasks focused on rational number concepts. Twenty-five second-grade students completed three fraction tasks in structured clinical interviews. The interviews and student work were analyzed using an interpretational analysis to examine the data for constructs, themes, and patterns that were useful in explaining children's mathematical practices. The results reveal that children used a variety of mathematical practices during the interviews to respond to the mathematical problems presented. Children's mathematical practices were both a product that they used to solve the mathematical situations, and a process that was developing during the interactions of the interview. The findings lead to new insights about how mathematical practices develop and what promotes their development.

A decade ago, the RAND Mathematics Study Panel (2003) identified *mathematical practices* as highly important and suggested that a better understanding of the nature of mathematical practices had the potential to have significant impacts on the improvement of student learning. One particular concern of the panel was that mathematical practices are often left implicit in curriculum and standards documents, which can lead to them being overlooked during instruction. Another concern was that there were many unanswered questions about mathematical practices, which has led to them being misunderstood by teachers and curriculum developers.

In 2010, the Common Core State Standards for Mathematics (CCSSM) brought mathematical practices to the fore by laying out eight *Standards for Mathematical Practice* (National Governor's Association Center for Best Practices, 2010). These practices are: a) make sense of problems and persevere in solving them; b) reason abstractly and quantitatively; c) construct viable arguments and critique the reasoning of others; d) model with mathematics; e) use appropriate tools strategically; f) attend to precision; g) look for and make use of structure; and, h) look for and express regularity in repeated reasoning. The CCSSM note that "The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (National Governor's Association Center for Best Practices, 2010, p. 6). In the CCSSM documents, descriptions are provided for each of the eight mathematical practices. But what do mathematical practices "look like" when students are employing them during mathematical tasks? Are there some mathematical practices that are more common or well-developed in children from an early age and other mathematical practices that take a great deal of time and experience to develop? While there are large bodies of research focused on specific mathematical content domains (e.g., number operations, fractions, geometry), there are virtually no studies that place emphasis on examining students' mathematical practices as defined in the CCSSM. The present study sought to begin the conversation on the CCSSM's Mathematical Practices that are employed by children during mathematical tasks. The purpose of this study was to better understand the nature of young children's mathematical practices through an exploratory examination of the practices of a group of second-grade students engaged in several mathematical tasks focused on rational number concepts.

Mathematical Practices as Product and Process

Our theoretical framework for this study was based on the idea that mathematical practices may be thought of as both a product and a process (Li, 2013). By being in the form of a product, we mean that children's mathematical practices may be well-developed and usable by the child; their mathematical practices may be in a well-formed state that can be accessed and employed like a tool for solving different types of problems. However, it is important to note that this "product" that may be employed during one mathematical task is constantly being reformed and reshaped by the individual in preparation for the "product" that is used during the next mathematical task. By mathematical practices being in the form of a process, we mean that children's mathematical practices are in a constant state of formation. This means that their mathematical practices are developing and changing while students are engaged in mathematical tasks and a new or revised mathematical practice may emerge as a result of engaging in each subsequent mathematical task.

Mathematical practices, either in their product or process modality, can be developed and can be developing through Piaget's (1972) idea of reflective abstraction. Reflective abstraction is identified as the mechanism of the development of intellectual thought and a key mechanism involved in learning (Dubinsky & Lewin, 1986). Similarly, we believe that mathematical practices develop and are shaped by reflective abstraction. Reflective abstraction was first introduced by Piaget to describe "the construction of logico-mathematical structures by an individual during the course of cognitive development" (Dubinsky, 1991, p. 95). While Piaget's use of reflective abstraction often focused on examples of children learning mathematics, we propose that reflective abstraction is also an important part of children's development of mathematical practices.

Piaget (1972, 1980, 1985) distinguished reflective abstraction from empirical abstraction. For example, empirical abstract is derived from external experiences with objects which can lead to some knowledge about the common properties of those objects. In contrast, reflective abstraction results when a child forms a generalization that is constructive. While empirical abstraction deals with objects, reflective abstraction deals with action by drawing properties from mental or physical actions. These ideas can also be related to the development of students' mathematical practices. Take for example the CCSSM mathematical practice of "using appropriate tools strategically." A student using a particular mathematical tool (e.g., a protractor) or a particular mathematical action (e.g., measuring an angle) will gain some knowledge about the properties and processes involved in that activity (empirical abstraction). However, to develop the mathematical practice of "using appropriate tools strategically" requires mathematical practices abstracted from mathematical activities and reflection on those activities, with mathematical practices becoming formed and developed through reflection on the use of those practices in mathematical activity (reflective abstraction). This means that to form a generalization that is constructive about the mathematical practice of "using appropriate tools strategically" students will undergo many experiences where they are using many different types of tools, they will find that some tools work better than others for particular types of mathematical situations, and they will begin to recognize the benefits of strategic tool selection. All of these processes (and many more) that come together for the individual will be involved in the process of developing this mathematical practice.

The idea of both the developed (product) and the developing (process) mathematical practice is most closely aligned with constructivist and embodied approaches to mathematical thinking and learning (Davis, Maher, & Noddings, 1990). For example, Nemirovsky, Kelton, and Rhodehamel (2013) propose that "mathematical learning consists of transformations in learners' lived bodily engagement in mathematical practices" (p. 376). Similarly, Clements and Sarama (2007) suggest that children actively and recursively construct knowledge. Aligned with these ideas, we suggest that children actively and recursively construct mathematical practices. Clements and Sarama's theoretical framework of hierarchic interactionalism is a synthesis of previous theoretical frameworks that represents the interaction of innate competencies, internal resources, and experience. In this theoretical framework, Clements and Sarama propose that "Mathematical ideas are represented intuitively, then with language, then metacognitively..." (2007, p. 464). We suggest that mathematical practices progress in a similar fashion ─ first intuitively, then with language, then metacognitively. Clements and Sarama describe hypothetical learning trajectories for children's construction of mathematical concepts. Just as there are hypothetical learning trajectories for learning within particular mathematical domains, there may be hypothetical learning trajectories for learning mathematical practices within or across mathematical domains. Just as learning particular mathematical content follows a hypothetical learning trajectory, the development of mathematical practices may also follow a similar type of learning trajectory. And just as reflective abstraction is important in the development of mathematical learning trajectories, reflective abstraction may also be key to children's developmental trajectories for mathematical practices (von Glasersfeld, 1995).

Until recently, there has been very little focus on the examination of students' mathematical practices. However, in 2013 Li proposed a conceptual framework for observing the status of mathematical practices and evaluating the growth of mathematical practices. In Li's framework, there are three overarching themes that underlie the mathematical practices that one would expect of learners. These include: (a) the level of learner engagement and commitment; (b) the learner's employment and development of knowledge, skills and strategies; and (c) the internalization and habituation of the mathematical practices. As Li (2013) notes, "...learner's mathematical practices exhibit a process-product duality: they make up both a means to and an end of mathematics learning and understanding" (p. 62). Therefore, in Li's framework, mathematical practices can be developing (a process) or developed (a product).

In the present study, we focus on Li's second theme underlying the mathematical practices by examining the learner's employment of knowledge, skills and strategies. Li (2013) includes in this category such actions as logical reasoning, justification techniques, applying knowledge, problem solving, and reasoning. We believe that this theme allows us to see children's mathematical practices in process (as they are forming their mathematical practices) and children's mathematical practices in product form (as they are employing a mathematical practice with which they are confident and knowledgeable).

The Current Project

The purpose of this project was to better understand the nature of young children's mathematical practices. We chose to examine children's mathematical practices using tasks in the domain of rational numbers. We selected rational numbers for our tasks because the Rational Number Project has a long history of research in this domain with well-developed tasks and instructional materials and sequences (Behr, Lesh, Post, & Silver, 1983; Behr & Post, 1992; Cramer, Wyberg, & Leavitt, 2008). Researchers in this domain have posited five subconstructs of rational number knowledge: part-whole relations, ratios, quotients, measures, and operations (Kieren, 1980), and three partitioning schemes have been identified (Lamon, 1996): (a) halving ─ an early developed partitioning action (Pothier & Sawada, 1983), (b) dealing ─ a primitive form of partitioning which generates equal shares by distributing in a cycle fashion until all shares are given out (Davis & Pitkethly, 1990), and (c) folding or splitting ─ where the number of pieces grow with the number of folds (Confrey, 1998; Kieren, Mason & Pirie, 1992).

We also know a great deal about young children's conceptions of the fraction ½ (Hunting, Davis, & Pearn,1996), and 3- and 4-year-olds' understanding of continuous and discrete quantities involving the fractions ½, ⅓, and ¼ (Hunting & Sharpley, 1988). For example, 4-year-olds can think about the *whole* or about the *parts,* but they cannot think about the two concepts simultaneously. This mental action requires the ability to perform two mental actions simultaneously as in the conservation of a whole partitioned into parts (Kamii & Warrington, 1999). This base of knowledge in the domain of rational numbers makes it an excellent site for the study of the CCSSM's mathematical practices within the domain. The following overarching research question guided our inquiry: What is the nature of second grader's mathematical practices (as defined in the CCSSM) during mathematical tasks that include a simple quantity, a discrete quantity, and three conflicting empirical models of one-half?

Methods

Participants

The participants in this study were 25 second-graders (one age 7, 22 age eight, two age nine) from two intact classes in a small rural elementary school in the southern United States. There were 15 females and 10 males; 17 African American an eight Euro American children. The children represented a range of abilities, as identified by their teachers. The children volunteered for the study, and parent permission was obtained. The children participated in the interviews during the spring of the academic year. The children had little formal instruction in rational number concepts, which allowed the focus of their responses to be based on the contextualized questions rather than on procedures learned during school instruction.

Procedures & Instruments

One way to examine children's mathematical practices is through interaction with others through verbal and nonverbal communication, as suggested by Vygotsky (1978) when he proposed that higher mental processes first occur on the social plane (i.e., between people). Therefore, the primary data collection instrument was a structured clinical interview protocol that was used in one-on-one interviews with each participating child. The structured clinical interview allowed students opportunities to explain their own mathematical ideas (Goldin, 2000). Providing children an opportunity to represent a mathematical problem in a way that is meaningful to them helps them to organize their thinking, making the problem more accessible. Each interview was audio taped, photographed, and documented through field notes and children's work samples. Children's responses to the tasks were transcribed along with the interviewer's notes on children's problem solving behaviours. The interviewer photographed children's work and collected work samples, including drawings and writings. Data for analysis included the transcripts, children's written and photographed work, and the interviewer's notes.

The interviews took place in the children's classrooms while other instructional activities were being completed by the classroom teachers, with the exception of two interviews that were conducted in the hallway right outside the classroom. The interviews included a series of structured questions followed by open-ended questions to probe for more information on the children's thinking. One researcher conducted all of the interviews. Prior to the interviews, the researcher had several formal and informal interactions with the children and their teachers during classroom teaching activities; therefore, the children knew and had interacted with the researcher prior to the project. By having the same person administer the interviews, each child essentially participated in the "same interview." This procedure leads to increased replicability of the interview itself, and a basis for drawing inferences from observations during the interview (Goldin, 1997).

Problem Solving Tasks and Materials

The content of the interviews focused on the part-whole subconstruct, which is based on the ability to partition a continuous quantity or a set of discrete objects into equal parts. Children were presented with three types of problems: (a) a simple continuous quantity (region models including circles and squares), (b) a discrete quantity (set models using counters), and (c) a task using three conflicting empirical models of one-half. Tasks were posed verbally using contextualized questions. The children were presented with models for partitioning the continuous (region models) and discrete (set models) quantities. The design of the interview tasks followed the recommendations for instruction in the Rational Numbers Project (Cramer, Behr, Post, & Lesh, 1997; Cramer, Behr, Post, & Lesh, 2009). Tasks were developed with the belief that children can solve a wide range of unfamiliar problems when they are able to represent the problems with objects, actions, and familiar situations (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993). We classified our tasks as easy, medium, and hard based on the expectations for Grade 2 in the CCSSM. These classifications are described below.

Task #1: A simple continuous quantity (region models including circles and squares). Children were asked to divide a circle region and a square region into ½, ⅓, and ¼. Based on the Grade 2 CCSSM, second-grade students should successfully complete all aspects of Task #1, therefore, we rated questions from this task as "easy."

Task #2: A discrete quantity (set models using counters). An important milestone for children's numerical development is understanding how to decompose numbers (Van de Walle, 1990). Children were given set models with 12 counters and with seven counters and asked to divide the counters in half. Based on the Grade 2 CCSSM, second-grade students would not have learned the fraction content of Task #2. However, students would have experience "pairing objects or counting them by two's" (National Governor's Association Center for Best Practices, CCSSM 2.OA.3, p. 19). These skills would help second graders solve the set model task with 12 counters, so we rated this task as "medium." Second grade students have experience identifying and pairing wholes and identifying fractions, but the CCSSM does not explicitly include mixed numbers until Grade 4. To divide the seven counters, identifying three pieces plus half of a piece is "medium" difficulty, while correctly naming the outcome "three and a half" was rated as "hard.

Task #3: A task using three conflicting empirical models of one-half. Children were asked to generalize the concept of ½ across different models. They were shown a paper circle cut in half, a paper square cut in half, and 12 counters divided in half from the previous tasks. The interviewer asked the children to name each of the pieces. After the children responded that each piece was "one-half," the interviewer asked, "These models do not look the same . . . Why are all of the models called one-half?" The aim of this task was to promote reflective abstraction through children generalizing the concept of one-half in the face of conflicting empirical models.

Based on the Grade 2 CCSSM, second-grade students should identify the half as part of a whole for each model ("medium" difficulty), but may not be expected to generalize across the three models ("hard").

We placed materials, including pictures of circles and squares, one set of fraction circles, crayons, scissors, cut-outs of circles and squares, 12 two-color counters, and extra paper and pencil, on the interview table. Children received materials for some tasks (e.g., shading or drawing a fraction portion on paper). For other tasks, children could select any of the materials on the interview table to provide an explanation. The interviewer proceeded through the tasks on the protocol and asked follow-up questions to encourage children to explain their thinking. Interviews ranged from 25-45 minutes depending on the length of each child's responses. From these interactions we identified children's mathematical practices.

Data Analysis

Interview transcripts were read and coded separately by three independent readers using an interpretational analysis to examine the data for constructs, themes, and patterns that were useful in explaining children's mathematical practices (Gall, Borg, & Gall, 1996). Researchers coded mathematical practices from the transcripts using a modified constant comparative method to identify behaviours and verbalizations that indicated the practices (Strauss, 1987). Themes were used to develop emic (those that were derived directly from the children's own words) and etic (those that were inferred from the children's responses) categories (Maxwell, 1996).

Microsoft Excel PivotTables were used to summarize, organize and sort categories of data, allowing the researchers to cross-reference and group large amounts of data in a summary format for ease of comparison and interpretation. There were nine main category codes for the interview tasks and these were shorthanded with the following abbreviated titles: Task 1a) Region:Circle:Half, 1b) Region:Circle:Third, 1c) Region:Circle:Fourth, 1d) Region:Square:Half, 1e) Region:Square:Third, 1f) Region:Square:Fourth; Task 2a) Set:12:Half, 2b) Set:7:Half; and, Task 3) Conflicting:Half. Overall, five of the CCSSM Mathematical Practices were employed in these children's interviews (National Governor's Association Center for Best Practices, 2010, pp. 9-10).

Results

The results that follow present a brief summary of the overall accuracy of the group on each of the interview tasks. Following this brief summary, we present examples from the transcripts that reveal the mathematical practices that students employed during the interviews.

Table 1: Students' Overall Percent Accuracy by Question Difficulty on Three Mathematical Tasks

	Accuracy on Each Portion of the Task		
	Easy Portion	Medium Portion	Hard Portion
Task 1a (Region:Circle:Half)	100	*na*	*na*
Task 1b (Region:Circle:Third)	28	*na*	*na*
Task 1c (Region:Circle:Fourth)	96	*na*	*na*
Task 1d (Region:Square:Half)	100	*na*	*na*
Task 1e (Region:Square:Third)	28	*na*	*na*
Task 1f (Region:Square:Fourth)	92	*na*	*na*
Task 2a (Set:12:Half)	*na*	100	*na*
Task 2b (Set:7:Half)	*na*	80	28
Task 3 (Conflicting Models:Half)	*na*	80	56

Note: na indicates "not applicable," meaning no portion of the task was in this category.

Overall Accuracy

Table 1 summarizes the overall percentage of the 25 students who were successful on each portion of the interview tasks. Some tasks contained only an "easy" portion (e.g., Task 1a to Task 1f), while other tasks contained portions with more than one level of difficulty (e.g., Task 2b and Task 3). Students seemed to have the most difficulty dividing a circle region and a square region into thirds. In contrast 100% of the children were successful dividing a circle region, a square region, and a set of 12 counters in half. Over half of the children (56%) said that a square region could be divided in half in more than one way. Children's high levels of accuracy on most of the tasks allowed us to delve more deeply into the mathematical practices that the children used in their explanations.

Mathematical Practices: Task #1 with a Simple Continuous Quantity (Region Models)

During Task #1, children divided circle and square region models into halves, thirds, and fourths. Pictorial representations of children's common solutions appear in Figure 1, showing children's facility at modelling the fraction amounts. Children who had difficulty with the "thirds" division often drew two parallel lines or divided the circle into fourths and said to ignore one of the pieces (see e.g., 1.b.2). Students said it was difficult to draw because thirds "was an odd number" or because they "had never seen one like that before." The comments showed that children were reasoning to

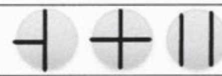
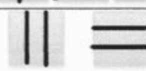
	Successful	Unsuccessful
Region:Circle:Half	1.a 	
Region:Circle:Third	1.b.1 	1.b.2 
Region:Circle:Fourth	1.c 	
Region:Circle:Half	1.d 	
Region:Circle:Third	1.e.1 	1.e.2 
Region:Circle:Fourth	1.f 	

Figure 1: Most Common Region Model Responses

find a solution. Four students, who drew two parallel lines in an unsuccessful attempt to divide the circle region into thirds (see e.g., 1.b.2.), were able to use the same two parallel lines to successfully divide the square region into thirds (see e.g., 1.e.1).

During the square region tasks, children divided squares into ½, ⅓, and ¼ by drawing lines and coloring the amounts. Students said the square could be divided in half in more than one way (see e.g., 1.d). Once again, these comments showed children's reasoning abilities. Some children had difficulty and divided the square into three unequal parts ─ one half and two fourths (see e.g., 1.e.2) ─ or they told the interviewer that the square could not be divided into thirds. They said it was difficult to draw because they could not "get the pieces to be the same size." When children are unable to divide a region into the specified quantity, it is known as nonexhaustive distribution, and is very common for children of this age. Comments about the importance of the pieces being the same size reveal children's understanding of equivalence and their attention to precision.

The most prominent mathematical practices in children's responses during Task #1 were modelling with mathematics (referring to models) and attending to precision. Children's reflections on familiar models allowed them to respond to the tasks with confidence; while children who could not access a familiar model had difficulty with the task. The following interview excerpts with four different children reveal students modelling with mathematics.

Interview Task #1 Example #1

Interviewer: Can you shade one-third in the next circle?

Student: (draws third of a circle accurately and shades one-third of the circle)

I: How did you know to draw it like that?

S: I know how to draw this one because my dad works for Mercedes.

Interview Task #1 Example #2

I: Let's say that we had this big cookie and we were going to share it between the four of us. How would you take your pencil and divide it to show fourths?

S: (draws fourths of a circle accurately)

I: And shade in your piece.

S: (shades in one-fourth of the circle); I though about an X.

I: You thought about an X?

S: And a circle around the X to make four.

Interview Task #1 Example #3

I: Do you know other ways that you might cut the square in half?

S: Like a sandwich (child uses a diagonal hand motion).

Interview Task #1 Example #4

I: What do you know about fractions?

S: Well, some come in halves.

I: Can you show me what you're talking about when you say some come in halves? (hands the child paper and pencil)

S: (child draws a circle with a line down the middle)

I: So you drew a circle and you put a line in it. What does that mean?

S: A half.

I: What part is the half?

S: This part here (pointing to half of the circle) and the other part too (pointing to the other half of the circle).

I: What if someone said to cut it into thirds and three people would share it?

S: It would look kind of like a peace sign. It wouldn't be exactly equal though.

As these examples from the interviews show, familiar models (eg., a sandwich, a peace sign, or a Mercedes emblem) played an important role in these young students' ability to model with mathematics. When the children were asked for a generalization that explained how to divide a circular or square region in half, they gave responses that fell into four categories: (a) use of text-based definitions, (b) the idea of two pieces, (c) use of the word "half" to describe the concept half, and (d) a focus on equality (see Table 2).

Table 2: Children's Responses When Asked How to Partition a Circle or Square Region

Response category	N	Sample response
Text-based definitions	11	"...the bottom number shows how many parts are in the circle...and the top number tells us how to shade."
Two pieces idea	9	"We have one piece and we cut it into two pieces."
The word "half" to describe half	9	"Because that's half of that and this is half of this."
Focus on equality	18	"Because if it's even, then it will make half." "If you make them half, make them both equal..."

Six children used one category in their explanation, and 19 children used more than one category in their explanation. As the summary in Table 2 shows, a majority of the children (18 of 25) were able to reason abstractly and construct a viable argument that focused on the concept of equality.

Mathematical Practices: Task #2 with a Discrete Quantity (Set Model)

During Task #2, children divided a set of 12 counters and a set of seven counters in half. The interviewer contextualized the tasks by asking children to think of the counters as pieces of candy to be shared between two people. Pictorial representations of children's common solutions appear in Figure 2. Overall, children's responses included spontaneous and prompted distributions (or systematic cyclic procedures) and nonsystematic procedures.

When second-graders divided 12 counters in half (Figure 2.a), responses focused on: (a) properties of the number (such as its "evenness," or that six plus six equals twelve), and (b) distribution or systematic cyclic procedures (such as dealing cards, "one for you, one for you, and one for you"). Examples of responses that focused on properties for the number included comments such as "This is even," "Because six plus six equals twelve," and "Because I know what's twelve take away six". Examples of responses that focused on prompted distributions included, "You get one and I get another one and I get another..." and "One, one, two, two, three, three, four, four, five, five, six, six" as the children distributed the counters. These responses show a variety of mathematical practices including making sense of the task, attending to precision, and reasoning quantitatively.

When second-graders divided seven counters in half, five children immediately told the interviewer to divide the seventh counter in half (Figure 2.b.3). The interviewer's follow-up questions encouraged the other children to persist and divide all seven counters in half. One child suggested remov-

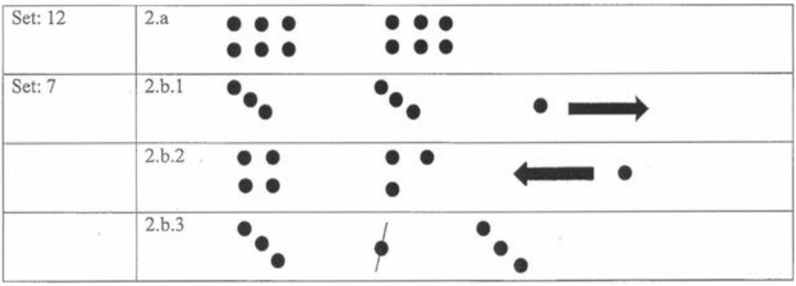


Figure2: Pictorial Representations of Most Common Set Model Responses

ing a counter (Figure 2.b.1) and giving the "piece of candy to someone else," and another suggested adding one counter (Figure 2.b.2.), both options that would produce an even number of counters. These solutions were all based on children's notions of "equal parts." Seven of 20 students used the terminology "three and a half" to correctly name the half portion.

The most prominent mathematical practices in children's responses during Task #2 were attending to precision, reasoning quantitatively, and making sense of problems and persevering in solving them. Their responses used words, drawings and hand gestures that communicated the importance that each half of the counters needed to be exactly equal. They reasoned aloud about what "half" would mean quantitatively for the given amounts of twelve and seven counters, showing a solid understanding of the concept of equivalence. Their mathematical practices were most evident when the children attempted to divide seven counters in half. The following interview excerpts with five different children reveal these practices.

Interview Task #2 Example #1

I: Let's pretend that you were sharing between two friends and they should get the same because they want to share.
S: Oh... (makes chopping motion and cutting sound)
I: What does that mean when you went like that?
S: Split it (pointing to the seventh counter).
I: Split it?
S: Right in the middle
I: Then how many would this friend get?
S: Four (the child calls the half-counter piece the fourth piece)
I: And how many would this friend get?
S: Four (the child calls the half-counter piece the fourth piece)

Interview Task #2 Example #2

I: What would happen if we just had seven pieces of candy and we had to share? How could we make half?

S: Okay, we could half the piece of candy in half.
I: What do you mean? Do you have three and I have three?
S: Mm-hmm (nodding head yes). And then we will have four if we half it. You will get a piece and I will get a piece.
I: So you will have four... (moving four counters toward the child) and then how will I have four?
S: Now we will half this piece... (moving one counter away from the four in front of the child). One half. Three wholes and one half.

Interview Task #2 Example #3

I: What's a way that we could share it (referring to the 7 counters/pieces of candy)
S: Break it.
I: What do you mean 'break it'?
S: Oh...okay (child grabs paper and begins drawing a circle with a line to divide it in half)
I: What are you drawing there?
S: A circle.
I: A circle. What does a circle mean?
S: That's that cherry candy.
I: And what does the line mean?
S: That's where you break it. Like, this is my half and that's your half. (child points to the two halves of the circle drawn on the paper)
I: So if we each get half of this (pointing to the child's drawing) and we also get these (referring to the six counters), then how many candy do we each get?
S: Three and a half.

Interview Task #2 Example #4

I: What happens if we have seven counters and we were going to share those in half? How would you share those?
S: (moves three counters to self and three counters to interviewer; picks up the seventh counter) That's a leftover one. You could break it in half.
I: What do you mean?
S: If it was like a cookie or something, you could break it in half.
I: Oh, okay.
S: And then we could both have three and a half.

Interview Task #3 Example #5

I: What would you do if we have seven pieces and we were going to share it so you get half and I get half?
S: Three (child pulls three counters toward herself)

I: Any my half would be?
S: Three (child pushes three counters to the interviewer)
I: What about this one? (pointing to the seventh counter)
S: Cut it in half.
I: What do you man?
S: Cut it down the middle and it would be two halves. I get a half and you get a half.
I: Then how many pieces would you have?
S: Three half.

As these interview examples reveal, although each child's language for three-and-a-half is still developing, they are all reasoning quite well quantitatively. Additionally, their drawings, verbalizations and hand gestures show that they are attending to precision when they explain "half" as two equal parts. In Interview Example #3, #4, and #5 we can see the children using tools strategically in their explanations when one child (#3) spontaneously grabs the paper and begins drawing an explanation and two other children (#4 and #5) begin moving the counters during the explanation.

Mathematical Practices: Task #3 with Conflicting Empirical Models of One-Half

During Task #3, children were presented with three conflicting empirical models of one-half and asked to generalize the concept across the different models. The interviewer showed the children the items from the previous tasks (a circle cut in half, a square cut in half, and the 12 counters divided in half), and posed the following dilemma: *All of these pieces are called one-half...These models do not look the same...Why are all of the models called one-half?* This question prompted the children to reason abstractly construct viable arguments, model with mathematics, and use appropriate tools strategically.

Overall, children's explanations for Task #3 were categorized in terms of the justification schemes identified by Sowder and Harel (1998) and adopted for elementary children by Flores (2002). (See Table 3.) These justifications included (a) externally-based proof schemes, (b) empirical proof schemes, and (c) analytic proof schemes. Children were placed in one category based on their most salient responses. Most children could be sorted into one category, with the exception of two students unable to articulate a proof.

There was a fairly even distribution of the children among the three justification categories. In the category of *external justification* schemes, children relied on information they had seen before, like an outside source of knowledge such as teacher, a parent, or a textbook. Children who provided *empirical justification* schemes constructed arguments based on specific

Table 3: Children's Justifications for Three Empirical Models of One-half

Justification Scheme	N	Sample Response
External	6	"Because my teacher showed me on the board." "I just remember what Mrs. X told me."
Empirical	9	Circle and square are half "because I cut it up into two halves" and candy is half because "I gave you six and I gave myself six."
Analytical	8	All of the models are called half because "each piece is the same size as the other." Half is "two out of one piece" that are the same.

examples. They focused on a procedure that applied to a specific problem or situation, and often provided a separate explanation for each task that did not generalize to all models. Children who provided *analytical justification* schemes offered a more generalized explanation of the concept of one-half that applied to all three models (circle, square, counters).

There were a variety of mathematical practices in children's responses during Task #3, including reasoning abstractly, constructing viable arguments, modelling with mathematics, and using appropriate tools strategically. For example, the following interview excerpts with four different children demonstrate their abstractions and justifications for why different shapes and objects can all be called "one-half."

Interview Task #3 Example #1

I: We took the cookie (a paper circle) and we cut it in half; and we took a piece of cake (a paper square) and we cut it in half. So we said this was half (pointing to the half circle) and this was half (pointing to the half square). But they don't look the same. How can they both be called half?

S: Well, they're both, like, they're, there was two pieces, like we cut this pieces, um, two pieces that would make one half.

I: So this is called half...

S: (child interrupts interviewer) We...because you because it's out of one piece into two pieces.

I: Why...

S: (child interrupts interviewer) Are you saying because why did I do it?

I: No, no, I'm saying why do we call it the word 'one-half'?

S: Because, like...Let me use these things for an example (child grabs eight crayons)

I: Okay.

S: Like we got...Let me see if I can think something out (counts the eight crayons). Like, I got this...all of these things are just one big chicken nugget put together.
I: One big chicken nugget put together?
S: Yeah (laughs)...and then we just split it. Like, this is the chicken nugget cutter and we just split it into two pieces (separates four crayons and four crayons), and you know two pieces is half.

Interview Task #3 Example #2
I: You told me that this is half of a circle and this is half of a square, what is half of the candy? (referring to the 12 counters)
S: It's half of a rectangle
I: Half of a rectangle? What do you mean?
S: This is a rectangle (child pushes the counters together to form a three by four rectangular array)... but when you split it in half it's two rectangles. (Child spontaneously draws a rectangle on a piece of paper.) I messed up (Child draws a second rectangle on the piece of paper.) (See Figure 3.)
I: So why did you draw this rectangle?
S: Because that's a rectangle (child points to the 12 counters in the three by four rectangular array).
I: Oh...okay...
S: When you divide it in half (child draws a line down the middle of the third rectangle that she drew on the piece of paper) it's not a rectangle, it's half a rectangle.
I: So when we take this (pointing to the 12 counters in the three by four-rectangular array) in two pieces like this (pointing to the child's picture of a rectangle with a line down the middle) it's half a rectangle?
S: It has like six pieces here and six pieces here (child moves the two groups of six counters onto the rectangle that she drew on the piece of paper).

As Examples #1 and #2 show, the children spontaneously used tools and models to help the interviewer understand what children were visualizing about the 12 counters. The children wanted the interviewer to understand that they saw the counters as a whole (e.g., chicken nugget or rectangle) and that they were relating that whole to the two region models as wholes (circle region and square region).

Interview Task #3 Example #3
I: My question is, let's take a look at all of the things we have here (referring to a paper circle and paper square cut in half and a group of 12 counters in half). This is a half of the circle, and this is a half of the

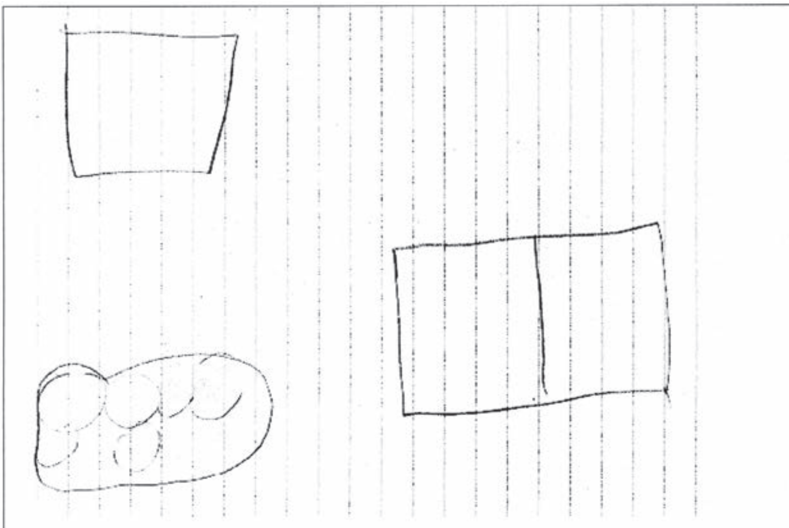


Figure 3: Child's drawing to explain 12 counters as a "rectangle."

square, and this is a half of the candy, right? Why are they all called half when they don't look the same?

S: Because the circle is the same piece (puts half paper circle on top of other half paper circle), the square is the same piece (puts half paper square on top of other half paper square), and these are the same pieces (puts half of counters on top of other half of counters), because I took all of the pieces of candy and set them on top of each other and they would equal the same.

I: Why aren't they called something different? Why are they each called one-half?

S: Because we cut them into halves.

I: What do you mean?

S: That these two are going to be the same (referring to the circle halves). They are going to be the same because this piece came from that piece (referring to the two circle halves) and that piece came from that piece (referring to the two square halves) and these from these (referring to the two groups of six counters each that she stacked on top of each other).

Interview Task #3 Example #4

I: This piece is called half (referring to the half paper circle), and this is called half (referring to the half paper square), an this is half of the candy (referring to half of the counters). Even though they are all called one-half, they look different.

S: This is red (referring to half of the counters) and this is white (referring to the half paper circle) and this is white (referring to the half paper square).

I: But look at how the shapes are ... this is just a piece of a circle and this is just a piece of a square and this is a bunch of circles. How can they all be called one-half?

S: Because you have two and they're cut. So that's a half and you have two of these (referring to the paper square and paper circle).

I: But there are six counters there... how can...

S: (child interrupts interviewer) They're not the same... but why they're all the same is because..., like this is two (touching the two circle halves with two hands), two (touching the two square halves with two hands), two (touching the two groups of six counters with two hands). They all have two parts. They all have partners.

I: They all have partners?

S: This is his partner...his partner...his partner (child matches up two half paper circles, two half paper squares, and two groups of 6 counters by pushing them together).

As Examples #3 and #4 show, the children tried to get the interviewer to understand the ideas that from one whole comes two one-half portions and two halves make one whole. As the interview examples from Task #3 show, children were reasoning abstractly and constructing a viable argument to explain why each different model was called one-half. In the last example (Example #4), when the child says "They're not the same..." the child is referring to the objects not being the same empirically; this is the empirical one-half. Next the child says, "but why they're all the same is because... they all have two parts." Here the child is referring to the abstract one-half. This example shows distinctly how the child is reflecting on the question to generalize ideas. This shows children's ability to see structual-similarity relationships, even though the examples of one-half are presented as dissimilar objects (Alexander, White, & Daugherty, 1997).

In task #3, children restructured their thinking to develop one method or concept for all models. An example of this response was when a student explained that all of the models were called one-half because "each piece is the same size as the other." The child in Example #2 drew a rectangle around the 12 counters to explain to the interviewer how to think of the set of counters as a whole. These analytical justifications demonstrated students' ability to generalize explanations that fit all of the models, demonstrating flexibility in understanding how to define a whole, and showing the ability to reason abstractly. Children's verbalizations and gestures demonstrate perceptual reasoning, in which the child is able to abstract the similarity among the representations of one-half and draw inferences about the meaning of one-half. Children's reasoning and justifications show that they have abstracted the concept of one-half.

Discussion

The purpose of this study was to better understand the nature of young children's mathematical practices through an exploratory examination of second-grade students engaged in several mathematical tasks. The results demonstrate that young children have and are developing mathematical practices from an early age. This exploration has brought us some new insights about children's mathematical practices, but it also leaves us with a variety of questions for future inquiry. Below we submit these insights as questions for future research.

Insights about How Mathematical Practices Develop

Mathematical practices are both a product and a process. The interviews show that children's mathematical practices are a *product* that they employ to solve the mathematical tasks. An example of this might be when the child adds or subtracts a counter to make the counters an even number in an effort to make sense of the task. Because the child has a well-establish idea of that would make sense, they use a sense-focused response in their explanation. The interviews also show that children's mathematical practices are a *process* that is developing and changing as they are solving the mathematical tasks. An example of this might be when the child employs different strategies (e.g., drawing or moving objects) to support their argument. This may show the process of developing mathematical practices such as learning to give evidence to make their argument more viable to the interviewer.

Children come to school with a foundation of mathematical practices that are inchoate. Clements and Sarama (2007) describe children's premathematical and general cognitive competencies and dispositions as "initial bootstraps" (p. 465). They describe these abilities in young children as supporting and constraining the development of mathematical knowledge, when experiences interact with the child's inborn capabilities. Throughout all of the tasks, students used concrete objects or pictures to conceptualize and make sense of tasks posed in the interviews. Students had access to manipulatives and space to illustrate, using them to demonstrate splitting the whole into equal parts in concrete and pictorial representations. Even without prompting from the interviewer on the set model tasks, students often mentioned cutting or dividing the last object to share it evenly. Providing tasks in a contextualized way helped children to make sense of the tasks. Perhaps children's mathematical practices are also in this initial bootstraps form and children's experiences allow their mathematical practices to develop beyond this initial form.

The development of mathematical practices may follow a hypothetical trajectory similar to mathematical learning trajectories. Just as Clements and Sarama (2007) have proposed that children's learning of mathematical concepts follows a hypothetical learning trajectory, the development of each mathematical practice may also follow a hypothesized path of learning. For example, very young children seem to intuitively know the importance of constructing a viable argument. They have had many non-mathematical experiences in which they have constructed a successful argument for family members long before they come to school. The development of constructing viable arguments begins prior to school and prior to formal school mathematics. Specific skills of constructing an argument are refined and, then, during schooling, children learn to apply this to the argument of a mathematical position or proof. Over many years of school mathematics children have multiple opportunities to develop and refine their expertise in the process of constructing a viable argument. Perhaps there are trajectories for learning each of the mathematical practices.

Different mathematical practices may be related to different points along the child's developmental trajectory. Our interviews in this study elicited six of the eight CCSSM Mathematical Practices. One explanation could be that the mathematical practices that children employ are connected with greater emphasis at different points along the child's mathematical development. Additionally, these practices may develop at different rates for different learners. This emphasis may be similar to the emphasis of mathematics content outlined in the NCTM content standards (NCTM, 2000). In this document, NCTM proposed that some of the mathematical content standards that students learn will be taught with increasing emphasis across the grade levels and some standards will be taught with decreasing emphasis across the grade levels. For example, in the NCTM Standards, Number & Operations, Measurement, and Data Analysis & Probability generally decrease in emphasis, and Algebra and Geometry standards generally increase in emphasis between Kindergarten and Grade 8. In our study we did not observe examples of two of the CCSSM Mathematical Practices: *look for and make use of structure* and *look for an express regularity in repeated reasoning.* Perhaps there is a greater emphasis on some practices over others during different points of development, with some practices more prominently used by children in earlier grades and other practices used rarely by younger children but developing and increasing over time among older children.

Insights about What Promotes the Development of Mathematical Practices

Mathematical tasks promote the development of mathematical practices. The children were not taught the mathematical practices that they used during the interviews. Children's mathematical practices emerged and developed during the interviews. One explanation for this occurrence is that the mathematical tasks, themselves, promoted the development of children's mathematical practices. Perhaps when children engage in a particular type of mathematical task, and this task is repeated, engagement in the task helps the child to develop different mathematical practices that can be used across a variety of mathematical situations.

We know from the literature on design-based research that excellent well-developed tasks promote mathematics learning (Diefes-Dux, Hjalmarson, Miller, & Lesh, 2008). Additionally Rau, Aleven, and Rummel (2009) suggest that multiple graphic representations of fractions, presented consecutively, can aid students' understanding of fractions, especially when combined with self-explanation. These examples and others point to the importance of presenting children with tasks that provide some cognitive dissonance. For example, children often have difficulty developing proportional reasoning skills, particularly with discrete units (Boyer, Levine, & Huttenlocher, 2008). Comparing the conflicting models was more challenging, as students were simultaneously presented with continuous and discrete wholes. Forcing children to examine conflicting models and to talk about this with an interviewer prompts more generalized understandings of concepts (Buschman, 2001). Perhaps excellent well-developed tasks promote mathematical practices.

Questioning promotes the development of mathematical practices. During the interviews, there were often responses provided by the children that could be characterized as surface answers. In these surface answers, the children rarely employed their mathematical practices. However, when the interviewer used follow-up questioning strategies, the children often employed a mathematical practice to respond to the interviewer and to think more deeply about the mathematics that was being discussed. Perhaps specific questioning techniques prompted mathematical practices, such as when the interviewer asked the children to explain something or asked the children to show the interviewer what they meant by a response. For example, Armstrong and Novillis Larson (1995) found that fourth, sixth, and eighth grade students who participated in clinical interviews discovered mathematical ideas as they constructed their responses to the interview questions.

Research has shown that students sometimes react to the context in which the question is posed. In this study, the interviewer often suggested that students think of the models as cookies, cakes, or candy, and students sometimes used this language in their explanations. In other instances, students suggested models such as apples or chicken nuggets to share, making their own real-world connections. Other studies support questioning in relatable contexts as a way to help hold students' interest while demonstrating the usefulness of the mathematics concepts (Barnes & Venter, 2011; Turtiainen, Glignaut, Els, Laine, & Sutinen, 2009). Perhaps the right questioning promotes children to employ and develop their mathematical practices.

Specific mathematical tasks and follow-up questions may align with specific mathematical practices. During the analysis of children's interviews it seemed that there were particular mathematical tasks and follow-up questions that promoted some mathematical practices more than others. For example, when children were asked to partition region models and to divide 12 and seven counters in half, this prompted children to focus on being precise in their partitioning. When children were presented with the three conflicting empirical models in the third task, this prompted them to focus on constructing a viable argument. When the interviewer's question asked the child to explain something more clearly, this promoted greater precision. Or when the interviewer asked the child to show what the child meant, this prompted the child to model with mathematics.

The tasks that we chose for these interviews clearly elicited six of the eight CCSSM Mathematical Practices with the children in our study. For example, when children were asked to divide seven counters in half this specific task seemed to elicit the mathematical practice of making sense of the problem; then the interviewer's repeated follow-up questions seemed to elicit children to persevere in solving the problem. When the task required students to make significant attempts to understand the problem, use concrete objects or pictures to conceptualize and solve the problem, and determine if the problem and the solution makes sense, this involved self-explanation, which is trying to make sense of a concept by forming one's own explanation (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Rau et al., 2009). Perhaps types of mathematical tasks and questions are more aligned with the type of mathematical practice that gets developed.

Certain mathematical practices may be more difficult to develop than others. Perhaps there are some mathematical practices that children develop easily, while other practices are only developed with a great amount of experience and support. For example, reasoning abstractly and quantitatively requires that students understand quantities and their relationships in a problem. They must be able to decontextualize and contextualize, understanding the meaning of the quantities and flexibly reason to represent their thinking about a problem. This can be difficult for students to develop, as Lee, Brown, and Orrill (2011) found that even teachers often struggle to flexibly develop and apply fraction knowledge and strategies.

Constructing viable arguments and critiquing the reasoning of others involves students' abilities to justify their solutions, as well as construct and evaluate conjectures. Nicolaou and Pitta-Pantazi (2010) state that justifications "provide an insight into students' understanding of fractions" (p. 3). Throughout the interview tasks, some students struggled to mathematically justify their answers, even when successfully modelling the given fraction. The interview tasks did not require manipulating symbolic representations of fractions in a decontextualized manner but some students showed developing contextualization skills. For example, one student explained a response to one fourth of a circle in Task 1 by saying, "the bottom number is four; that tells me how many pieces are in the whole thing." Students also referred to equal or fair portions in all tasks, though explanations in tasks involving set models mostly focused on operational thinking using counting by twos or adding six and six to make twelve (Mack, 1995). Researchers report that whole number knowledge can be used effectively to develop fraction knowledge by connecting key conceptual understandings between the two number systems (Olive, 1999; Steffe, 1992, 2002; Steffe & Olive, 1996, 2010). Perhaps some mathematical practices are more difficult to develop and children need more help developing some of the mathematical practices than others.

Conclusion

Children's problem solving strategies develop in the toddler and preschool years and allow children to solve problems of increasing complexity (Clements & Sarama, 2007). These developing strategies can lead to mathematical practices that become polished and refined for specific purposes in later mathematics development. The children in this study employed six mathematical practices in response to our tasks. For example, students used their understanding of whole numbers for discrete quantity tasks for *making sense of problems.* Students used both empirical evidence and analytical thinking to *construct arguments.* When *modeling with mathematics,* students were able to model different representations of one-half and created a variety of models on their own. Students *reasoned* by making connections among the three conflicting models. *Attending to precision* proved difficult for children as they struggled to use their knowledge of dividing a circle or square in half and apply it to dividing a circle or square in thirds; as when the child recognized that the responses of a "peace sign" would not be precise. And children *using tools strategically* was evident when they spontaneously grabbed paper and pencil to draw an explanation or began moving counters to clarify their thinking to the interviewer. These results demonstrate that the children in this study had a foundation of mathematical practices that they developed from formal and informal experiences, and that they were able to apply these practices to the mathematical tasks in this study. However, our investigation has left us asking many more questions than when we began. Perhaps ending with a number of questions is a good place to begin to understand the complexity of students developing mathematical practices and teachers developing them in their students.

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