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Mathematics Teaching: Listening, Probing, Interpreting and Responding to Children's Thinking

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Abstract

The perception of what a teacher says s/he does in the classroom may or may not match the reality of their actual teaching practice. This case study considers one second-grade teacher's instructional methods and pedagogical decisions when teaching number sense and her perception of what informed her teaching practice. This teacher supported students' development of mathematical strategies, valued debriefing time, and students' sharing mathematical strategies. Additionally, she listened to students purposely by probing their thinking. Her experience in a mathematics professional development program helped her be consistent in her beliefs about mathematics learning, her perception of her teaching, and the observed practice.

Teaching students so they learn and know mathematics conceptually along with understanding how to move students' mathematical knowledge forward is a complex enterprise (Kazemi & Franke, 2004; Hill, Rowan, & Ball, 2005). However, many elementary school teachers did not learn mathematics or observe mathematics being taught as National Council of Teachers of Mathematics (NCTM) or the Common Core State Standards in Mathematics (CCSS-M) advocate. Most teachers learned mathematics through direct instruction that emphasized fact-based, low-level questions, and rote memorization (Raymond, 1997; Spielman & Lloyd, 2004). Many teachers now use this as their model of how to teach (Stigler & Hiebert, 1999) and therefore struggle to understand what it means to teach mathematics as the NCTM *Standards* (1989, 1991, 2000) or CCSS-M (2010) envisioned (King, 2011; Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012).

All too often, reform in mathematics instruction (i.e., curriculum, teaching, and assessment) is implemented with little or no support for teachers, which leaves them to interpret reform measures on their own (Ball & Cohen, 1996).Teachers then "risk constructing 'lethal mutations' in their classrooms, as they modify practice, or extend it and unintentionally violate rudiments of the reforms theoretical base" (McLaughlin & Mitra, 2001, p. 307). If teachers cannot picture the type of instruction promoted by NCTM because they have never experienced it, then how can we expect teachers to apply this vision effectively? Thus, professional development experiences must inform and transform teachers' beliefs about learning mathematics and instructional practices for teaching mathematics. For this research study, I examined the alignment between one teacher's beliefs on mathematics instruction, her perception of her teaching practice, the observed practice, and the professional development experience that influenced her thinking about teaching mathematics.

Theoretical Perspective

Mathematics Teaching Practices

Mathematics instruction advocated by NCTM (1989, 1991, 2000) asks teachers to: (a) use mathematical activities that help students develop a conceptual understanding of the mathematics involved, (b) encourage students to use a variety of mathematics strategies to solve problems effectively and flexibly, (c) discuss and examine mathematical strategies through classroom discourse for the underlying mathematical concepts, and (d) determine what students understand by listening to and probing their students' thinking.

Mathematics activities that help students develop conceptual understanding often require students to develop their own strategies for solving problems (Kamii, 1993; Lampert 2001). Children's intuitive or prior mathematical knowledge, along with planning and facilitation from a teacher, enables children to construct strategies for solving mathematical problems, which can lead to a deeper understanding of the mathematics they are learning (Carpenter, Fennema, & Franke, 1996; Heaton, 2000).

When children share their strategies for solving problems they participate in mathematical discourse with their peers and teacher (Sherin, Mendez, & Lois, 2000; Heaton, 2000). Kazemi and Stipek (2001) advocated that children should share and discuss reasoning that justifies the answer, rather than just give the answer itself or a summary of the procedural steps. This social interaction of sharing and justifying their mathematical reasoning and strategies with their peers is what Vygosky calls socially constructing new knowledge–they are developing the cognitive skills needed to learn new information (Wertsch, 1984). Sharing strategies also enables other students in the classroom to become flexible thinkers because they are now aware of other ways to solve a problem. These alternative strategies may be more efficient, easier to perform, or simply present a different method than the student had first considered (Jacobs & Ambrose, 2008).

One way teachers learn about their students' mathematical understanding or strategies is by asking them to explain their mathematical thinking through high level, probing, and/or pressing questions, and mathematical discourse (Kazemi & Stipek, 2001). Listening to students' thinking is vital for mathematics instruction to be effective. Without understanding their students thinking, teachers cannot deepen their students' mathematical knowledge (Ferrini-Mundy, 1996). Using the information gathered from these discussions, teachers then can decide their next instructional moves based on the interpretations for their students' current understanding of the subject material (Franke, Kazemi, & Battery, 2007).

Jacobs, Lamp, and Philipp (2010) conceptualized the term *professional noticing of children's mathematical thinking* as: (a) attending to children's strategies in which teachers focus on details of children's strategies using "meaningful ways to discern patterns and chunk information in complex situations" (p. 172); (b) interpreting children's mathematical understanding that is based on the details of the specific child's strategies and the related "research on children's mathematical development" (p. 173); and (c) deciding how to respond based on the child's understanding. The authors argued that there are several ways to move a child's thinking forward, but any direction should be grounded in the child's way of thinking.

Teacher Beliefs and Perceptions

Teachers' beliefs and values about teaching and learning mathematics affect how they teach (Bray, 2011; Philipp, 2007; Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1992). Such beliefs are based on their previous experiences learning mathematics and learning about teaching mathematics (Brown & Borko, 1992; Lampert & Ball, 1998; Raymond, 1997). These beliefs determine what content is important to teach and the way in which it will be taught. "Teachers will develop a repertoire of teaching methods that they believe are in tune with the ideas they believe are important for students to learn" (Gudmundsdottir, 1990, p. 47). Teacher beliefs also provide the lens through which they perceive and evaluate their teaching practice and mitigate their awareness for deciding if they are effective as teachers (Pajares, 1992; Wilson & Cooney, 2002).

Researchers need to examine the link between teachers' beliefs about learning and their actual instructional practice (Darling-Hammond, Hammerness, Grossman, Rust, & Shulman, 2005; Thompson, 1992). The perception of what a teacher says he or she does in the classroom may or may not match the reality of her/his actual teaching practice (Cohen, 1990; Desimone, Smith & Frisvold, 2010). Research articles show consistencies between teachers' beliefs or perceptions about teaching and their teaching practice (Leatham, 2006; Peterson, Fennema, Carpenter & Loef, 1989) while others describe the inconsistencies (Cohen, 1990; Raymond, 1997).

Teacher Learning through Professional Development

To implement teaching methods that reflect the recommendations of the NCTM *Principles and Standards for School Mathematics* (2000) and the Common Core State Standards for mathematics (National Governors Association Center for Best Practices & Council of Chief State Schools Offices, 2010), teachers need professional development experiences that include opportunities for teachers to actively learn and enhance content knowledge and pedagogy through models of effective classroom practice, collaborative activities, analysis of student work and thinking, reflection, and connecting the experience to teachers' daily work life (Garet, Porter, Desimone, Birman, & Yoon, 2001; Hill, 2004). One way to facilitate of reexamination of their instruction is to have teachers analyze case studies of classroom episodes that reveal contrasts between traditional mathematics instruction and alternative approaches that probe students' thinking about mathematical concepts thereby creating a chance for discussion and disequilibrium (Lampert & Ball, 1998; Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010; Schifter, 1996).

Teachers need to experience a disequilibrium that challenges their initial beliefs or perceptions about their teaching (Ambrose, 2004). Having teachers conduct action research studies on their students' mathematical understanding can inform teachers' beliefs on what their students know and understand (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). Wilson and Berne (1999) found that when teachers focused on understanding students' thinking during professional development experiences, teachers were more likely to change their beliefs and instructional strategies to listen to children's thinking. Franke and Kazemi (2004) also documented that when teachers focused on students' work and understanding students' mathematical thinking, they were better able to recognize their students' mathematical competencies and developed appropriate learning benchmarks for moving the students forward mathematically.

Context

This case study was conducted in a Washington state school district in which more than 60% of students did not meet the mathematics standard on the state's fourth grade assessment of student learning. To respond to the immediate need to raise test scores, the superintendent instructed all elementary teachers to teach a common mathematics curriculum, *Investigations in Number, Data and Space (Investigations)* (Pearson Education, 2008). This standards-based mathematics curriculum helps students explore and develop strategies for solving mathematical problems and understanding mathematical concepts. The district participated in an NSF grant that used the *Developing Mathematical Ideas* (DMI) (Schifter, Bastable, & Russell, 1999a, 1999b) professional development program to enhance teachers' mathematics knowledge and to understand students' thinking.

DMI employs a series of classroom-based case studies as a pedagogical approach for teachers to examine their mathematics knowledge, their teaching practice, and their understanding of how students learn mathematics. In these professional development courses teachers investigated major ideas about the base-ten structure of our number system and operation on numbers. Teachers also examined how children develop those ideas.

The number sense courses were taught over 16 three-hour classes. Teachers were involved in an ongoing learning experience that enabled them to develop an understanding of how students learn our base-ten structure and arithmetic strategies. Appendix A shows an example of a daily synopsis for *Session 4* of the professional development course. Teachers came to the classes having read the case studies of classroom episodes from *Building a System of Tens* (Schifter, Bastable, & Russell, 1999a) or *Making Meaning of Operations* (Schifter, Bastable, & Russell, 1999b). These case studies are based on case-teachers' investigations of issues that result from implementing alternative instructional approaches to teaching mathematics. These instructional approaches probe students' thinking of mathematical concepts and then base instructional decisions on students' knowledge of mathematics. In small groups of three or four, teachers solved mathematical problems using either the different strategies students used in the cases or their own strategies.

The DMI program further required teachers to watch videos showing children creating their own algorithms for computational problems and articulately sharing their thinking. Appendix B provides the transcript of one of these video vignettes. Teachers were requested to analyze the children's thinking presented in the case studies and videos. The class sessions were grounded in discussions of the issues presented in the cases and videos, and how they may related to the teacher's own students. Teachers discussed the mathematics concepts that the students' either understood or struggled to understand in the cases and/or videos. Sometimes teachers addressed the learning goals they might have for the students in the case and/or further questions that the teacher (in the case) could have asked to learn more about the students' thinking.

Teachers wrote reflections after each class on the mathematics studied, students' mathematical thinking, and/or connections to their own teaching strategies. Examples are presented in Appendix C. As part of each course, teachers also interviewed a small group of students (or an individual student) to learn how their students were making sense of the base-ten system or numerical operations. The teachers then wrote their own case study about their students' mathematical understanding and shared that experience with their colleagues to gather informative feedback.

Cohen (2005) analyzed the effectiveness of DMI with elementary school teachers. She reported that DMI increased teachers' awareness of the power and complexity of children's ideas and understanding of number sense and that teachers employed instructional practices that used children's thinking as a basis for pedagogical decisions. Bell, Wilson, Higgins and McCoach (2010) found the DMI participants increased their mathematics knowledge for teacher (MKT) significantly. They also found that teachers increased their specialized content knowledge (SCK) and knowledge of content and students (KCS) as a result of participating in DMI.

Methods

Participant

BethAnne is a second-third grade multi-age teacher, who has worked at her elementary school for four years. The school is comprised of middle to high SES students in which 47% of the population are students of color. During mathematics time, BethAnne traded students with another multi-age teacher to teach second grade mathematics with 19 students. During the four mathematics lessons observed, BethAnne followed the *Investigations,* second grade curriculum materials, *Putting Together and Taking Apart* (Economopoulos & Russell, 1998). She specifically taught lessons from the second investigation, *Working with 100* (Economopoulos & Russell, 1998, pp. 40-71).

Each of the four mathematics lessons observed followed the same pattern. BethAnne began the mathematics lessons by placing students in a semi-circle at the front of the room. She involved students in a discussion that reviewed some of the mathematical ideas they had learned the previous day. Next, she introduced the new content to be learned. The students then went back to their desks, arranged in groups of four, to complete an activity. BethAnne walked around the room asking questions of her students about the mathematics they were doing. Near the end of their mathematics lesson, she had the students share the different mathematical strategies they used to solve their problems. She concluded the lessons with a class discussion about the mathematical concepts they had just learned. BethAnne was warm and welcoming to her students and guest throughout the lessons. She exuded enthusiasm in her voice and actions as she taught. Most of the students sat eagerly and worked diligently on the tasks she provided, with only one or two students struggling to remain on task.

Data Sources

The case study on BethAnne took place over one academic year. Data sources included (a) 48 hours of observations of the DMI courses and four hours of classroom observations, (b) six semi-structured interviews (each over one-hour long), and (c) review of course handouts, project documents, and district memos.

Interviews. Interviews with BethAnne provided rich information about her experiences as a participant of DMI; what she perceived she learned; what she planned to teach during the observations; and what she believed were the salient topics in the lessons. I conducted follow up interviews after observing the mathematics lessons and through gathered data on incidents that occurred during the observed lessons and her reflections on those incidents. All interviews were audio-taped and transcribed.

During the semi-structured interviews, a general interview protocol served as a guide (Yin, 1994). I employed Shulman's (1986) notion of teacher cognition to ascertain what BethAnne perceived she learned from the DMI courses, and how that related to her work with students. Having BethAnne reflect on her pedagogical decisions uncovered her thinking as to why she made those choices and what informed her thinking in those pedagogical decisions. Shulman claimed, "to understand adequately the choices teachers make in classrooms...and the cognitive processes through which they select and sequence actions...we must study their thought process before, during, and after teaching" (p. 23).

Observations. I observed the seminars based on DMI number sense courses. As a direct observer, I gathered descriptive information (Merriam, 1988; Yin, 1994) about the professional development sessions. I also observed BethAnne teaching mathematics lessons to second graders. Observing both the professional development experiences and BethAnne's mathematics teaching methods provided insight into what she learned about her students' thinking in number sense and how she enacted that knowledge.

Documents. Handouts from professional development experiences helped explain what BethAnne learned about mathematics and from listening to students as they discussed their thinking. Other document data, such as newsletters and memos sent from the district mathematics specialists, revealed what BethAnne was told about teaching the *Investigations* curriculum.

Data Analysis

I reviewed the interview transcripts, observation field notes, and documents by underlining the central ideas from the text. Then, I returned to those underlined passages, wrote marginal notes about the main ideas, and wrote reflective memos. I also used the marginal notes to assign codes that reflected by theoretical perspective and common themes that emerged from the data (Merriam, 1988; Miles & Huberman, 1994). I condensed initial codes into thematic units that are presented in the findings section. Examples of data and coding from a data matrix are found in Appendix D.

Interview data on the professional development formed the basis for what BethAnne perceived she learned about students' thinking and understanding about number sense concepts. Course handouts and observations from the DMI professional development experiences contributed to identifying what BethAnne was taught about number sense and children's thinking about number sense. Observation and interview data on the teaching of *Investigations* lessons revealed what BethAnne actually did in her classroom. Triangulation occurred from these multiple sources.

Finally, I required member checks for the data and findings. BethAnne reviewed the data for accuracy. I also asked two people–a participant and a facilitator from DMI–to review the content of my findings and conclusions for verification (Merriam, 1988; Miles & Huberman, 1994).

Limitations

Case study analysis is appropriate for an intensive in-depth examination of a situation (Goetz & LeCompte, 1984). This one teacher does not allow the data to be generalized to a larger population. This study was not about the whole district or a comprehensive evaluation of DMI, rather, the in-depth examination of one teacher's experience, her perception of learning about teaching mathematics, and her implementation of this learning with her students. Through this one case, I was able to describe in detail what BethAnne perceived she learned and how she enacted that learning in her classroom. These details are often lost when multiple cases or participants are discussed. I focused my attention on a single teacher to illustrate this teacher's pedagogical approaches to teaching mathematics after completing the DMI number sense modules. This case study provides readers with in-depth descriptions of a teacher listening to and probing students' thinking in number sense and numerical operations, and how she made pedagogical decisions based on what information.

The original request of BethAnne was to learn how the *Investigations* curriculum was being taught. Although the observations reported in this study were limited to the teaching of an *Investigations* lesson, DMI rose as a central influence in the way that BethAnne ultimately implemented the *Investigations* curriculum.

Findings and Discussion

The findings and discussion presented in this section will show an alignment between how BethAnne perceived she taught mathematics and her actual teaching. The discussion will also show a connection between what she was taught at the DMI professional development sessions and her self-report of what she learned. According to BethAnne, the DMI professional development experience informed her understanding of how students learn and think about number sense and numerical operations, and how to navigate students' thinking. She reported, and the observational data confirmed, that DMI helped her to value debriefing time, sharing of mathematical strategies, and listening to students more purposely by probing their thinking.

Sharing, Listening, and Probing

When students share mathematical thinking and a teacher actively listens to the strategies the child shares, the teacher can then determine what the student knows and understands. She or he can then ask questions of the child that further push the student's thinking forward or probes the child's thinking so that a better diagnosis of the student's gaps can be made. These three aspects of teaching (sharing, listening, and probing students' mathematical thinking) also help students build conceptual understanding (Jacob, Lamb, & Philipp, 2010).

On the first day of the DMI professional development course *Building a System of Tens* (Schifter, et al., 1999a), teachers were asked to mentally add 57 + 24 and share their strategies. Later they mentally subtracted 83-56 and again shared the different methods for solving the problem. This activity illustrated the different strategies people used to solve the problem when paper and pencil were absent. BethAnne commented,

> The first time you ever do that mental math thing--and everybody does it and shares out--the first time we ever did that I was totally floored. And honestly, I don't know what I thought! I guess I thought everybody thought the same way, which seems dumb now.

This idea that people can solve problems in different ways connected with BethAnne. She incorporated that belief into her teaching by having students share different strategies for solving problems. In all the lessons observed, she had students develop their own strategies then share those strategies. As students shared their thinking, BethAnne listened and probed their thinking to learn what they understood. BethAnne continually referred to her experience with DMI as to why she taught this way. Before an observed mathematics lesson, she explained that she would have students share strategies at the end of math time. Then, she continued to explain why she believed that this was important for her students.

I think sharing the strategies is a big part of the [*Investigations*] books.... But the more I do it, the more I realize that that's how kids learn new strategies is from listening to other kids....[I]t seems like if you can repeatedly expose kids to listening to other kids and seeing what they're doing, and how they're counting, then eventually they'll want to try it themselves and then they can pick it up faster and more accurate strategies.

One example of BethAnne inviting students to share strategies then listening to and probing their thinking is evident in the following excerpt from her class. During a mathematics lesson, BethAnne had written on the board:

100 - 20 = _____

20 + _____ = 100.

After her students solved the problems individually, BethAnne asked them to share their strategies. The questions in bold are examples of BethAnne asking higher-order thinking questions that enable the student to explain their thinking and for her to interpret their thinking.

BethAnne: What did people get?

S1: I got 80.

BethAnne: How did you get that?

S1: I went to the hundreds chart and counted by 10s, see 10 [points to 30], 20 [points to 40], 30 [points to 50], 40 [points to 60], 50 [points to 80], 70 [points to 90] 80 [points at 100].

BethAnne: Someone else?

S2: I got 80 too. [Student 2 comes up and writes on the overhead] I wrote 10 tens 'cause 10 tens make 100 and crossed two out for 20 and got 80.

BethAnne: That really works. You both picked strategies that work. Someone else?

S3: 20 + 80 = 100 because 2 + 8 = 10.

BethAnne did not stop there, but asked the girl (S3) to clarify the relationship between 20 + 80 = 100 and 2 + 8 = 10–she is continuing to learn the depth of her student's thinking of the base-ten system.

BethAnne: Why does that work?

S3: Because there's 10 tens in 100 so 2 + 8 = 10.

BethAnne: So why does 2 + 8 = 10 work for 20 + 80 = 100?

S3: Because there's 2 tens in 20 and 8 tens in 80.

BethAnne: So we could think 2 tens + 8 tens = 10 tens. (BethAnne wrote on the overhead.) Sounds like we have a bunch of different strategies for solving this problem.

Many teachers would accept the answer that 2 + 8 = 10 for the problem 20 + 80 = 100 as a valid solution. However, BethAnne pressed the student to make the mathematics connection explicitly between the tens place and the ones place. In having the student explain why 2 + 8 = 10 represented 20 + 80 = 100 for the strategy the student shared BethAnne helped the class consider how they could use their knowledge of adding ones as a means to add groups of ten. When I questioned BethAnne as to why she continued to probe this student's thinking, she replied:

> So I think I was trying to figure out if these kids knew this works because really we're talking about 2 tens plus 8 tens equals 10 tens.... When kids can look at a number like this and tell me, how many tens and how many ones, to me that means they really understand the number system. So, I guess that's what I'm getting at with the two plus eight equals ten. So twenty plus eight equals one hundred. Does that kid really only know ones, or does that kid also understand tens? So the kid who was able to answer that questions, knows tens.

In a post-observation interview, she reported that the DMI seminars helped her see the value of debriefing the *Investigations* lessons and providing opportunities for students to share their strategies for solving problems.

> I value [the discussion time] a lot more since I've started taking [DMI]. And I try to make the kids aware of that too. Like, why are we sitting her listening to kids come up and tell about their math strategies? What good is that? It's so they can learn from each other, and see how other people do it.

Having students share strategies enables them to learn from each other and be exposed to different strategies, thus creating flexible mathematical problem solvers. BethAnne's attribution to DMI for her commitment to this process underscores the power of this professional development experience on her perception of good mathematical teaching practices.

The vignette below demonstrates another example of BethAnne having students share their strategies for solving addition problems, actively listening to her students' thinking, then asking probing questions to further understand the depth of her students' thinking. The class discussion begins after students individually solved the problem, "You have a box of 100 paper clips. Forty-three have fallen out. How many do you have left in the box?"

> BethAnne: Who can share a strategy?
>
> S1: 43 + 7 = 50, 50 + 40 - 90, 90 + 7 + 3 = 100
>
> BethAnne: So then how did you figure out how many paper clips were in the box?
>
> S1: I added 7 + 3 = 10, 40 + 10 = 50, 50 + 7 = 57
>
> BethAnne: Who had a different strategy?
>
> S2: I used the numbers chart, I knew there was 50 left.
>
> **BethAnne: Did you count by ones or 10s? Come up and show me.**
>
> S2: I counted by ones to 100 (comes up to overhead and points as he counts to 100 pointing to 44, 45, 46, etc by saying 1, 2, 3)
>
> S3: I started at 44 and counted to 50 then counted by 10s to 100.
>
> **BethAnne: Why did you start at 44 and not 43?**
>
> S3: Because we already counted the 43.

BethAnne: If we put the paper clips down what number am I going to land on?
S3: 43
BethAnne: So the number 43 is covered by a paper clip? Is that why you started at 44?
S3: Yes
S4: I started at 43 then counted 10 (pointing to 53) 20 (pointing to 63), 30 (pointing to 73), 40 (pointing to 93, 50 (pointing to 93). Then I counted on 1 (pointing to 94), 2 (pointing to 95), 3 (pointing to 96), 4 (pointing to 97), 5 (pointing to 98), 6 (pointing to 99), 7 (pointing to 100),
BethAnne: Some people counted by the ones first and then other people counted by 10s first. Both are good strategies.

The two probing questions bolded above illustrate incidents where a teacher can determine the depth of a student's knowledge and if a gap exists in her or his understanding. In the post-observations interview BethAnne discussed why she concluded her lesson in such a fashion.

> We did an interview for DMI the other week, and it was like, take one kid, spend 20 minutes with them. There was like a picture--the iceberg, they show an iceberg....[G]oing to DMI is like realizing that kids have that iceberg in math. A little mistake a child makes is just a hint of really what they're thinking, or maybe a huge misconception, or maybe some great, great thinking! ...So, I think getting kids to explain it is a big part of this [*Investigations*] curriculum. Because without knowing that, you can't know what they need in their next step.

Listening to student's thinking enabled BethAnne to understand how her students were making sense of the mathematics and determine which direction to move in their mathematical thinking.

BethAnne reported that she also valued listening to students' thinking about mathematics, and she believed that her professional development experience with DMI helped her listen to students' thinking more intently.

> It totally made me more aware of what kids are thinking. I think a lot of times in the past, let's say I'm collecting all the strategies for something, and kids are telling me, "Well, I took this number and I split it..." if it's really obvious, like, I took thirty two and made it into thirty and two, I'm like, "Okay." But if they start telling me, "I took the seven out..." It's taking too long. But now, even if I don't always understand it, a lot of times, after seeing a lot of clips, I recognize it, like, OH! I think I know what they're trying to do. They're trying to take something from here and patch it here, or they're trying to round it, but they're not calling it that...so it just made me a better listener.

She found that by listening to her students' thinking she could learn how her students understood a mathematical concept. With this information, she could then ask questions that would further develop the students' mathematical understandings.

The interaction below is between BethAnne and one student. It demonstrates how she listened to and probed that student's thinking in how he solved the problem, then helped him become more aware of the strategy he used. The interaction is during the individual solving time students had for the problems, "You have a box of 100 paper clips. Forty-three have fallen out. How many do you have left in the box?"

> **BethAnne: Tell me how you got your answer.**
>
> Student: I guessed.
>
> **BethAnne: How did you guess?**
>
> Student: I counted to 100.
>
> **BethAnne: How did you count to 100?**
>
> (Student counts by ones to one hundred from 43.)
>
> **BethAnne: But how did you get your answer so quickly? Counting to 100 is how you checked.**

Student: Well, (thinks) 40 to 100 is 60 and it's 43 so I took away 3 from 60 is 57. In this example, BethAnne did not accept the answer, "I guessed." Many times when students solve questions quickly, because the numbers are easy for them to use, they do not realize what strategy they used to solve the problem. These students often get frustrated later when the problems become more difficult. Helping this student articulate his thinking through careful questioning made him aware of the strategy he used. In doing so, that strategy will be available to him when he is faced with more complicated problems.

In a follow up interview to this observation, BethAnne explained her thinking about this interaction with her student.

> This is a kid who knows a lot about how to hold numbers in his head and take chunks from one place and put them in another. But he doesn't know that he knows that. He thinks he has everything memorized....He always thought these answers just came to him out of the blue. So he really understands conservation of number. [Another] thing I think he understands is number constancy, that you can take a number and move it and hold this piece here but then you have to put it back later....So for him I think a big step is learning to talk about them so he can figure out how all these different things he knows are connected....

By probing and listening to this student's thinking, BethAnne learned he understood conservation of numbers and could use an invented algorithm to solve the problem rather than his initial response of counting by ones. If she simply accepted his original answer her pedagogical decision would be based on a limited understanding of his knowledge. She also encouraged this student's meta-cognition of the strategy he employed through the pressing questions she asked. This vignette of the 43 paper clips and her reflection illustrates how she attended to student's strategy, interpreted what the child knew, and then determined a way to move him forward.

In explaining why she had students explain their reasoning and asked probing questions BethAnne reported,

> I think that is something that I've picked up from the DMI module [*Building a System of Tens*]. That if a kid gives a wrong answer, maybe the best thing is not to step in right away and say, ohhh, I'm not sure about that. But see how they got that answer, to see, number one, if they can self correct it, because a lot of kids do. And, number two, if it is the wrong answer and you say it's wrong right away, then their answer will be contrived....So it's kind of like in reading, when you want kids to self monitor, and you say, does that sound right. And you're supposed to say that when there's nothing wrong, and sometimes when there is something wrong. So, I think I've been trying in math to ask kids strategies and things even when it's right or wrong, so I can see what they were thinking. And see which parts of it are wrong, and which parts of it are right. Maybe [the issue] is just computational, or maybe it's a bigger problem.

She also discussed within the interview the importance of flexibility, efficiency, and children understanding the algorithm they use. In talking about solving two-digit addition or subtraction problems, she compared the idea of using different strategies for building conceptual understanding of the content with students who learned the traditional computational algorithm.

> The biggest difference I think is in the double-digit addition and subtraction. It used to be kids get the [traditional] algorithm or they don't get it. And the kids who get it can do whatever problem you put in front of them. And the kids who don't get it, can't do any, or could do a whole page and it's all wrong. But now, every kid can do those problems, but it's more a matter of is your strategy as efficient as someone else's. So there's not that big gap. Also before, a lot of kids could do [the traditional algorithms], but they could not explain to you. And I think a lot of kids got [the concept of adding and subtracting], but could not do the algorithm, or could not remember how to make [the traditional algorithm] work and would get frustrated....Because they wouldn't know when they had to borrow or something like that.

BethAnne understood the power students feel in learning computational strategies that are based on their own strategy.

> I think everybody can think more deeply about it. And I think that kids feel more powerful and more excited about it when they can come up with their own strategy and explain how they did their thinking. Rather than just being told, well, here's how you do it.

She also demonstrated her knowledge of how students progress in their understanding of arithmetic problems. In DMI *Making Sense of Operations,* teachers focus extensively on how students develop an understanding for operations.

> But now, it's really--in my mind I see it as more of a progression, like moving from ones to tens, and moving from concrete to more abstract. Like maybe you're in ones, and you need all these cubes. Now maybe you're in ones, but you can draw lines. Now maybe you're in ones, but you can count on, and now maybe you know a little more about tens, and you can pull the tens out. And now maybe you're even beyond that, and you can totally manipulate the numbers by thinking, this is similar to another problem.

BethAnne is describing the development trajectory for children understanding of operations that Carpenter, et al. (1996) discuss. This research is covered in the DMI course *Making Sense of Operations.* She is drawing on her knowledge of research to support children's learning. DMI helped her use knowledge from research to move her students' mathematical thinking forward.

Conclusion

BethAnne had children develop and share strategies to solve problems with operations. During the debriefing time of the mathematics lesson, BethAnne listened to her students and attended to their strategies by probing their thinking about the mathematics topic. For example, BethAnne probed a student's thinking to learn that he solved the problem not by counting-on by ones as he originally said but rather by applying a more sophisticated strategy that involved an invented algorithm that was based on derived mathematical facts. When students shared effective and noneffective strategies, BethAnne pressed their thinking to learn about how and why they solved the problem. By probing students' thinking she made the debriefing time more valuable for student learning. In doing so, she provided opportunities for her students to become more conscious about their strategies and mathematical ideas.

For BethAnne, the professional development experience of DMI enabled her to support her students' learning more effectively. Her own case study projects with her students, along with the systematic analysis and discussion of the DMI case studies involving children's mathematical thinking, enhanced her MKT and KCS. She also learned, from DMI, that when teachers probe a student's thinking they must listen to the student and consider that information in order to ask other questions that further illuminate what that student understands. The DMI professional development experience gave her the teaching strategies for gathering information about what her students know and not know, and moving them forward in their mathematics knowledge. As a result, BethAnne is able to enact mathematics teaching as the NCTM *Standards* and the CCSS-M envisions for number sense and operations.

References

Ambrose, R. (2004). Initiating change in prospective elementary school teachers' orientation to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education, 7,* 91-119.

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is--or might be--the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(9), 6-8, 14.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Towards a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco, CA: Jossey-Bass Publishers.

Bell, C. A., Wilson, S. Higgins, T., & McCoach, D. B. (2010). Measuring the effects of professional development on teacher knowledge: The case of developing mathematical ideas. *Journal for Research in Mathematics Education, 41,* 479-512.

Brown, C. A. & Borko, H. (1992). Becoming a mathematics teacher. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209-239). New York: Macmillan.

Bray, W. S. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for Research in Mathematics Education, 42*(1), 2-38.

Carpenter, T. P., Fennema, E., & Frankie, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics education. *The Elementary School Journal, 97*(1), 3-17.

Cohen, D. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12,* 311-329.

Cohen, S. (2004). *Teachers' professional development and the elementary mathematics classroom: Bringing understanding to light.* Mahway, NJ: Lawrence Erlbaum Assoc.

Darling-Hammond, L., Hammerness, K., with Grossman, P, Rust, F., & Shulman, L. (2005). The design of teacher education programs. In L. Darling-Hammond & J. Bransford (Eds.), Preparing teachers for a changing world: What teachers should know and be able to do (pp. 390-441). San Francisco, CA: Jossey-Bass/Wiley.

Desimone, L. M., Smith,T. M., & Frisvold, D. E. (2010). Survey measures of classroom instruction: Comparing students and teacher reports. *Educational Policy, 24,* 267-329.

Fennema, E., Carpenter, T. P., Franke, M. L., Levin, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27,* 403-434.

Ferrini-Mundy, J. (1996). Mathematical thought-in-action: Rich rewards and challenging dilemmas. In D. Schiefter (Ed.), *What's happening in math class?* (pp. 77-86). New York: Teachers College Press.

Franke, M. L., Kazemi, E., & Battery, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research in mathematics teaching and learning* (Vol. 1, pp. 225-256). Charlotte, NC: Information Age Publishing.

Garet, M. S., Porter, A. C., Desimone, L. Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38,* 915-945.

Goetz, J. P. & LeCompte, M. D. (1984). *Ethnography and qualitative design in educational research*. Orlando, FL: Academic Press.

Gudmundsdottir, S. (1990). Values in pedagogical content knowledge. *Journal of Teacher Education, 41,* 44-52.

Heaton, R. (2000). *Teaching mathematics to the new standards: Relearning the dance.* New York: Teachers College Press.

Hill, H. C. (2004). Professional development standards and practices in elementary school mathematics. *Elementary School Journal, 104*(3), 215-231.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects for teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Jacobs, V. R., & Ambrose, R. C. (2008). Making the most of story problems. *Teaching Children Mathematics, 15*(5), 260-266.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41,* 169-202.

Kamii, C. (1993). Primary arithmetic: Children inventing their own procedures. *Arithmetic Teacher, 41,* 200-203.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203-235.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal, 102,* 59-80.

King, J. E. (2011). *Implementing the Common Core State Standards: An* *action agenda for higher education.* Retrieved from: http://www.eric.ed.gov/PDFS/ED540255.pdf

Lampert, M. (2001). *Teaching problems and the problems of teaching.* New Haven, CT: Yale University Press.

Lampert, M., & Ball, D. L. (1998). *Teaching multimedia and mathematics: Investigations of real practice.* New York: Teachers College Press.

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9,* 91-102.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, P. W. (2010). *Designing professional development for teachers of science and mathematics* (3rd ed.). Thousand Oaks, CA: Corwin Press, Inc.

McLaughlin, M., & Mitra, D. (2001). Theory-based change and change-based theory: Going deeper, going boarder. *Journal of Educational Change, 2,* 301-323.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach.* San Francisco, CA: Jossey-Bass Publishers.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook.* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Murata, A., Cofferding, L., Pothen, B. E., Taylor, M. W., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal of Research in Mathematics Education, 43*(5), 616-650.

National Governors Association Center for Best Practices & Council of Chief State Schools Officers. (2010). *Common Core State Standards for Mathematics.* Retrieved from http://www.corestandards.org

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teacher's pedagogical content beliefs in mathematics. *Cognition and Instruction, 6,* 1-40.

Philip, R. (2007). Mathematics teachers' belief and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Reston, VA, National Council of Teachers of Mathematics.

Raymond, A. M. (1997). Inconsistency between beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28,* 550-576.

Pearson Education (2008). *Investigations in number, data and space.* New York: Author.

Schifter, D. (1996). *What's happening in math class?* New York: Teachers College Press.

Schifter, D., Bastable, V., & Russell, S. J. (1999a). *Building a system of tens: Facilitator's guide.* Parsippany, NJ: Dale Seymour Publications.

Schifter, D., Bastable, V., & Russell, S. J. (1999b). *Making meaning for operations: Facilitator's guide.* Parsippany, NJ: Dale Seymour Publications.

Sherin, M. G., Mendez, E. P., & Louis, D. A. (2000). Talking about math talk. In M. J. Burke & F. R. Curcio (Eds.), *Learning mathematics for a new century: NCTM 2000 yearbook.* Reston, VA: The National Council of Teachers of Mathematics.

Shulman, L. S. (1986). Paradigms and research programs in the study of teaching; A contemporary perspective. In M. C. Wittrock (Ed.), *Handbook of Research on Teaching, 3rd Edition.* New York: Macmillian.

Spielman, L. J., & Lloyd, G. M. (2004). The impact of enacted mathematics curriculum models on prospective elementary teachers' course perceptions and beliefs. *School Science and Mathematics, 104*(1), 32-44.

Stigler, J. W., & Hiebert, J. (1999) *The teaching gap: Best ideas from the world's teachers for improving education.* New York: The Free Press.

Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacyGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education, 17,*213-226.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.). *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Wertsch, J. V. (1985). *Vygotsky and the social formation of mind.* Cambridge, MA: Harvard Press.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge. An examination of research on contemporary professional development. *Review of Research in Education, 24,* 173-209.

Wilson, M., & Cooney, T. (2002). Mathematics teacher change and development. In G. C. Leder, E. Pehkomen, & G. Torner (Eds.), Beliefs: A hidden variable in mathematics education? (pp. 149-159). Netherlands: Kluwer Academic Publishers.

Yin, R. K. (1994). *Case study research: Design and methods.* (2nd ed.). (Vol. 5). Thousand Oaks, CA: Sage Publications.

Appendix A. Example of Daily Synopsis for Session 4 of DMI Building a Systems of Tens

Small Groups Discussion of Student Interviews - 30 minutes
To share what they learned, have teachers return to the same small groups in which they planned their interviews. Remind them that they will have 30 -minutes for this sharing and they should be aware of using the time equitably.

Listening to each group, take note of the points that are being raised so that you can refer to them when leading the whole group discussion.

Begin the whole group discussion by asking each group to offer a comment of interest from its discussion.

Some teachers may have been uncomfortable as they became aware of a child's misconception and may be wondering what they are supposed to do now that they have this information. Other teachers may begin to realize that it is likely that many of their students, have the same confusions but that they (as teachers) never realized it before. For these reasons, the interview assignment and this discussion can become emotional.

In grade level groups have teachers share what they learned about their student's understanding of mathematics AND have teachers discuss what they would do with their student next. The answer I don't know is not good enough. Teachers are to think of activities, strategies or questions they would like to ask their student.

Take notes during the discussion so that you can raise points or questions that arise in the groups. Some teachers may have become uncomfortable as they become aware of a child's misconception and may be wondering what to do next with this information.

Whole Group Discussion of Student Interviews- 20 minutes
You need to listen to teachers' ideas about teaching and interviewing so that you can find elements of strength in their ideas that can be highlighted and leveraged to help them reconsider some of their own notions about teaching.

Case discussion in pairs (30 minutes)
The cases in this chapter have a different character from those in the first three chapters because they are written by a single teacher in two different years. Through the mathematics of these cases, participants revisit ideas first presented in Chapter 1 (case 3) regarding students' own procedures for addition and subtraction.

This chapter helps teachers clarify and solidify their own methods for combining and breaking apart numbers.

Focus questions 1 and 2 provide a means to work on the mathematics themselves. Q3 raises more general mathematical principles.

The cases also raise a charged issue--to teach or not teach the traditional algorithm. Q4 has teachers address this matter

It's important for teachers to think about and talk about the role of traditional algorithms as they make decisions about their own teaching practice.

The whole group discussion will take place after watching the video.

Viewing the video for session 4 (25 minutes)
The video shows children creating their own procedures for mathematics problems. It also shows interviews of teachers describing their reactions to their children's methods.

Examples of Children's strategies from video
Sean
39 - 17
17 = 9 + 8
39 - 9 = 30
30 - 8 = 22

Victoria
40 + 20 = 60
Splits the 8 into two numbers 5 and 3
5 + 5 = 10
60 + 10 = 70
70 + 3 = 73

Laura
35 - 16
10 + 10 + 10 + 5 = 35
10 + (4+6) + 10 + 5 = 35
4 + 10 + 5 = 19

4th grader
1000 - 359
1000 - 300 = 700
700 - 50 = 650
650 - 9 = 641

Anthony
999 - 359 = 640
640 + 1 = 641

Whole-group discussion of the video and print cases (45 minutes)
Start by talking about Q3 from the focus questiouns sheet.

What questions have these cases, both in the casebook and on the video, raised for you?

Discussion of movement towards creating alternative algorithms

How do you think you move students to getting them to invent their own algorithms after the readings we have done for addition and subtraction?

What questions have these cases, both in the casebook and on the video raised for you?

Discuss the teachers' statements on the following lines:

Lines 7 – 25 & 40 – 45

What are these trading games that Lynn talked about?

Why might students not be making connection between the trading games and adding numbers greater than 10?

Lines 53 – 59

What role does the standard algorithm play in the curriculum?

Lines 163 – 172

What is the teacher seeing in her students that she feels is different from last year? Why is she seeing those things?

Mental Mathematics (10 minutes)

I am going to ask you to solve some mental math problems. (no paper)

Pay attention to how they are working on the problem, because I will be asking for strategies when we share out.

69 + 23 132 - 85 29*6 16*3 153*2

Ask Ts to solve each problem individually then share out strategies.

Be sure to write the numbers horizontally to see how teachers add or subtract the numbers in their head. Do they move the numbers to a vertical position? Do they group by 10s first? Do they group by ones first?

Write out the strategies on flip chart paper, so teachers can examine the variety of strategies used. Ask teachers to pay attention to how they are working on the problem. Ask them why they solved the problem the way they did. What was their mathematical strategy? Why did they use it? Be sure to label the different strategies that teachers share. Every strategy is valued, it's important to stress that, it's the strategy that worked for them.

Exit Cards and Homework (15 minutes)

Lesson plan outline provided by (DMI, 1999a)

Appendix B. Transcript of Session 1 Video from Building a System of Tens (1999a)

The tape begins with a second grade student reading, "Kira has 48 cents in her pocket. Her big brother gave her 25 cents for running an errand. How much does money does she have?

Teacher: Our task today is to figure out how much Kira has.

Students to to the board to write and explain.

Student 1: 40 + 20 = 60

60 + 8 = 68

68 + 5 = 73

Student 2: 48 + 20 = 68

68 + 2 (I took the 2 from the 5) = 70

70 + 3 (from the 5) = 73

Student 3: I counted 48 on the number board. I started from the 48 (points to 48) and then I counted 25.

Teacher: Go ahead and count.

Student 3: (She points to each number as she counts by ones) and ended up with 73.

as she counts by ones) and ended up with 73.

Appendix C. Examples of Professional Development Course Assignments

Collecting Children's Work Samples Collect work samples from three children. Choose one whose work you think is strong and two whose work is not so strong. Explain why the first sample satisfies you. What is your analysis of the other two? What are your learning goals for each of the three children? Please bring three copies of the work samples and your written analysis to our second session to share with colleagues (Schifter, Bastable, & Russell, 1999a, p. 25).
Thinking about base ten For the first two sessions of this course, you have been exploring issues about the base-ten structure of the number system in two ways: (1) through examining classroom episodes that bring aspects of children's thinking to light, and (2) through doing mathematics activities yourself and reflecting on your own strategies and insights. For the next session, please respond to the following questions: 1. What ideas about numbers and the base-ten number system have been highlighted for you by these readings, videos, discussions, and activities? 2. What questions do you have? (Schifter, Bastable, & Russell, 1999a, p. 51)
Doing a student interview One major thread of this course is the examination of the ways children think as they build an understanding of the number system. One way to learn about children's thinking is to listen carefully to individuals as they articulate their thoughts while doing mathematical tasks. The math interview is one strategy for obtaining evidence about children's thinking. Doing interviews helps you develop a sense of the kinds of response that most students at a certain age or grade might give. An interview also helps you delve into the thinking of an individual child, to get underneath the surface of the child's responses to better understand his or her reasoning. Your assignment for the next session will be to plan, conduct, and write about an interview with a student. In your interview, explore the student's ideas about the number system, drawing on what we have done in this seminar. Ask the student to perform some tasks, which might include questions about reading and writing numbers, counting, adding, or subtracting. You might want to use activities or questions from the cases we have read, from the videotapes we have watched, or from activities we have done together in class. These will give you a place to start, but feel free to add your own tasks and questions. Although you will need to plan questions and tasks in advance, you will also need to follow carefully what the student does and says during the interview, so that you can follow up with questions or problems that seem appropriate. Keep in mind that your job in the interview is to find out as much as you can about the student's ideas, *not* to try to teach the student anything. Think of it almost as an assessment of the student's knowledge. Tape-record the interview so you will be able to listen to it later. Include the following in your write-up: what questions you planned to investigate, what tasks you chose, what happened during the interview, what did the child say or do, what you learned (or didn't learn) about the student's ideas, what surprised you or confused you, what questions you are left with, and what you learned from conducting this kind of interview. Give enough information so that the reader can understand the highlights of what happened and what you learned (Schifter, Bastable, & Russell, 1999a, p. 62).
Thoughts on teaching mathematics

In the cases you have been reading, the teacher-authors have been presenting the conversations of their students in detail. In the course, we have used these student discussions to analyze the children's mathematical ideas. Turn now from the course to think about your own teaching. 1. In what ways does your understanding of your own students' ideas help you in your work as a teacher? 2. What is the importance or usefulness to the students of expressing and discussing their own and their peers' mathematical ideas in class? Please use one or two examples from the casebook, the videos, your own course experiences, or from your own classroom as you write about these two questions (Schifter, Bastable, & Russell, 1999a, p. 43).
Invented strategies vs traditional algorithms In the first few classes, you have seen and read about children inventing their own strategies for performing arithmetic computations. Consider your responses to the readings, the videos, and the discussions to explain your ideas about the following questions. 1. What mathematics do children need to understand in order to be able to invent their own strategies and approaches? 2. What is the relationship between these invented strategies and the traditional algorithms? What questions does this work raise for you? (Schifter, Bastable, & Russell, 1999a, p. 71)
Changes in your classroom The class so far has really focused on the central ideas that underlie number sense and on students' thinking as they encounter those ideas. We haven't spent a tremendous amount of time focusing on the teachers' actions but I would you to respond to the following two questions. 1. Have these sessions affected how you think about teaching? Have you noticed any shift in what you are thinking as you work with your students in your mathematics class? 2. Are there ideas from reading the cases or watching the videos that your are planning to implement in your classroom. If so what are they? (Schifter, Bastable, & Russell, 1999a, p. 81)
A case from your class For the last seven classes, you have read cases written by teachers describing events in their own classrooms and examples of student thinking. To have an opportunity to examine your own teaching practice, you will write a case based on your own class. This case should present the mathematical thinking of one or more of your students. To make the narrative understandable to your colleagues, include details such as dialogue and what you were thinking as you worked with the students. To capture dialogue, some teachers find it helpful to tape-record a class session, in addition to taking notes during a class discussion. Your write-up should include your analysis of the mathematical thinking of the student(s) and questions it raises for you. We will meet in small groups according to case theme to read and discuss each other's cases. Please bring three copies of you case to class to discuss with your colleagues (Schifter, Bastable, & Russell, 1999a, p. 100).
Examining your curriculum In this assignment, you will examine your curriculum materials to determine what mathematical ideas it could raise for your students and to consider how you might highlight those ideas. Read over your materials and choose one activity to focus on. You should not feel limited by the particulars of the lesson. By focusing on its mathematical and pedagogical potential, consider how you could use this material with your own students.

Once you are familiar with the lesson/activity, consider the following questions:

1. When using this activity, what mathematical ideas would you want your students to work through?
2. How would you work to bring that mathematics out?
3. How would you modify the lesson to make it more accessible or more challenging for your students?
4. What questions might you ask as you watch your students work?
5. What might you learn about their understandings by listening to them or by observing them?

At our next class, you'll have a chance to share ideas with your colleagues. Please bring three copies of your portfolio writing to class and a copy of the curriculum activity upon which you are basing your lesson. (Schifter, Bastable, & Russell, 1999a, p. 96)

Reflecting on the cases

Go through the previous cases we have read for this course so far. Pick one case that has had an impact on your thinking about mathematics, about learning mathematics, or about teaching mathematics. Describe that impact and your ideas. What made the case evocative for you? You will share these ideas with your colleagues at Wednesday's class (Schifter, Bastable, & Russell, 1999a, p. 105).

Appendix D. Example of Data and Coding from Data Matrix

Code	Participant	Background	Data
Investigations Lesson – Sharing Strategies	BethAnne	Sex: Female Age: 25-30 Years Teaching: 5 Census: Asian-American Leader Status: Building Teacher Leader Courses taken: Building a System of Tens and Making Sense of Operations School: Spruce Elementary School Grade: 2-3 Multiage Teacher Teaching 2nd grade mathematics	B: 50 + ____ = 100 S: 50, you can put sticks of 10 together and count them. *B: Who used a different strategy?* S: 5 tens here, 5 tens there = 10 tens
Investigations Lesson – Sharing Strategies – **Probing student's thinking**			B: 60 marbles in a jar, how many to 100 S1: 60 + 50 = 100 **B: What kind of strategy did you use to figure that out?** S1: I mean 60 + 40 = 100 **B: What kind of strategy did you use to figure that out? (same tone in voice as last question)** S1. I added 4 tens.
Investigations Lesson – Sharing Strategies – **Probing student's thinking**			B: 75+ ________=100 S2: 25 **B: What makes you think that?** S2: two tens here and 5. [student points on 100s chart to 90 and 100.] B: What 5 are you pointing at? S2: 76, 77, 78, 79, 80 *B: Can someone explain what S2 is doing?* S3: Goes up to 75 and points to 85, 95 and counts 96, 97, 98, 99, 100
Investigations Lesson – Sharing Strategies – **Probing student's thinking**			B: Who can share a strategy for figuring out how many paper clips were in the box if your dropped 43? S: 43 + 7 = 50, 50 + 40 = 90, 90 + 7 + 3 = 100 **B: So then how did you figure out how many paper clips in the box?** S: I added 7 + 3 = 10, 40 + 10 = 50, 50 + 7 = 50
Investigations Lesson – Sharing Strategies – **Probing**			*B: Who had a different strategy?* S2: I used the numbers chart? I knew there was 50 left. **B: Did you count by ones or 10s? Come up and show me.** S2: I counted by ones to 100(comes up

student's thinking			to overhead and points as he counts to 100 pointing to 44, 45, 46, etc but saying 1,2, 3)
Investigations Lesson – Sharing Strategies – **Probing** **student's thinking**			S3: I started at 44 and counted to 50 then counted by 10s to 100. **B: Why did you start at 44 and not 43?** S3: Because we already counted the 43.
Investigations Lesson – Sharing Strategies			S6: I started at 43 then counted 10(pointing to 53) 20 (pointing to 63), 30(pointing to 73), 40(pointing to 83), 50 (pointing to 93). Then I counting on 1(pointing to 94), 2(pointing to 95), 3(pointing to 96), 4 (pointing to 97), 5 (pointing to 98), 6 (pointing to 99), 7(pointing to 100), B: Some people counted by the ones first and then other people counted by 10s fist. Both are good strategies.