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Redefining the whole: Common errors in elementary preservice teachers' self-authored word problems for fraction subtraction

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Abstract

A goal of this study was to examine elementary preservice teachers' (PSTs) ability to contextualize and decontextualize fraction subtraction by asking them to write word problems to represent fraction subtraction expressions and to choose prewritten word problems to support given fraction subtraction expressions. Three themes emerged from the data: (a) subtraction problems were represented by an incorrect redefinition of the whole; (b) the type of unit chosen for the whole (e.g. cups, gallons, pounds vs. pizzas, pies) influenced the success of PSTs in representing Separate (Result Unknown) context problems for subtraction; and (c) the structure of the problem influenced PSTs' performance in writing subtraction word problems.

Introduction

"Students learn mathematics through the experiences that teachers provide. Thus, students understanding of mathematics, their ability to use it to solve problems, and their confidence in, an disposition toward, mathematics are all shaped by the teaching they encounter in school. The improvement of mathematics education for all students requires effective mathematics teaching in all classrooms" (NCTM, 2000, 16-17). It is fairly well accepted that teachers need pedagogical content knowledge to teach effectively (Shulman, 1986). In more recent years, mathematical knowledge for teaching (MKT) has been explicated by Ball and her colleagues (Ball & Bass, 2000). While they have sought to define the mathematical knowledge for teaching in general, the task still remains to define what that knowledge should include for specific mathematical concepts, in this case fraction operations. Most would agree that it is not enough to just be able to compute with fractions. The question remains, what must teachers be able to do to convey the concept of fraction operations? Is it enough for teachers to be able to select appropriate word problems for representing operations for fractions or must they also be able to create problems to support such expressions? As expectations of students are increased in light of the Common Core State Standards (CCSS) (National Governors Association (NGA) & Council of Chief State School Officers (CCSSO), 2010) do the expectations of teachers need to be adjusted as well? The purpose of this research was to explicate preservice elementary teachers' knowledge with respect to these aspects. What follows are results of a qualitative study describing elementary preservice teachers' (PSTs) struggles related to selecting and writing appropriate word problems to support fraction subtraction along with a description of scenarios that inhibited or supported success with this task and the research implications for the classroom.

Background

Shulman (1986) brought to the forefront of mathematics education literature the need to describe and examine the pedagogical content knowledge of teachers. Shulman examined the cognitive research on learning and shifted the focus from learner to teacher, describing the knowledge the teacher needs to possess to be an effective teacher of mathematics. Pedagogical content knowledge research examines this specialized type of knowledge which is a need specific to the teaching of content, not necessarily needed directly for either the practitioner or the learner of the subject matter content. Likewise, Ball and Bass (2000) have worked throughout the last two decades to further define pedagogical content knowledge as it relates specifically to mathematics with what they term Mathematical Knowledge for Teaching (MKT). Through their work with elementary school preservice and inservice teachers, they have come to define specific types of knowledge for mathematics in general, namely Knowledge of Content, Knowledge of Students; however, the research is limited when it comes to defining these types of knowledge within specific mathematics domains. This research aims to expand on these efforts to define what that knowledge should include for teaching fraction operations, specifically subtraction of fractions in context.

In the same manner as Shulman, we shift the focus of a known area of mathematics education research from the learner's perspective to that of the teacher. Research on the assessment of the depth of mathematical conceptual knowledge possessed by students has been examined through the evidence of students' abilities to author their own word problems that utilize the operations being studied (Alexander & Ambrose, 2010; Alibali et al., 2009). Extending this idea to teacher education, it is reasonable to use the same strategies to assess PSTs' conceptual understanding of mathematical concepts (Drake & Barlow, 2007; Whittin & Whittin, 2008).

When *students* are able to create their own word problems related to a new mathematical concept it has a positive influence, not only on their understanding, but also on their problem solving skills and disposition towards mathematics (Barlow & Cates 2007). In addition, original student-authored word problems can reveal a variety of misunderstandings that students may hold (Alexander & Ambrose, 2010). Researchers from the Wisconsin Center for Education Research (Alibali, Brown, Stephens, Kao, & Nathan, 2009) conducted a survey using student-authored story problems focused on middle school students' understanding of equations. The results suggested that middle school students have substantial difficulty generating stories to correspond with algebraic equations. Not surprisingly, students who were more successful generating stories were also more successful solving such equations. Our research sought to apply this research to the arena of *preservice teacher preparation* by focusing on "student"-authored word problems based on fraction subtraction with the "students" being PSTs.

Friske (2011) used student-authored word problems to assess her sixth-grade students' understanding of fraction operations. Through the process of examining her data related to her students' abilities to write fraction word problems, Friske realized that she did not take into account the variety of problem structures for fraction word problems. She actually was not even aware that there were different types of problems to be represented. Her own misconception, which was connected to the structure of the problem, had inadvertently been conveyed to her students. We wondered, as researchers, if PSTs had similar misconceptions related to fraction operation word problems that might then be inadvertently conveyed to their future students.

Ball (1990) found that PSTs "demonstrated that they wanted to give the pupils what they considered to be meaningful answers, but often they could not do so because their subject matter knowledge ... was insufficient to act on that commitment" (p. 142). In order to improve a teacher's ability to provide multiple explanations, varied instructional strategies, and in the case of this research, varied problem structures, the PST's own knowledge must be improved (Shulman, 1987).

Ma (1999) reported findings related to the pedagogical content knowledge of 23 U.S. teachers and 72 Chinese teachers who were asked "to compute 1 ¾ ÷ ½, and to represent meaning for the resulting mathematical sentence" (p. 55). Only 43% of participating American teachers could calculate this division of fractions problem accurately; additionally, most of them could not accurately represent division of fractions in a word problem. Ma's research supports the notion that many teachers in the United States are unable to provide contextual support for dividing fractions. Is this also the case for other fraction operations, particularly subtraction of fractions?

In examining contextual support for subtraction, it is also relevant to examine the problem structures elucidated by the research of Carpenter et al. (1999) through their work with Cognitively Guided Instruction (CGI). When the PSTs create context problems to model subtraction of fractions, will the problem type make a difference in their ability to create accurate story problems? Are some problem types more problematic?

Purpose

Assessing student learning and deciding on a means of instruction is one of the most critical decisions a teacher has to make (Hiebert et al., 1997). Students' mathematical proficiency is shaped by the learning experiences that result from the tasks assigned by their teachers. When designing these tasks, teachers must create activities that are relevant to their students' lives and believable to students (Gravemeijer, 2004). But more importantly, teachers need to be able to make sense of the mathematics they teach on a level that is deep enough for them to understand their students' thinking and the mathematical activities that support that thinking. "Quality of instruction is a function of teachers' knowledge and use of mathematical content, teachers' attention to and handling of students, and student's engagement in and use of mathematical tasks" (NRC, 2001, p. 424). Ball and Bass (2000) argue that teachers must be able to "work backwards from mature and compressed understanding of the content to unpack its constituent elements" (p. 98).

One goal of this study was to investigate elementary PSTs' conceptual understanding of subtraction of fractions and how those understandings influenced the PSTs' ability to write word problems for subtraction of fractions (for a discussion of all of the fraction operations see Dixon and Tobias, 2013). If the teacher is not confident in his/her own understanding of this topic, s/he may have the tendency to address fraction operation concepts devoid of context. Context is vitally important for students to make sense of the mathematics they are learning in the classroom (Gravemeijer, 2004). If the teacher is not able to create, or interpret, relevant and accurate contexts, the students are less likely to make sense of the mathematics in meaningful ways. A second goal of this study was to explore types of problem structures aligned with the Cognitively Guided Instruction model to determine if particular types were more supportive of making sense of fraction subtraction than others (Carpenter et al., 1999; Carpenter, Fennema, & Franke, 1996). This research sought to answer the following questions:

1. What conceptually-based errors occur when preservice elementary teachers write word problems to support subtraction of fractions?
2. What contexts and problem structures are helpful in writing word problems for subtraction of fractions?

For the purposes of this study, prior research related to the use of student-authored word problems coupled with identified difficulties elementary school teachers tend to encounter with fraction operations guided the research design. Elementary PSTs' knowledge and understanding of fraction subtraction was identified and explicated through the use of PST-authored word problems. What follows is a detailed description of the methodology, data, and analysis.

Procedures

This study consisted of 19 PSTs who were enrolled in a graduate elementary mathematics methods course in a large, urban university in the Southeastern United States. This course was designed for graduate students with degrees in fields other than education and who chose to pursue teaching elementary school. The instructor for the course was an experienced professor of mathematics education and a member of the research team. The course was designed to focus on both methods and content for teaching elementary school mathematics with the methods in the foreground and content knowledge for teaching in the background. Students were frequently situated in small groups, each working on the same tasks. Once students had explored the tasks in small groups, the class would meet as a whole group to discuss their solutions and solution strategies. The instructor facilitated all small- and whole-group discussions. The class met once per week for three hours each week. The major topics of the class were guided by learning trajectories within the content areas of whole number concepts and operations, fraction concepts and operations, geometry, and linear and area measurement.

Instructional Design

The use of learning trajectories in mathematics is not new, but has gained momentum with the introduction of the CCSS and the work of the Center on Continuous Instructional Improvement (CCII). In the document, *Learning Trajectories in Mathematics: A Foundation for Standards, Curriculum, Assessment, and Instruction* (Daro, Mosher, & Corcoran, 2011), learning trajectories are described as

> empirically supported hypotheses about the levels . . . of thinking, knowledge, and skill in using knowledge, that students are likely to go through as they learn mathematics and . . . reach or exceed the common goals et for their learning. Trajectories involve hypotheses both about the order and nature of the steps in the growth of students' mathematical understanding, and about the nature of the instructional experiences that might support them in moving step by step toward the goals of school mathematics (p. 12).

Trajectories help the teachers clearly identify interim goals for students in learning a particular mathematical concept and help to define formative assessments relative to the trajectory rather than solely in comparison to their peers. Trajectories provide models of student thinking, which can assist teachers in making sense of student work and providing experiences that develop conceptual understandings of mathematics. Students' experiences with fraction concepts and whole number operations need to be solidified before students start to operate on fractions (Common Core Standards Writing Team, 2011). Using these types of trajectories give teachers informal and formal assessment data to drive instruction; however, at the same time, it requires a higher level of teacher knowledge of curriculum, content, and pedagogy. "It is one thing to talk theoretically about learning trajectories and a whole other thing to understand how to transfer the knowledge from learning trajectory research to practice in a way that teachers can embrace it" (Daro, Mosher, & Corcoran, 2011, p. 35).

When developing fraction operation concepts, a teacher's instruction should build off of previously learned whole number operation concepts. For example, in the case of division of fractions, Sharp and Adams (2002) found that students naturally applied concepts of division of whole numbers to division of fractions. By using similar contexts to whole number division, students were able to invent their own common-denominator algorithm for the division of fractions. It was one goal of the course to focus on the use of trajectories by modeling through instruction, which focused on correcting predictable errors and misconceptions and sense making on the part of the PST.

As with operations with whole numbers, students need to be aware of the units that accompany the given quantities with fraction operations. As indicated in the *Progressions for the Common Core State Standards in Mathematics (draft), 3-5 Number and Operations - Fractions* (Common Core Standards Writing Team, 2011), students should attend to precision in examining addition and subtraction of fractions by attending to the underlying unit quantities. "In order to formulate an equation of the form A + B = C or A - B = C for a word problem, the numbers A, B, and C must all refer to the same (or equivalent) wholes or unit amounts" (p. 7). For example, ¾ of a pizza + ½ of a pizza = 1 ¼ of *two* pizzas. In the latter example, the addends each refer to one pizza as the whole and the sum refers to *two* pizzas as the whole or unit. If students are expected to attend to this unit quantity, teachers must also be able to attend to the same precision and to identify what units are appropriate for fraction addition and subtraction contexts. The knowledge that teachers should have must not only be focused on fraction operations procedures but on listening to students' mathematical thinking while they construct or solve word problems as it is paramount in understanding and gauging their conceptual knowledge; however, this is not easily accomplished (Fennema et al., 1996; Tirosh, 2000). Through the instructional design of this course, it was hoped that PSTs would experience this type of learning for themselves and then eventually be able to translate that learning back into their own classrooms. Throughout the course, whenever word problems were examined, the work associated with Cognitively Guided Instruction was examined, particularly the problem structures described.

Cognitively Guided Instruction

The work of research associated with Cognitively Guided Instruction (CGI) included a clear description of problem structures for word problems along with examples of how students solved word problems with these various structures. The aspect of CGI that was used in this study was the problem structures for addition and subtraction word problems. Word problems dealing with addition and subtraction can be sorted into four classes: Join, Separate, Part-Part-Whole, and Compare (Carpenter et al., 1999). Within the Join and Separate classes there are three types: Result Unknown, Change Unknown, and Start Unknown. Within the Part-Part-Whole class there are two types: Whole Unknown and Part Unknown and within the Compare class there are three types: Difference Unknown, Compare Quantity Unknown, and Referent Unknown. When examining word problems dealing with addition and subtraction it is helpful to be familiar with the different types of problems and how their structure is related to how children solve the problems (Carpenter et al., 1999). Because of this, problem types played an important role in discussion around word problems for whole number and fraction operations. In our study when the PSTs were asked to write word problems they were often directed to write problems with specific structures.

Participants

Of the PSTs who agreed to participate in the study, 17 completed a fraction survey, which required them to select a context to support a mathematical expression involving fractions. The survey served as a pretest for knowledge of contexts to support operations with fractions prior to the unit on fractions and was developed by the researchers with attention to including various problem structures, each of the four operations, and several types of contexts. Results from this survey assisted the researchers in developing an interview protocol, which was adaptive to the participants' responses to the survey items and focused primarily on fraction subtraction due to PST's overwhelming difficulty with this area.

A sample of the PSTs was selected to participate in an interview based upon the participants' willingness and availability to schedule an interview. Nine PSTs were interviewed prior to the unit on fractions. The survey results for the interviewed participants were representative of the overall group. The survey results assisted the researchers in explicating the participants' thinking during the interview.

Each participant interviewed was asked to write a word problem for 4/5 -1/2 using a situation involving pizza. They were then asked to write a word problem for 5/6 - 1/3 using a situation involving gallons of iced tea. We required the specific contexts of pizza and gallons of iced tea to examine if using a standard unit of measurement (a gallon instead of a pizza) might affect the results. Following writing the word problems, interviewees were asked to revisit their responses to the two survey items that dealt with subtraction (see Figure 1). For each item, the interviewee was asked to provide a justification for his/her answer choice.

Observation field notes were collected during class instruction on portions of the fraction unit focused on operations with fractions. Prior to this portion of the unit, instruction had focused on using context, manipulatives, and drawings to make sense of fractions in sharing situations; to model equal parts of the whole using area, linear, and set models; and in examining the relationship between the defined whole and the corresponding parts.

Fraction operations were explored by having groups of PSTs write word problems to represent given expressions. The word problems were discussed with the whole class so that misconceptions could be identified and resolved. Resolution of misconceptions often involved using drawings to illustrate solutions to the word problems and contrasting those drawings with those representing solutions to the fraction expressions. The fraction unit spanned two three-hour class sessions, with one class session devoted to fraction concepts and the other to fraction operations. The researchers audio recorded the class session on fraction operations in order to gather data related to the results of the survey and interviews. At the end of the course, the PSTs were administered a final examination. Students were required to

4/5 – 1/2
a) Kenny had 4/5 of a chocolate chip cookie leftover. Christopher ate 1/2 of the leftover chocolate chip cookie. What fraction of the cookie is left now? **b) There was 4/5 of a cup of sugar in the pantry. Mom used 1/2 of a cup of sugar from the pantry to make chocolate chip cookies. How much sugar is in the pantry now?** c) Lulu played 4 innings at her softball game last night. A softball game is 5 innings long. Chloe played 1/2 the innings Lulu played. Write a fraction that represents the part of the game that Chloe played. d) Martha was traveling from New York to San Francisco, 4/5 of the way from New York to San Francisco is Phoenix, Arizona. Martha made it to Phoenix on day two of her trip. On day one, she went 1/2 that distance. What fraction of the trip did she make on day two?
7/8 – 1/4
a) Mark ate 7/8 of a medium pepperoni pizza. Kenny ate 1/4 of a medium cheese pizza. How much more pizza did Mark eat than Kenny? b) Mark had 7/8 of a cake leftover from a party. For dinner he ate 1/4 of his leftovers. How much cake does Mark have left? c) Steven was able to get seven-eighths of his house painted on Saturday. On Sunday he was able to do one-fourth of the amount of painting that he did on Saturday. How much of the house is left to be painted? d) Emmett ran 7/8 of mile. Timmy ran 1/4 of a kilometer. How much further did Emmett run than Timmy?

Figure 1: Survey Subtraction Items (correct response in bold type)

respond to 5 of 7 tasks. One task they could select was related to fractions and asked them to "Write a Separate (Result Unknown) word problem for 6/8 - 2/3."

The research team consisted of five mathematics education doctoral students and two mathematics education faculty members. Data from surveys, interview transcripts, class transcripts, class observation notes, and the final examination were analyzed by the researchers using the Constant Comparative Method (Glaser & Strauss, 1967). At least two researchers separately analyzed each type of evidence and differences in analysis were discussed and resolved. The researchers convened as a team to review the analyses. Three themes emerged from the data: a) subtraction problems were represented by an incorrect redefinition of the whole; b) the type of unit chosen for the whole (e.g. cups, gallons, pounds vs. pizzas, pies) influenced the success of PSTs in representing Separate (Result Unknown) context problems for subtraction; and c) the structure of the problem influenced PSTs' performance in writing subtraction word problems. Each theme is further discussed in the results and findings.

Results

Results of the initial survey indicated that PSTs experienced difficulty with fraction subtraction, particularly in comparison type problems (see Table 1). Additionally, multiplication and division of fractions was difficult, but this was likely compounded by confusion between multiplication and subtraction contexts as indicated by PSTs during interviews.

Table 1: Results of Survey of Selecting Context to Support Operations with Fractions (N = 17)

Problem	Context	Percent of Correct Responses
5/6 + 2/3	Part-Part-Whole (Whole Unknown)	71%
4/5 – 1/2	Separate (Result Unknown)	59%
7/8 – 1/4	Compare (Difference Unknown)	29%
1/3 x 3/4	Part of a Part	53%
6/7 ÷ 3	Sharing/Partitive Division	65%
3 ¼ ÷ ¾	Measurement Division	41%

For the addition expression, 12 of the PSTs chose the correct context. However, for the two subtraction expressions, 10 or fewer of the PSTs were able to choose the correct context. Most notably, for the Compare (Difference Unknown) problem, only five PSTs chose the correct context and nine of the participants chose the same incorrect representation of Separate (Result of Unknown). This is the only problem for which more respondents chose the same incorrect context compared to the correct context. For the multiplication expression, nine PSTs selected the correct context. And lastly, for the division expressions, the results were different based on the type of problem. Eleven participants chose the correct context for division by a whole number while seven of them chose the correct context for division by a fraction. These results guided the choices for interview tasks (see Figure 1) and subsequent focus of data collection. As the data were analyzed, three themes emerged which are explicated below.

Subtraction problems included an incorrect redefinition of the whole. When asked to write subtraction word problems for 4/5 - 1/2 and 5/6 - 1/3, of the nine PSTs interviewed, only one PST was successful. The other eight PSTs all chose to write problems that represented Subtraction (Result Unknown) but were unsuccessful in modeling the given expression. As described in *Progressions for the Common Core Standards in Mathematics (draft), 3-5 Number and Operations, Fractions* (Common Core Standards Writing Team, 2011), when representing subtractions as A - B, A and B must refer to the same size whole or unit amount. Similarly, when representing such expressions with meaning in context, the quantities must also refer to the same size whole. What we observed in transcripts of student responses was that the whole was actually redefined from the minuend to the subtrahend such that A (the minuend) referred to a whole and B (the subtrahend) represented a scaled version of A with the scale factor of B. In essence, when the eight PSTs attempted to write a word problem to represent A - B, they actually wrote a word problem to represent A - B*A. As such, the expression could have been represented as A - D where both A and D *did not* refer to the same size whole or unit amount. As a result the whole was redefined.

A common incorrect response for 4/5 - 1/2 was:

> "Sean had 4/5 of a pizza leftover from yesterday. He ate half of his leftover [pizza] for lunch today. How much pizza is left for the dog?"

This word problem may seem to represent subtraction but it does not represent the *given* subtraction expression. The word problem as it was written above incorrectly interprets the operation taking place by beginning with Sean's 4/5 *of a pizza* and subtracting 1/2 *of his leftovers.* In order to subtract quantities, the units must be the same. The story problem written above subtracts *leftovers* from *pizza.* The units are not the same. The unit or whole of a *pizza* was redefined as *leftovers* from a pizza. The word problem, as written, actually represents a multi-step subtraction problem of 4/5 of a pizza minus 1/2 of 4/5 of a pizza, or 4/5 - (1/2 x 4/5).

A representative incorrect response for 5/6 - 1/3 was:

> "Beth has 5/6 of a gallon of iced tea. If she drinks 1/3 of that, how much will she have left?"

As with the earlier example, this word problem depicts 5/6 - (1/3 x 5/6) rather than the given expression.

During class, at the beginning of the fraction subtraction lesson, PSTs were working in small groups to write a word problem for 4/5 - 1/2. This occurred prior to formal instruction on fraction subtraction. The instructor asked students from each group to share their word problems. Remarkably, every single small group made the same error in their word problems. In each case, the word problem actually depicted 4/5 - (1/2 x 4/5). Notice in the transcript that follows how students were able to determine that their problems were similar in structure from one to the next but they did not notice their error.

PST from First Group: *OK. Ours was, I had 4/5 of a cupcake left. I ate a half of my remaining cupcake, how much cupcake will I have left?*

Instructor: *OK. You guys [pointing at another group].*

PST from Second Group: *Ours was..., hum, Johnny has 4/5 of cake left over from his graduation. He ate half. How much cake is there left over?*

Instructor: *OK, same or different? [Silence] Same, right? [nods and expressions of agreement] Next group.*

PST from Third Group: *I had 4/5 of a pizza. If I eat 1/2 of that pizza, how much is left?*

Instructor: *Same or different? [brief pause] Same. Isn't that interesting? [nods and expressions of agreement] Ok, go ahead.*

PST from Fourth Group: *Mine is the same with more cake.*

Instructor: *And yours, [pointing to the last group] is it the same?*

PST from Fifth Group: *Yes, but with pizza.*

Instructor: *Well, do you notice all the problems are the same? Isn't that interesting?*

...

Instructor: *So, they are all wrong. [there is a paused silence and nervous laughter] Let's look at this one..."*

PST: *So they are all wrong? I can't believe they are all wrong. [there is murmured agreement.]*

The students shared all of their responses and analyzed them to determine that they were similar in structure without realizing that the responses were actually incorrect. They had represented the minuend as a part of the whole and the subtrahend as a part of the part. However, with subtraction both the minuend as a part of the whole and the subtrahend as a part of the part. However, with subtraction both the minuend and the subtrahend must be based on the same unit. Consider the first word problem shared, *"I had 4/5 of a cupcake left. I ate a half of my remaining cupcake, how much cupcake will I have left?"* It would have needed to be worded something like, *"I had 4/5 of a cupcake left. If I eat half of a cupcake from what I had remaining, how much cupcake will I have left?"* The difference would be to change the "half of my remaining cupcake" to "half of a cupcake from what I had remaining." Grammatically, the sentence seems to have changed very little, but mathematically, the two expressions have completely different meanings.

As students made sense of their error, there was a need to reinvent their understanding within the context of fraction subtraction. It was interesting to observe how they were engaged in constructing viable arguments and critiquing the reasoning of others as described in the Standards for Mathematical Practice of the CCSS (NGA & CCSSO, 2010). Through discussion, PSTs engaged in dialogue with each other related to redefining the whole in subtraction problems. However, they continued to struggle with incorrectly redefining the whole as opposed to keeping the whole consistent as illustrated in the following transcript.

Sarah: *Well, 4/5... We were all talking about it... like 4/5 of a cookie and then we started talking about taking ½ of a cookie...*

Instructor: *Ok, ½ of what?*

Sarah: *Take ½ of the 4/5 of the cookie, if we do that then we would take 4/5 and divide it into 2 equal parts instead of subtracting ½ of a cookie.*

Instructor: *So, what does it mean?*

Jessica: *I think mistake is that we are dividing 4/5 into half instead of 5/5*

Instructor: *So, is this a division problem? So dividing 4/5 in half instead of 5/5 in half. What do you mean by that?*

Jessica: *It is supposed to be, there is 4/5, are we taking half of that? Or is it half of a whole?*

Instructor: *That is the question. So, what do you mean? What are you saying?"*

Zachary: *I think I know what Jessica is saying . . . are we . . . like we are dividing that 4/5 into half rather than subtracting half?*

Instructor: *OK, you guys are saying similar things. How are they the same?*

Zachary: *I think that we are subtracting half of that 4/5 instead of 1/2 of the whole.*

Instructor: *And that is what you just said too [pointing to the first students]. So...*

Sarah: *Clearly it's wrong; still, the problems are wrong . . . I am sorry . . . because since the problem is giving you 4/5, then the 4/5 will be the whole. But, I mean, I could be wrong...*

Instructor: *What do you mean?*

Jessica: *I mean, if you are given a number, why would you assume it is another number? That the half refers to another number. Why would we assume it is 5/5, when we are given 4/5?*

Alex: *I know what they mean, wouldn't they say 1 – 1/2 if they wanted us to subtract it from the whole rather than saying from 4/5?*

Dialogue continues and concludes with:

Instructor: *Let's go back to those first problems that you wrote: 'I have 4/5 of a pizza. If you eat ½ of my 4/5, how much is left?' Every one of you wrote a word problem that beautifully represented the wrong problem. You represented this problem [pointing to 4/5 – (1/2 x 4/5)'. Isn't that amazing?*

...

Sarah: *So, what you mean, that when we are subtracting 4/5 – ½, the whole is still the pizza, it is not the 4/5?*

Instructor: *This problem is 4/5 of a whole minus ½ of a whole.*

PSTs had difficulty identifying their own errors; however, through facilitation by the instructor, PSTs came to the correct conclusion that the whole must remain constant throughout the subtraction problem. The class session related to fraction operations occurred in the 10th week of a 15-week semester. At the end of the semester, PSTs were to answer five of seven given problems on the final examination. One of those tasks was to write a Separate (Result Unknown) word problem for 6/8 – 2/3. Eighteen out of 19 PSTs provided responses to this prompt, demonstrating their confidence with the prompt. Eight of them wrote incorrect word problems, however, and of those eight, 7 incorrectly redefined the whole as in the following example: "I have 6/8 of a pizza in my fridge. I ate 2/3 of the pizza in my fridge. How much pizza do I have left?" This demonstrates that, although more than half of the PSTs demonstrated the ability to write a Separate (Result Unknown) problem, misconceptions related to incorrectly redefining the whole persisted.

When examining the incorrect and correct representations for fraction subtraction in context, it became evident that the choice of unit influenced the ability to make sense of the Separate (Result Unknown) problem structure. This is the second theme identified through analysis of the data.

Units of measure influence success with representing subtraction in context. While students continued to have difficulty using the Separate (Result Unknown) problem structure, the type of unit chosen for the whole (e.g. cups, gallons, pounds vs. pizzas, cupcakes, cookies, tanks of gas) seemed to influence the success PSTs had in writing word problems. Consider the following dialogue, which is a direct continuation from the dialogue in the previous section. Notice how students used the unit of measure, in this case a cup, to assist them in avoiding redefining the whole.

Paige: *So, if I have 4/5 of a cup of milk and I use 1/2 of a cup of milk. That would be this.*

Instructor: *Would that be this (referring to the expression 4/5 – 1/2)? Say it again.*

Paige: *If I have 4/5 of a cup of milk and I use 1/2 of a cup of milk, so I could say, half of a cup of milk, not just half.*

Instructor: *Half of the milk is what you said earlier.*

Paige: *Half of the milk.*

Instructor: *Right, I do not use half of the milk but* 1/2 *of* ***a cup of*** *milk, so that keeps them separate, that is how I see it in my head at least. Does that help you? [Noticing a student who seemed confused]*

Alex: *A little.*

Instructor: *Which one makes more sense to you, the pizza or the cup of milk?*

Alex: *Say the pizza like this?*

Instructor: *OK, say the pizza like this.*
Alex: *Well I think I got it!*
Instructor: *So teach this class right now!*
Alex: *OK, I have 4/5 of a pizza. I ate half of [Prolonged pause here] does it have to be food?*
Paige: *I don't think you can do that with food, can you?*
Instructor: *OK, milk is food. So what is the difference between the pizza and the milk?*
Paige: *Mine has a measurement like a cup of milk.*
Instructor: *OK.*
Paige: *And the pizza is pizza, it is not like a cup of pizza.*
Instructor: *OK, so what is your whole?*
Paige: *I can't figure that out.*

Even when the students seemed to make sense of their errors in redefining the whole, they had trouble with the language used to define it. Students were more successful representing problems when they used a standard unit of measure such as a cup rather than a unit that was less clearly defined, such as pizza, because they could visualize the whole more clearly. Making sense of the whole is crucial as it is at the center of the concept of fraction subtraction. NCTM posits that, "the concept of 'unit' is fundamental to the interpretation of rational numbers" (2010, p. 19). According to Smith (2002), this is due to a detachment between paper and pencil representations of fractions and mental visualization because, "both are acts of mental construction, but only one is visible" (p. 6).

Students were actually shocked when they finally came to the understanding of the need to refer to the same unit for the subtrahend and minuend. Notice how the students are able to begin to verbalize the need for the same whole and the surprise at this understanding in the transcript that follows.

Sarah: *I have 4/5 of a gallon of iced tea. I am thirsty so I am going to drink ½ of a gallon of iced tea. How much iced tea will I have left?*
Instructor: *Does that work?*
Sarah: *Yes.*
Instructor: *Why does it work?*
Sarah: *Because I minus, the same*
Instructor: *Because I subtract from the same whole. Any questions about that?*
Sarah: *Oh my gosh! [indicating surprise and understanding]*

On the survey and during the interviews prior to class instruction on fraction subtraction, students did not experience success with fraction subtraction regardless of the type of unit used. However, on the final examination, of the ten PSTs who correctly represented 6/8 – 2/3 with a Separate (Result

Unknown) structure, seven of them used a standard unit of measure (gallons, pounds, and cups) while three PSTs used units that would not be considered standard (pizza, pie, and cake). The following is representative of a response using standard units: *I had 6/8 lbs. of fudge. I ate 2/3 lb. of that fudge. What fraction of a pound of fudge was left?* Of the eight PSTs who incorrectly represented the same problem, seven used a nonstandard unit of measure while only one PST used a standard unit of measure. This provides additional support that the type of unit chosen for the whole seemed to influence the success PSTs had with writing Separate (Result Unknown) problems since a majority of the PSTs who wrote the problem correctly used standard units. The structure of the word problem also seemed to be a factor in success with writing problems as described in the next theme.

Problem structure influences success with writing problems. The PSTs in this study associated the operation of subtraction of fractions with the Separate (Result Unknown) structure rather than with the Compare (Difference Unknown) structure. The only question on the survey that included the Compare (Difference Unknown) problem section was 7/8 - 1/4 (see Figure 1). In this task, the correct word problem to support 7/8 - 1/4 was actually the Compare (Difference Unknown) problem. This question on the survey had the least amount of correct responses at 29%. The PSTs overwhelmingly chose the response that contained the Separate (Result Unknown) structure even though it was incorrect. This option received 53% of the responses. During the post survey interviews, we asked the PSTs to explain the reason behind their choice between the correct answer - the Compare (Difference Unknown) structure - and the most popular distractor - the Separate (Result Unknown) structure - without identifying to them whether they were correct or not. Only one PST switched from the incorrect answer to the correct answer and based that change upon a realization of the difference in problem structures between the Compare (Difference Unknown) and the Separate (Result Unknown), seemingly using her earlier experiences making sense of these problem structures with whole numbers to inform her work with fractions. All the other PSTs persisted with their misconception. A representative response from a PST who selected the incorrect answer was "I chose b (indicating incorrect answer). You know why I chose it? Because it was clear to me that it was 7/8 - 1/4 because it says, 'how much cake does Mark *have left*?' and up here (indicating the correct answer) I remember, how much more pizza did Mark eat than Kenny. I wasn't sure that it was 7/8 - 1/4 but this one (referring to the wrong answer) was very clear that it was 7/8 - 1/4." With these students, the work with problem structures in whole numbers did not seem to transfer to work with fractions. It might be because their earlier school experiences focusing on key words was a stronger influence on their thinking making the Separate (Result Unknown) structure more dominant for them.

This theme of problem structure and keyword association was confirmed when we interviewed PSTs and asked them to create word problems for fraction subtraction expressions. PSTs were asked to write word problems for 4/5 - 1/2 using a context having to do with pizza and 5/6 - 1/3 using a context having to do with gallons of iced tea. Out of the nine PSTs, only one PST used the Compare (Difference Unknown) structure. This PST created word problems for both expressions using this problem structure (see Table 2) and wrote them correctly. All others used the Separate (Result Unknown) structure for both expressions and wrote them incorrectly.

Table 2: Correct Responses for fraction subtraction during interview

Expression	Word Problem
4/5 – 1/2	Bob ate 4/5 of a large pepperoni pizza. Billy ate ½ of a large mushroom pizza. How much more pizza did Bob eat than Billy?
5/6 – 1/3	Reenee bought two gallons of flavored ice tea. One gallon was peach flavored. The second gallon was green tea flavored. Reenee drank 5/6 of the gallon of peach flavored ice tea. Her husband Scott drank 1/3 of the gallon of green tea flavored ice tea. How much more ice tea did Reenee drink than Scott?

Even more interesting, the PST who used the Compare (Difference Unknown) structure as opposed to the Separate (Result Unknown) structure was the only one whose word problems correctly reflected the given expressions. All of the PSTs who used the Separate (Result Unknown) structure wrote incorrect word problems. These word problems that the PSTs created are reflected in the sample provided in Table 3.

Table 3: Sample of Incorrect Responses for fraction subtraction during interview

Expression	Word Problem
4/5 – 1/2	Sammie had 5/4 of a pizza. She got really hungry and ate ½ of that pizza. How much pizza was *left*?
5/6 – 1/3	Amy loves iced tea and had 5/6 of a gallon. She pored a large glass and ended up drinking 1/3 of her iced tea. How much iced tea was *left*?
4/5 – 1/2	Erick had 4/5 of a pizza left, he then ate ½ of the pizza. How much pizza does he have *left*?
5/6 – 1/3	Erick was thirsty. He had 5/6 gallons of iced tea in the refrigerator but Gary drank 1/3. How much is there *left* for Erick to drink?
4/5 – 1/2	Mark had 4/5 of a pizza and he gave ½ of it to Karen. How much pizza does he have *left*?
5/6 – 1/3	Candy had 3 gallons of iced tea. She drank some and now she has 5/6 of the 3 gallons left. If she drinks 1/3 more, how much will she have *left*?
4/5 – 1/2	Sean had 4/5 of a pizza leftover from yesterday. He a half of his left over for lunch today. How much pizza is *left* for the dog?
5/6 – 1/3	Sean has 5/6 of a gallon of iced tea in the fridge. I drank a third of the tea. How many gallons of tea is *left* for Sean?

All but one of the PSTs who wrote Separate (Result Unknown) word problems used the key words "left" or leftover" in their responses.

During instruction on whole number addition and subtraction, PSTs explored problem types according to those described in Cognitively Guided

Instruction (Carpenter et al., 1999). Problem types continued to play an important role in discussion around word problems for fraction operations. When the instructor drew PSTs' attention to the problem type, they were able to change their problems to both reflect the Compare (Difference Unknown) structure as well as the given expression.

Instructor: *All of you wrote a separate result unknown problem. Remember those? What if you wrote a compare problem?*

...

Riley: *Ella has 4/5 of a pizza. Ray has* 1/2 *of a pizza. How much more pizza does Ella have than Ray?*

Instructor: *Does yours sound like that?*

Alex: *Similar*

It was interesting that once one student was able to make this jump from incorrectly representing the subtraction expression 4/5 - 1/2 with the Separate (Result Unknown) structure to the Compare (Difference Unknown) structure, the rest of the class seemed to follow.

Discussion

According to the CCSS for Mathematics, "students must be given the opportunity to reason abstractly and quantitatively" (NGA & CCSSO, 2010, p. 6). This includes the need to decontextualize a problem situation as well as to recontextualize computations and solutions. Making connections with context inherently requires that there be a context from which to begin. The context could be generated by commercial resources or be developed by teachers and/or students. Regardless, the teacher's role must include making sense of the context. "Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects" (NGA & CCSSO, 2010, p. 6). Based on the results of this study, this level of reasoning may be difficult for teachers to facilitate without particular attention given to their own knowledge relative to contextualizing and decontextualizing fraction subtraction. In the case of this study, PSTs demonstrated difficult with creating word problems (contextualzing) given an expression as well as with identifying appropriate contexts from a list of options to support a given expression (decontextualizing). When contextualizing, there seemed to be an overreliance on the Separate (Result Unknown) word problem structure. In applying the Separate (Result Unknown) word problem structure, PSTs neglected the need to maintain the unit. Decontextualizing was assessed through the PSTs' need to examine word problems and mathematize them to determine if they correctly represented the expression given. Our conclusions are supported by the findings of Tobias (2009) and Dixon & Tobias (2013) who studied PSTs ability to decontextualize word problems and found that PSTs had difficulty determining if word problems represented fraction multiplication or fraction subtraction. In our study, in addition to difficulties contextualizing and decontextualizing, as PSTs reasoned quantitatively, it became clear that attention to the unit involved and focusing on precision were either problematic or held a key to success with the given tasks.

Similar to research with student-authored word problems (Alexander & Ambrose, 2010), misunderstandings that PSTs held were revealed through their self-authored word problems. These difficulties could inhibit their ability to facilitate student engagement with Standard for Mathematical Practice 2: Reason Abstractly and Quantitatively (NGA & CCSS0, 2010) as it did in Friske's (2011) case. According to Friske, her understanding of Compare (Difference Unknown) was sufficient but as the teacher, she also needed to be able to create contexts supporting other problem types. As such, and consistent with Shulman (1987) and Ball (1990), teachers' pedagogical content knowledge related to teaching mathematics must include an ability to both contextualize and decontextualize fraction subtraction. This supports similar research related to fraction division (Ma, 1999) and also must include attention to problem structures (Carpenter et al., 1999) for fractions specifically.

Conclusion

According to NCTM (2000), "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn well" (p. 16). Additionally, teachers must provide students with the opportunities to demonstrate each of the Standards for Mathematical Practice specified in the CCSS document. For fifth grade, students must add and subtract fractions with unlike denominators and solve problems involving addition and subtraction of fractions (NGA & CCSSO, 2010). When students are only presented with traditional algorithms for adding or subtracting fractions (i.e. finding a common denominator), it is unlikely the students will have the conceptual understanding of fractions called for in the standards. This kind of deep understanding is only developed when teachers provide rich, meaningful learning activities that are deliberately chosen to meet the goals set by the teacher (Hiebert et al., 1997). The preservice teachers in this study had considerable difficulty selecting and authoring correct word problems to represent given subtraction contexts. Given this difficulty, and without intervention, it is unlikely that preservice teachers with these sorts of misconceptions will become teachers who will adequately support student engagement in reasoning abstractly and quantitatively or attending to precision regarding fraction subtraction. It is unlikely that connections to real word problems will be part of the teachers' planning and instruction. Without a deep understanding of both fraction subtraction procedures *and* situations that accurately model fraction subtraction, teachers are not able to both provide appropriate learning activities and assess student learning in meaningful ways. A goal, then, is to increase focus on PSTs self-authored word problems to support fraction subtraction as a means of making sense of common misconceptions and preparing PSTs for their future facilitation of students' reasoning abstractly and quantitatively.

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