Developing Mathematical Knowledge for Teaching Teachers:
A Model for the Professional Development of Teacher Educators

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What is needed to prepare teachers to effectively teach mathematics has been the subject of considerable debate for at least two decades (Brown, Cooney, & Jones, 1990; Conference Board of the Mathematical Sciences, 2001, 2012), with the focus shifting back and forth between the number of mathematics courses that teachers need to the mathematics content that teachers need to know (Ball, Thames, & Phelps, 2008). Building upon previous work related to teacher knowledge (Shulman, 1986), some researchers have reconceptualized mathematics content knowledge and have argued that teachers need to know not only the ways that mathematics is used in applied contexts and other professions, e.g., using percentages to compute amounts of discounts, but also the ways required exclusively for teaching, e.g., evaluating the validity of the mathematics in solution methods (Ball et al., 2008; Hill & Ball, 2004). Nevertheless, we know little about what mathematics teacher educators, the individuals who are primarily responsible for the mathematical preparation of teachers, should know.

Building on our work as part of a two-year professional development project for mathematics teacher educators, and drawing from research

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on features of high-quality professional development programs, we propose a model for the professional development of teacher educators who teach mathematics content courses as part of elementary teacher preparation programs. We illustrate this professional development model by drawing on our ongoing work as part of the Mathematical Knowledge for Teaching Teachers (MKTT) Project, which is a National Science Foundation[1] funded professional development project designed to provide opportunities for mathematics teacher educators to develop their own understandings of mathematical knowledge as it influences their work with preservice teachers in mathematics content courses. While the proposed professional development model has been used only in the context of our project, our aim is to provide insight into the nature of professional development activities designed specifically for teacher educators so that we can develop ways to support teacher educator learning in broader contexts.

Background

Features of High-quality Professional Development

While there is limited research that demonstrates the impact of professional development activities on teachers' practice and, ultimately, student achievement, there is some research that offers guidance for the design of high-quality professional development programs. Smith (2001) and Stein, Smith, and Silver (1999) propose several features of high-quality professional development programs that have the potential to enhance participant learning. Although these features are presented in the context of professional development programs designed for K-12 teachers, given the parallels between the work of K-12 teachers and teacher educators, they have broad applicability to the design of professional development for teacher educators. For example, as we discuss later, just as K-12 teachers need to consider ways to support and foster students' mathematical understanding, so too do teacher educators; they need to consider ways to support preservice teachers' mathematical understanding. That is, both K-12 teachers and teacher educators need to support learners' understanding of mathematics. Drawing from Smith (2001) and Stein et al. (1999), we discuss features of high-quality professional development that hold promise for the professional development of teacher educators.

The first feature of high-quality professional development is that participants' learning is grounded in the content of teaching and learning. That is, teacher educators have opportunities to work on and engage with the mathematics concepts and ideas that they use with preservice teachers as well as to address the related instructional challenges that they encounter in their daily work. The second feature is that the activities create some disequilibrium for teacher educators. In other words, the professional development activities need to challenge teacher educators' assumptions about what preservice teachers need to know in a math content course and how they can learn the content with understanding. The third feature is that professional development activities need to encourage collaboration among participants and support the development of communities of professional practice. For teacher educators, this implies that their professional development experiences need to allow for collaborations with other teacher educators around the work of teaching mathematics to preservice teachers.

A fourth, and final, feature of high-quality professional development that has informed our work with teacher educators is that participants' learning is embedded in or directly related to the work of teaching teachers. In this way, the professional development activities are situated in the actual practice of teaching mathematics to preservice teachers and are, thus, tied directly to what teacher educators do every day in their own institutional contexts. Taken together, these four features have informed the design of our professional development activities and frame the nature of the learning opportunities that we have designed for our participants. To situate these features in the context of professional development for mathematics teacher educators, however, we need to first consider what constitutes the work of teaching teachers and what is entailed in teaching mathematics to preservice teachers.

Mathematical Knowledge for Teaching Teachers

Although most scholars and educators believe that mathematics teachers at all levels need to have a thorough knowledge of the content that they teach (e.g., Kilpatrick et al., 2001; RAND Mathematics Study Panel, 2003), there is less agreement about the precise nature of the mathematics content that teachers should learn in teacher preparation programs. Building upon previous work related to teacher knowledge (Shulman, 1986), some researchers have reconceptualized mathematics content knowledge and have argued that teachers need not only to know mathematics content, or *common content knowledge*, but also that they need to know mathematics in ways needed for teaching, or *specialized content knowledge* (Ball et al., 2008). While common content knowledge refers to the knowledge that bankers or retailers, for example, have to know, e.g., computing percentages, multiplying multi-digit numbers, specialized content knowledge refers to the mathematics knowledge that is specific to teaching, e.g., evaluating students' conjectures, anticipating unusual solution methods, and more closely resembles what teachers have to know and do with students in the classroom.

Arguably, understanding such specialized content knowledge entails a different conception of what mathematics is and how it can be learned, a conception that may be unfamiliar, given preservice teachers' prior mathematical experiences. Having to anticipate unusual solution methods, for example, assumes that mathematics tasks can be solved using a variety of methods and not simply using one "right" method. Similarly, having to evaluate students' conjectures assumes that students will make conjectures about mathematical relationships and will have to justify and explain their thinking, which are mathematical practices in which preservice teachers may not have engaged throughout their previous coursework, including their own K-12 schooling experiences. Thus, if they are expected to support students as they investigate mathematical concepts, preservice teachers need to have a strong understanding of mathematical knowledge for teaching, which includes both common and specialized content knowledge.

Broadly speaking, the nature of what both K-12 teachers and teacher educators have to do as part of their work is to consider ways to support learners' understanding of mathematics, and we posit that the challenges that these two groups face are similar. For example, as discussed above, recent advances in research on teacher content knowledge have emphasized changes in the nature of the mathematics knowledge that preservice teachers need to be effective in the classroom (Ball et al., 2008). This changing nature of mathematics knowledge can be instructionally challenging, as many preservice teachers may not consider analyzing student solutions or revising mathematical definitions, for example, as part of the domain of mathematical knowledge that they need to learn as a teacher. This implies a need for mathematics teacher educators to be able to foster such changes in the ways in which preservice teachers learn and understand the mathematics needed for teaching.

If preservice teachers need to know how to compute percentages, analyze common student errors, and connect representations to underlying ideas, for example, so, too, do mathematics teacher educators need to possess such mathematical knowledge. To support preservice teachers' thinking at a high level of cognitive complexity, mathematics teacher educators not only need to know the content that preservice teachers need to know, but they also need to know the ways in which preservice teachers engage with such content to anticipate the questions, misconceptions, and challenges that preservice teachers may have with learning this content.

As Nipper and Sztajn (2008) suggest, however, while the general instructional relationships involved in their work are quite similar for both

K-12 teachers and teacher educators, the specific nature of the content to be learned and the preconceptions and prior knowledge that learners bring to the class are what differentiate the work of K-12 teachers from the work of teacher educators. While K-12 students need to learn mathematics with understanding, preservice teachers need to learn mathematics in ways specific to teaching, which has different implications for the work of K-12 teachers and teacher educators. Many teacher educators may have limited understanding of mathematical knowledge for teaching and what learning such knowledge entails. Without such knowledge, teacher educators cannot effectively support preservice teachers' learning during instruction. Moreover, many teacher educators, in teaching content courses for preservice teachers, focus their teaching almost solely on the development of common content knowledge with too little emphasis on the development of specialized content knowledge (RAND Mathematics Study Panel, 2003). Thus, teacher educators must not only understand mathematical knowledge for teaching themselves, but they also must know how to use such knowledge in their work with preservice teachers. Nevertheless, relatively few teacher educators have had opportunities to learn about and develop such specialized content knowledge in their institutional contexts.

In his work on teacher educators, Mason (1998, 2010) suggests that the work of mathematics teacher educators is similar to that of teachers and that, in addition, the work of teacher educators involves helping preservice teachers recognize how to relate what they are learning to teaching. Specifically, Mason (1998) suggests that the work of teacher educators involves developing and enhancing different levels of awareness in preservice teachers as opposed to simply helping them learn the content that needs to be learned. "Teaching is fundamentally about attention, producing shifts in the locus, focus, and structure of attention" (p. 244). Mason (1998) argues that preservice teachers need to know how to engineer instructional situations in which students experience a shift in their attention where they (i.e., students) become aware of ideas and concepts of which they were previously unaware. Consequently, the work for teacher educators is to develop preservice teachers' understanding of certain ideas and concepts and to develop preservice teachers' awareness of how to connect what they are learning to teaching.

Mason (1998) further argues that the work of mathematics teacher educators is challenging because preservice teachers often enter their coursework with a procedural focus and, sometimes, a negative attitude toward mathematics due to their past experiences in learning mathematics. Thus, they may not always be focused on connecting what they are learning to teaching. This implies that teacher educators themselves also have to be aware of what they are attending to as a means to know what they need to make explicit or make aware to preservice teachers in the ways needed for teaching. For example, teacher educators must engage preservice teachers in mathematical explorations in ways that emphasize how their learning of mathematics, both common and specialized content knowledge, might influence their future work of teaching students. In addition, mathematics teacher educators must not only develop preservice teachers' ability to evaluate the transparency of mathematical ideas in mathematical representations for themselves as learners but also support preservice teachers in recognizing why evaluating the transparency of mathematical representations is important for planning lessons and selecting representations that will support the development of students' understandings. Such practices of teacher educators have the potential to enhance preservice teachers' awareness and to connect their learning to teaching practice.

In short, the features of quality professional development programs, together with a general understanding of what is entailed in teaching mathematics to preservice teachers, provide a foundation for designing professional development opportunities for teacher educators. In the sections that follow, we propose a model for the professional development of mathematics teacher educators. We illustrate this professional development model by drawing on our ongoing work as part of the MKTT project.

MKTT Project Overview

Project Background

The MKTT project is a two-year program whose aim is to design professional development materials for mathematics teacher educators and to implement these materials as part of a professional development project. The goals of the MKTT project are not to evaluate teacher educators' learning in terms of their preservice teachers' achievement in their coursework but, rather, to understand the nature of teacher educators' conceptions of their work as teachers of prospective elementary teachers and the extent to which their conceptions shift throughout the project. Thus, the project's focus is on understanding the work of the participating teacher educators. The main project activities include (a) designing and implementing professional development workshops based on the identified practices and accompanying artifacts; and (b) examining shifts in participating teacher educators' conceptions of what is involved in the work of teaching mathematics to preservice teachers. In this article, we focus only on the first project activity.

The professional development project included six three-hour-long workshops implemented over a year and a half. Project participants included six teacher educators (two male, four female) from various two- and four-year institutions in and around the Chicago, Illinois, area. All participants were selected because they not only taught content courses for preservice elementary teachers, but they also expressed an interest in discussing their practice with other teacher educators in the local area. They received a modest stipend for their participation in the project. In addition, all participants had at least two years' experience teaching math content courses for preservice elementary teachers at their institutions. Similar to the group surveyed by Masingila, Olanoff, and Kwaka (2012) about the design of their coursework, the teacher educators involved in the project had diverse backgrounds; they were adjunct, clinical, and tenure/tenure-eligible faculty and, thus, were typical of those who teach content courses for preservice elementary teachers in the United States. Moreover, the participants represented expertise and experience from three categories described by Bergsten and Grevholm (2008): mathematicians with mathematical sophistication, teacher educators with pedagogical expertise, and teacher educators who themselves have classroom teaching experience at the elementary, middle, and high school levels.

Design Rationale of Professional Development Model

An assumption that underlies the design of the six professional development workshops is that teacher educators who teach preservice teachers are often insufficiently prepared to support preservice teachers in developing mathematics knowledge in the ways needed for teaching and, indeed, may not clearly understand for themselves what such knowledge entails. Thus, teacher educators who teach preservice teachers need to understand mathematical knowledge for teaching for themselves and to think carefully about how to engage preservice teachers in ways that support their development of such knowledge. The workshops are designed around tasks of teaching that require specialized content knowledge, a type of knowledge that is specifically required in the work of teaching mathematics (Ball et al., 2008). The aim is to provide opportunities for participants to discuss aspects of preservice teacher learning, to examine teacher educator practices that are supportive of preservice teachers' development of mathematical knowledge for teaching, and to reflect on their own practice and collaborate with other teacher educators.

The topic of each of the workshops was based on our analysis of teacher educator practices, with a particular focus on the work involved in implementing the tasks of teaching that require specialized content knowledge, such as representing mathematical ideas using different representations, mapping between mathematical representations, and understanding students' alternative strategies (see Table 1). Both authors collaboratively facilitated the workshop discussions; we led the whole group discussion and made in-the-moment decisions about the focus of the discussion. Workshops were offered twice each semester during the project. All participants attended each workshop meeting, with the exception of two participants who each missed one workshop.

The design of the workshops was drawn from a multimedia database, developed across five years, regarding a required mathematics content course for preservice teachers. The database includes detailed lesson plans, which provide both a description of the evolving course content as well as rationales for design and instructional decisions; PowerPoint slides used during each lesson; over 200 hours of videotaped class sessions; over 100 hours of audiotaped small group discussions; videotaped clinical interviews with preservice teachers; transcripts of all video and audio data; photographs of preservice teachers' board work generated during class; digitized copies of preservice teachers' classwork, exams, and course notebooks; and audio recordings of planning meetings. All data were collected from content courses taught by the authors as well as their colleagues.

Using the multimedia database, we took a top-down approach rather than a bottom-up approach in regard to the workshop design. First, the design of the workshop started with a particular task that requires specialized content knowledge that the authors posit is instructionally challenging for teacher educators to implement. Second, with the focus

Table 1
Workshop Schedule and Topics

01-20-2012	Workshop 1: Mapping Between Representations
03-02-2012	Workshop 2: Analyzing Student Errors
09-07-2012	Workshop 3: Formulating and Revising Mathematical Definitions
11-01-2012	Workshop 4: Justification and Proof for Preservice Elementary Teachers
04-26-2013	Workshop 5: Evaluating Mathematical Explanations
09-20-2013	Workshop 6: Multiplication and Division of Fractions

on specialized content knowledge task of teaching identified, mathematics tasks relevant to the task of teaching were retrieved from the multimedia database. Related artifacts, such as student work samples and classroom videos, were similarly identified and retrieved from the database. Based on the overall quantity and quality of the artifacts for each task, artifacts from one or two tasks were subjected to a detailed examination, referred to as artifact analysis. The artifact analysis included both a student work analysis and classroom video analysis. The goals of the student work analysis were to (a) identify preservice teachers' common solution strategies and common misconceptions, and (b) select representative student work samples that have potential to elicit productive discussions of and facilitate teacher educator learning around the specialized content knowledge task of teaching. Meanwhile, the goals of the classroom video analysis included (a) identifying and analyzing teacher educators' teaching practices of teaching specialized content knowledge, and (b) identifying teaching practices that appear to be particularly challenging for teacher educators.

The *Mapping Between Representations* workshop provides a useful example. First, we selected mathematics tasks that could be solved by using different mathematical representations, which include the Candy Box Problem and the Cake Problem. Then, we retrieved student work samples and videos related to the two tasks and then chose to focus on the Candy Box Problem because the nature of the student work samples and the video artifacts made visible various aspects of preservice teacher thinking and misconceptions. Specifically, for the student work artifacts, both class notes and assignments were used because the class notes reveal the thinking processes involved in solving the problem, and the assignments included completed solutions to the problem. We coded the student work in terms of two strategy categories that emerged from the data: an algebraic approach and a pictorial approach as well as forward thinking and backward thinking. For the classroom video artifacts, episodes that involved a teacher educator's facilitating small-group problem solving and whole-group discussions of solution strategies were analyzed, guided by the following questions: What is the teacher educator doing? What aspects of specialized content knowledge are involved? How is what the teacher educator does related to the specialized content knowledge task of teaching?

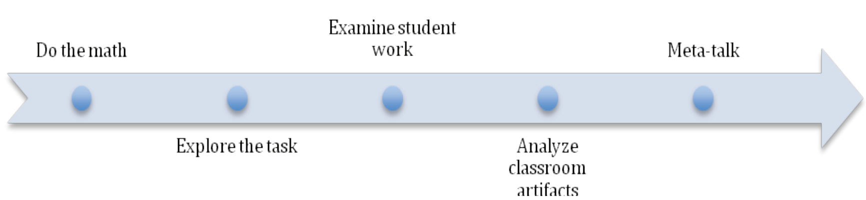
Structure of the Workshops

Each workshop lasted about three hours and consisted of two major components: *Case Study* and *Collaborative Lesson Planning*. The *Case Study* component comprised about two hours of time during the workshop, and the *Collaborative Lesson Planning* component comprised about an hour and was often extended to post-workshop meetings among participants and in their classrooms. The activities for the Case Study component included: *Do the math, Explore the task, Examine student work, Analyze classroom artifacts*, and *Consider meta-talk* (see Figure 1). *Do the math* and *Explore the task* activities build the foundation for the discussion of student work and classroom artifacts, while *Consider[ing] meta-talk* is an opportunity for teacher educators to think about how to connect the mathematics content from the previous activities to teaching practice.

The *Case Study* component began with *Do the math*. First, participants were invited to solve the mathematics task related to the case study, which served as the foundation for the entire workshop; then, participants moved to *Explore the task*, for which they explored the task in terms of different mathematical and pedagogical aspects, e.g., what mathematical goals could be accomplished with this task; anticipate how preservice teachers might solve this task. All of the mathematics tasks used in the workshops were selected from the database. The mathematics content of the tasks was generally topic areas with which preservice teachers struggle conceptually, e.g., fractions. Overall, the tasks used in the workshops encourage multiple solution strategies and/or elicit preservice teachers' misconceptions.

The activities in *Do the math* and *Explore the task* provided participants with opportunities to engage in mathematical problem solving; to consider the mathematical task goals, key concepts, and challenges of the task relative to their preservice teachers' knowledge and skills; and to anticipate preservice teachers' solution strategies and misconceptions. In this way, the design of the *Do the math* and *Explore the task* activities reflects two of the aforementioned features of high-quality professional development: grounding participants' learning in the content of teaching and learning, and embedding participants' learning in the work of teaching preservice teachers.

Figure 1
Workshop Activities



Following these two workshop activities, participants were invited to analyze student, i.e., preservice teacher, work and scrutinize selected student work samples and/or video clips of interviews in the *Examine student work* activity. We chose student work samples from multiple data sources to illustrate preservice teachers' common strategies, alternative strategies, and/or strategies that reveal unusual mathematical thinking. These data sources included preservice teachers' notebooks, homework assignments, and videotaped clinical interviews.

Understanding preservice teachers' mathematical thinking is an important aspect of teacher educators' practice. The *Examine student work* activity provided opportunities for participants to investigate preservice teachers' mathematical thinking, to find evidence of learning, and to raise questions about the nature of preservice teachers' understanding. Specifically, with the written work samples, participants had opportunities to focus on understanding the mathematical ideas represented in the work sample, while, with the videotaped interviews, participants had opportunities to listen to preservice teachers explain their thinking. In these ways, the design of the *Examine student work* activity is guided by one of the aforementioned features of high-quality professional development: create some disequilibrium for participants. The disequilibrium in the *Examine student work* activity is generated from preservice teachers' alternative strategies or mathematical ideas that are communicated in their explanations of their thinking. Moreover, the use of student work provides a context in which teacher educators can collectively analyze an aspect of teaching practice (Kazemi & Franke, 2004).

During the *Analyze classroom teaching* activity, teacher educators were invited to investigate particular aspects of teaching practice as it unfolds in an actual content course. We selected video clips to illustrate teacher educators' work with preservice teachers, such as a teacher educator's interacting with a small group of preservice teachers around a task; a teacher educator's facilitating a whole-group discussion of a task; and a teacher educator's commenting and highlighting aspects of preservice teachers' mathematical explanations. In addition, the video clips that we select exemplify good models of teaching preservice teachers; some may present an unexpected teaching moment, e.g., the emergence of an unanticipated solution strategy and the subsequent instructional moves, and some clips may show potentially missed teaching opportunities. Indeed, video can be a useful medium for promoting discussions about teaching practice (Borko, Jacobs, Eiteljorg, & Pittman, 2008).

Analyzing teaching practices and considering the influence on student learning is a critical aspect of professional learning and can be accomplished using a variety of tools and records of practice (Elliott, Kazemi, Lesseig, & Mumme, 2009; Kazemi, & Franke, 2004; LeFevre, 2004). Teacher educators may reflect frequently on their own teaching practice but may have minimal opportunities to look inside other teacher educators' practices or to share their thinking about others' teaching practices. The *Analyze classroom teaching* activity is designed to provide opportunities for teacher educators to examine classroom teaching cooperatively, reflect on a sequence of instructional moves, and suggest alternative ways to facilitate preservice teachers' work. This activity reflects the aforementioned features of connecting to participants' teaching practice and encouraging collaboration among participants.

The discussion of instructional artifacts with the focus on mathematical thinking and instructional moves leads into the *Consider[ing] meta-talk* activity. Meta-talk is focused on making knowledge of and about mathematics explicit, and connecting mathematical knowledge to the work of teaching children (Zopf, 2010). Thus, this activity is an opportunity for teacher educators to consider how to help preservice teachers connect the work on the task to teaching practice. Unlike for K-12 teachers, meta-talk is unique to the practice of teaching teachers. Teacher educators not only have to develop preservice teachers' knowledge of the mathematics content, but they also have to make explicit connections between the content taught and the work of teaching children. In this way, the *Consider[ing] meta-talk* activity is designed to reflect the feature of embedding participants' learning in the work of teaching teachers.

When engaging in the second main component of the workshops, *Collaborative lesson planning*, teacher educators worked with a partner or in small groups to identify and plan for a lesson(s) that is directly related to the topic in the *Case Study* component (see Table 1). During this component, the goal for each participant was to generate a lesson plan for use in his or her own content course, which is accomplished in consultation with other participants. First, each teacher educator selected a relevant task or activity that he or she would be using in his or her content course in the upcoming month. For example, in Workshop I: *Mapping Between Representation*, teacher educators, in consultation with the workshop facilitators and other participants, selected tasks that involved the use of multiple representations from their course textbooks and materials. Then, teacher educators considered and discussed the selected task or activity based on the questions from a modified version of the Thinking Through a Lesson Protocol (TTLP; Smith & Bill, 2004), which requires, among other things, teacher educators to pay careful attention to the issue of "meta-talk" in their teaching practice. The lesson plan generated is based on the structure of the TTLP. Teacher educators were encouraged to try out the written lesson and to bring back artifacts from their content course to share their experiences and challenges with other participants in subsequent workshops. In this way, teacher educators not only received feedback from other teacher educators on the initial development of their lesson plans but also received feedback on their implemented lesson plans as they debriefed their lessons with teacher educators in subsequent workshops. This activity reflects the aforementioned features of encouraging collaboration among participants and embedding participants' learning in the work of teaching teachers.

Looking Inside a Workshop: Mapping Between Representations

Mapping between representations is an important mathematical practice for teaching mathematics. The workshop *Mapping Between Representations* is designed to make explicit this practice for teacher educators and to support teacher educators in supporting preservice teachers in mapping between different mathematical representations.

Do the math and explore the task. The task for the workshop is the Candy Box Problem (see Figure 2). The task is mathematically rich and embedded in a real-world context for exploring fractions. Most importantly, the task is designed to elicit multiple solution strategies and representations.

Teacher educators first were asked to solve the task in as many different ways possible and to compare their different solution strategies and representations with other participants. For example, in the Candy Box Problem, teacher educators can use algebra and set up an equation to find the answer, or they can solve the problem by representing the problem with a drawing. The pictorial representations can include continuous rectangular models, e.g., Figure 3 is a continuous rectangular model, continuous circle models, or discrete models.

Figure 2.
Mathematics Task

The Candy Box Problem

There was a box of candy on the table. Jenny was hungry because she hadn't had breakfast, so she ate half the candy. Then, Shannon came along and noticed the candy. She thought that it looked good, and had not packed a lunch, so she took two-thirds of what was left in the box. Katina came by and decided to take three-fourths of the remaining candies with her to her next class. Then, Rhonda came dashing up and took one piece of candy to munch on. When Liliana looked at the candy box, she saw that there was just one piece of candy left. "How many pieces of candy were there in the box to begin with?" she asked Jenny suspiciously.

Then, the teacher educators are invited to examine the task with the guidance presented in Figure 4, such as considering the goals of the task as well as preservice teachers' potential strategies, misconceptions, and errors. The guiding questions seen in the figure are based on the TTLP (Smith & Bill, 2004). For example, one participant pointed out that one of the critical ideas in this task is changing the referent whole of the different fractions. If a preservice teacher is not aware of this, he or she might write 2/3x to represent the portion taken by Shannon.

Figure 3
Forward Strategy and Pictorial Representation

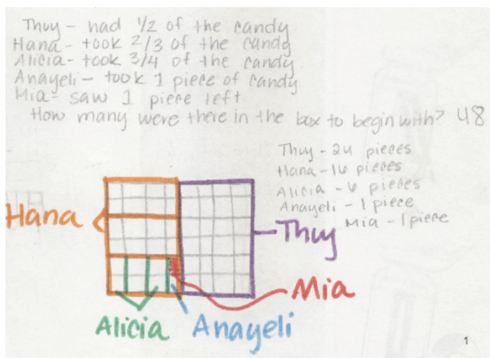


Figure 4
Guiding Questions for Exploring the Task

1. What are the mathematical goals that can be achieved with this task?
2. What definitions, concepts and/or ideas do preservice teachers need to know to begin work on the task?
3. What are the possible ways preservice teachers may solve this task?
   a. Which of these methods do you think your preservice teachers would use?
   b. What misconceptions might preservice teachers have?
   c. What errors might preservice teachers make?
   d. What are the challenges associated with the specialized content knowledge aspects of this task that preservice teachers may face?
4. What would your mathematical expectations be for your preservice teachers who work on this task?

Examine student work. After exploring the task, teacher educators have an opportunity to examine student work from the Candy Box Problem. Eight student work samples were selected to include forward strategies, backward strategies, pictorial representations and algebraic representations, correct answers, and partially correct answers. Student work in Figure 2 features a forward strategy with a pictorial representation, while the student work in Figure 5 features a forward strategy with an algebraic representation.

Guiding questions for examining student work (Figure 6) direct teacher educators to compare preservice teachers' strategies as well as to consider what strategies to share and in what order. These questions are also based on the TTLP (Smith & Bill, 2004). One participant stated that she would have preservice teachers present the pictorial strategies first, given that the diagram can provide a visualization for finding the algebraic solution.

Analyze classroom teaching. A classroom episode for the Candy Box Problem (see Figure 7) was presented to participants in the workshop. In this episode, the teacher educator in the video clip was facilitating a whole-class discussion of three different strategies that preservice teachers presented on the blackboard. The video clip provides opportu-

Figure 5
Forward Strategy and Algebraic Representation

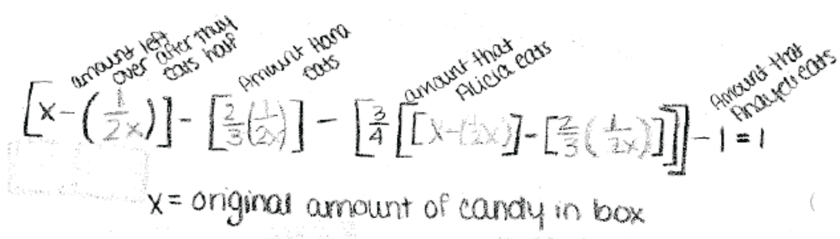


Figure 6
Guiding Questions for Examining Student Work

1. What is similar about the different solution methods? What is different?

2. Which solution methods would you want to have shared during the class discussion? In what order will the solutions be presented? Why?

3. What specific questions would you ask so that preservice teachers make connections between the different solution methods that are shared in class?

nities for participants to consider the theme of the workshop, *Mapping Between Representations*.

Teacher educators were provided the transcript for the classroom teaching episode and guiding questions (see Figure 8) to analyze the classroom teaching depicted in the video clip. The guiding questions for this particular episode mainly focus on the teacher educator's questioning practice. By analyzing the questions asked by the teacher educator, participants have an opportunity to reflect on the ways in which one might support preservice teachers in mapping between mathematical representations.

Consider strategies for "meta-talk." A meta-talk discussion around this task provides opportunities for preservice teachers to reflect on the Candy Box Problem: What is the mathematics involved in the task? How is the idea of mapping between representations related to teaching children? The *Consider[ing] meta-talk* activity in the workshop is designed to prepare participants to lead reflective discussions in their

Figure 7
Screen Shot of a Classroom Teaching Episode

Figure 8
Guiding Questions for Analyzing Classroom Teaching

As you watch the video clip, consider the following questions:

1. What questions did the instructor ask preservice teachers?
2. What are the mathematical purposes of these questions?
3. What are the pedagogical purposes of these questions?

own content courses. The guiding questions in Figure 9 are intended to elicit teacher educators' strategies to engage in meta-talk around the Candy Box Problem. For example, one participant mentioned that preservice teachers might be able to solve the Candy Box Problem, but they might not know explicitly what knowledge of fractions they used to solve the problem, such as representing fractions with different wholes or finding a fraction of another fraction. The meta-talk discussion is crucial in a content course, as it requires preservice teachers to reflect on the mathematics they do and how it connects to teaching practice. Further, this type of activity can provide opportunities for teacher educators to engage in collective reflection on the mathematical knowledge and aspects of teaching practice that can be connected to the task as well as the strategies that they can use that can help preservice teachers make such connections.

Conclusion

While there is considerable research on preservice teacher education in mathematics, including the nature of preservice teachers' understanding in different mathematical domains and what they learn in methods and content courses, much less is known about what mathematics teacher educators do as they develop preservice teachers' mathematical knowledge in ways needed for teaching these courses. Specifically, we need to know what types of professional development experiences teacher educators need to develop specialized knowledge of mathematics needed for teaching teachers. We posit that mathematics teacher educators need to understand mathematical knowledge for teaching for themselves and should be knowledgeable about ways to

Figure 9
Guiding Questions for Meta-Talk

As you watch the video clip, consider the following questions:

1. How will you discuss the structure of the mathematical work that underlies the Candy Box problem?

2. How will you make a mathematical point explicit?

3. How will you connect the mathematical ideas in the task to the work of teaching mathematics to children?

4. What rationale will you provide for the task that connects it to teaching children?

connect preservice teachers' mathematical learning to the practice of teaching K-12 students. Building on our ongoing work as part of the MKTT project, we propose a model for the professional development of teacher educators that provides opportunities for teacher educators to develop and refine precisely these aspects of their work. Drawing from professional development research, our model has the following features central to its design: learning is grounded in the content of teaching and learning, activities should create disequilibrium for teacher educators and encourage collaboration among teacher educators, and learning is embedded in or directly related to the work of teaching teachers.

Although our ongoing work with teacher educators is situated within a specific group of institutions, we have learned a considerable amount from participating teacher educators about their work, which has potential implications for the larger community of teacher educators who teach mathematics content courses for preservice teachers. For example, prior to starting the workshops, all of the teacher educators in the project had very different conceptions of the practice of connecting preservice teacher learning of content to teaching practice. Such conceptions included the following: "Connecting preservice teacher learning of content to teaching should not be the focus of content courses because preservice teachers need mathematics content to understand the practice" and, "Connecting preservice teacher learning of content to teaching is a catalyzer of content learning." By the end of the six workshops, four of the participating teacher educators reported engaging in different ways of connecting content learning to teaching practices within their own practice, such as showing videos of K-12 classrooms, providing personal anecdotes of classroom teaching, and using K-12 curriculum materials as part of the content course curriculum.

Further, by the end of the workshops, three of these teacher educators indicated that they struggle to find other means by which to meaningfully connect what preservice teachers are learning to K-12 teaching practice. This sentiment suggests that connecting content learning to teaching practices is an important part of the work of some of the teacher educators. We posit that the practice of connecting learning content to future teaching is one of the practices of teacher educators, perhaps beyond those participants in our project, and, thus, considering ways of supporting other teacher educators' learning about this particular practice and its need appears crucial. Moving forward as part of the project, then, will involve revising certain aspects of the workshop components based on what we have learned, e.g., developing workshops around certain common content knowledge topics that are challenging for preservice teachers, and developing new workshops focused on other specialized content knowledge tasks. Doing so also may necessitate revisions to the professional development model.

We hope that the model we have outlined here for the professional development of teacher educators will begin a broader discussion about how to structure professional development for mathematics teacher educators and what knowledge is entailed by the work of teaching teachers. This is particularly important, as the phrase "mathematics teacher educators" is often used to refer to individuals who teach content courses for preservice teachers, which includes mathematicians, graduate students, mathematics educators, and classroom teachers who not only have different professional backgrounds but who are often not always professionally prepared for the work of teaching preservice teachers.

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